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Recent results on well-posedness and optimal control for a class of generalized fractional Cahn–Hilliard systems*

by

Pierluigi Colli[1], Gianni Gilardi[1] and Jürgen Sprekels[2]

[1]Dipartimento di Matematica "F. Casorati", Università di Pavia
and Research Associate at the IMATI – C.N.R. Pavia
via Ferrata 5, 27100 Pavia, Italy
pierluigi.colli@unipv.it, gianni.gilardi@unipv.it
[2] Department of Mathematics
Humboldt-Universität zu Berlin
Unter den Linden 6, 10099 Berlin, Germany
and
Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstrasse 39, 10117 Berlin, Germany
sprekels@wias-berlin.de

Dedicated to Günter Leugering
on the occasion of his 65th birthday with best wishes

Abstract: In this paper, we give an overview of results for Cahn–Hilliard systems involving fractional operators that have recently been established by the authors of this note. We address problems concerning existence, uniqueness, and regularity of the solutions to the system equations, and we study optimal control problems for the systems. The well-posedness results are valid for a wide class of fractional operators of spectral type and for the typical double-well nonlinearities appearing in the Cahn–Hilliard system equations, namely the classical differentiable, the logarithmic, and the nondifferentiable double obstacle potentials. While this also applies to the existence of optimal controls in the related optimal control problems, the establishment of first-order necessary optimality conditions requires imposing much stronger assumptions on the admissible class of fractional operators. One main reason for this is the necessity of deriving suitable differentiability properties for the associated control-to-state mapping. Nevertheless, it turns out that also in the singular case of logarithmic potentials, the first-order necessary optimality conditions can be established under suitable assumptions, and a "deep quench" approximation, based on the results derived for logarithmic nonlinearities makes even the case of double obstacle potentials accessible.

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded, and connected set with Lipschitz boundary Γ and outward normal derivative $\partial_{\mathbf{n}}$, let $T > 0$ be a final time, and let $H := L^2(\Omega)$ denote the Hilbert space of square-integrable real-valued functions defined on Ω, endowed with the standard inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, respectively. We set $Q_t := \Omega \times (0,t)$ for $0 < t < T$ and $Q := \Omega \times (0,T)$. We investigate in this paper an abstract system of variational (in)equalities, namely, we look for functions (μ, y) with the regularity

$$y \in H^1(0,T;V_A^{-r}) \cap L^\infty(0,T;V_B^\sigma) \quad \text{and} \quad \tau\partial_t y \in L^2(0,T;H), \tag{1.1}$$

$$\mu \in L^2(0,T;V_A^r), \tag{1.2}$$

$$f_1(y) \in L^1(Q), \tag{1.3}$$

which satisfy

$$\langle \partial_t y(t), v\rangle_{A,r} + (A^r\mu(t), A^r v) = 0 \quad \text{for every } v \in V_A^r \text{ and a.e. } t \in (0,T), \tag{1.4}$$

$$\begin{aligned} &(\tau\partial_t y(t), y(t) - v) + (B^\sigma y(t), B^\sigma(y(t) - v)) + \int_\Omega f_1(y(t)) \\ &\quad + (f_2'(y(t)) - u(t), y(t) - v) \le (\mu(t), y(t) - v) + \int_\Omega f_1(v) \end{aligned}$$

$$\text{for every } v \in V_B^\sigma \text{ and a.e. } t \in (0,T), \tag{1.5}$$

$$y(0) = y_0 \quad \text{in } \Omega. \tag{1.6}$$

Here, it is understood that $\int_\Omega f_1(v) = +\infty$ whenever $f_1(v) \notin L^1(\Omega)$. The precise meaning of the involved quantities and spaces will be given below. Notice that (1.4)–(1.6) is a generalized version of the evolutionary system

$$\partial_t y + A^{2r}\mu = 0, \tag{1.7}$$

$$\tau\partial_t y + B^{2\sigma} y + f_1'(y) + f_2'(y) = \mu + u, \tag{1.8}$$

$$y(0) = y_0. \tag{1.9}$$

Here, $\tau \ge 0$ is a constant, $f_2 : \mathbb{R} \to \mathbb{R}$ is a smooth function, and $f_1 : \mathbb{R} \to [0, +\infty]$ denotes a convex and lower semicontinuous function with $f_1(0) = 0$, whose *effective domain* $\mathrm{dom}(f_1) := \{r \in \mathbb{R} : f_1(r) < +\infty\}$ is an interval in $\mathbb{R}$ (possibly $\mathbb{R}$ itself). The linear operators A^{2r} and $B^{2\sigma}$, with $r > 0$ and $\sigma > 0$, denote fractional powers (in the spectral sense) of the operators A and B. We will give a precise definition of such operators in Section 2 below. Denoting by $D(L)$ the domain of any linear operator $L : D(L) \subset X \to Y$ between two linear spaces X and Y, we generally assume:

(A1) $A : D(A) \subset H \to H$ and $B : D(B) \subset H \to H$ are unbounded, monotone, and selfadjoint linear operators with compact resolvents.

This assumption, which is satisfied by many standard differential operators with appropriate boundary conditions, implies that there are sequences $\{\lambda_j\}$ and $\{\lambda'_j\}$ of eigenvalues (ordered by their magnitudes) and orthonormal sequences $\{e_j\}$ and $\{e'_j\}$ of corresponding eigenvectors, that is,

$$Ae_j = \lambda_j e_j,\ Be'_j = \lambda'_j e'_j, \text{ and } (e_i, e_j) = (e'_i, e'_j) = \delta_{ij}, \text{ for } i, j = 1, 2, \ldots, \tag{1.10}$$

$$0 \le \lambda_1 \le \lambda_2 \le \ldots, \text{ and } 0 \le \lambda'_1 \le \lambda'_2 \le \ldots, \text{ with } \lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \lambda'_j = +\infty, \tag{1.11}$$

$$\{e_j\} \text{ and } \{e'_j\} \text{ are complete systems in } H. \tag{1.12}$$

1.1. Physical background

The state system (1.7)–(1.9) (and thus its weak version (1.4)–(1.6)) can be seen as a generalization of the famous Cahn–Hilliard system originally introduced in Cahn and Hillard (1958) and first studied mathematically in Elliott and Zheng (1986) (for a large list of references on the original Cahn–Hilliard system, see Heida, 2015). The Cahn–Hilliard system models the physical evolution of a phase separation process taking place in the container Ω. In this connection, typically the operators A^{2r} and $B^{2\sigma}$ are replaced by standard Laplace operators with zero Neumann or Dirichlet boundary conditions, and τ is a nonnegative relaxation parameter, where for the classical Cahn–Hilliard system one has $\tau = 0$ (the *nonviscous* case), while $\tau > 0$ corresponds to the *viscous* case. The unknown functions y and μ stand for the *order parameter* (usually a scaled density of one of the involved phases) and the *chemical potential* associated with the phase separation, respectively. Moreover, $f := f_1 + f_2$ denotes a double-well potential. Typical and physically significant examples for f are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$f_{\text{reg}}(r) := \frac{1}{4}\left(r^2 - 1\right)^2, \quad r \in \mathbb{R}, \tag{1.13}$$

$$f_{\log}(r) := \begin{cases} (1+r)\ln(1+r) + (1-r)\ln(1-r) - c_1 r^2 & \text{if } r \in (-1,1) \\ 2\ln(2) - c_1 & \text{if } r \in \{-1,1\} \\ +\infty & \text{if } r \notin [-1,1] \end{cases} \tag{1.14}$$

$$f_{\text{obs}}(r) := I_{[-1,1]}(r) + c_2(1 - r^2) = \begin{cases} c_2(1 - r^2) & \text{if } |r| \le 1 \\ +\infty & \text{if } |r| > 1 \end{cases} \tag{1.15}$$

where $I_{[-1,1]}$ denotes the indicator function of the interval $[-1, 1]$, defined by $I_{[-1,1]}(r) = 0$ if $r \in [-1, 1]$ and $I_{[-1,1]}(r) = +\infty$ if $r \notin [-1, 1]$. The constants c_1

and c_2 are chosen in such a way that $f_{\log}$ and f_{obs} are nonconvex; the reader can easily verify that this is the case if $c_1 > 1$ and $c_2 > 0$. Observe that obviously $\mathrm{dom}(f_{\mathrm{reg}}) = \mathbb{R}$, while $\mathrm{dom}(f_{\log}) = \mathrm{dom}(f_{\mathrm{obs}}) = [-1, 1]$.

Recently, in Colli, Gilardi and Sprekels (2019a), Theorems 2.6 and 2.8, it was shown that (1.4)–(1.6) admits a solution (μ, y) satisfying (1.1)–(1.3), where the admissible nonlinearities include all of the three cases (1.13)–(1.15). In the analysis, it turned out that the first eigenvalue λ_1 of A plays an important role. Indeed, the main assumption for the operators A, B (besides **(A1)**) was the following:

(A2) Either

(i) $\lambda_1 > 0$

or

(ii) $0 = \lambda_1 < \lambda_2$, and e_1 is a constant function that also belongs to the domain of B^σ.

The existence proof in Colli, Gilardi and Sprekels (2019a) was based on Moreau–Yosida approximation. It turned out that the second solution component y and the expression $A^r\mu$ are always uniquely determined, while this is not necessarily the case for μ. Below, we will give sufficient conditions under which also μ is uniquely determined.

REMARK 1 *The condition* **(A2)**(i) *is satisfied for many standard elliptic operators with zero Dirichlet boundary conditions (however, also zero mixed boundary conditions could be considered, with proper definitions of the domains of the operators). For instance, A can be the Laplace operator $-\Delta$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ or the bi-harmonic operator Δ^2 with the domain $D(\Delta^2) = H^4(\Omega) \cap H_0^2(\Omega)$. The second case* **(A2)**(ii)*, in which the strict inequality means that the first eigenvalue $\lambda_1 = 0$ is simple, arises in both of the following important situations: $A = -\Delta$ with zero Neumann boundary conditions, where $D(-\Delta) = \{v \in H^2(\Omega) : \ \partial_{\mathbf{n}} v = 0 \ \text{on } \Gamma\}$, or $A = \Delta^2$, with the boundary conditions encoded in the definition of the domain $D(\Delta^2) = \{v \in H^4(\Omega) : \ \partial_{\mathbf{n}} v = \partial_{\mathbf{n}} \Delta v = 0 \ \text{on } \Gamma\}$.*

1.2. Optimal control problems for smooth potentials

Besides the results, concerning the well-posedness of the system (1.4)–(1.6), we are interested in control problems for this system. More precisely, we consider the following distributed control problem:

(CP) Minimize the tracking-type cost functional

$$\mathcal{J}(y, u) := \frac{\beta_1}{2} \|y(T) - y_\Omega\|^2 + \frac{\beta_2}{2} \int_0^T \|y(t) - y_Q(t)\|^2 \, dt + \frac{\beta_3}{2} \int_0^T \|u(t)\|^2 dt \tag{1.16}$$

over the admissible set

$$\mathcal{U}_{\mathrm{ad}} := \left\{ u \in H^1(0, T; L^2(\Omega)) : |u| \le \rho_1 \ \text{ a.e. in } , Q, \ \|u\|_{H^1(0,T;L^2(\Omega))} \le \rho_2 \right\}, \tag{1.17}$$

and subject to the state system (1.4)–(1.6).

For the analysis of **(CP)**, the general assumptions **(A1)** and **(A2)** are not sufficient. Indeed, in order to be able to prove that the control-to-state operator $u \mapsto y$ is Fréchet differentiable between suitable Banach spaces, it seems to be indispensable to be able to show that the solution component y almost everywhere attains its value in a compact interval $[a, b]$, contained in $\operatorname{int}(\operatorname{dom}(f_1))$, the interior of the effective domain of f_1. In Colli, Gilardi and Sprekels (2018c), Examples 1-3, three situations were presented, in which such a property of y can be guaranteed. One of these cases is the logarithmic potential (1.14), for which one needs to separate y away from the critical arguments ± 1, which turns out to be possible under proper assumptions. In Colli, Gilardi and Sprekels (2018c), it was shown that then, under additional conditions, the Fréchet differentiability of the control-to-state operator $u \mapsto y$ between suitable Banach spaces can be proved; this eventually led to the derivation of proper first-order necessary optimality conditions. We will address these results in Section 4.2.

1.3. Optimal control problems for the double obstacle potential: the "deep quench" approximation

A different situation arises in the optimal control problem (which in the following will be denoted by $(\mathbf{CP}_0)$) for the double obstacle potential (1.15). In this case, we have $f_1 = I_{[-1,1]}$ in (1.5), and the condition (1.3) only yields that the possible states must satisfy $y \in [-1, 1]$ almost everywhere in Q (and thus $\int_\Omega f_1(y(t)) = 0$ for almost every $t \in (0, T)$ in (1.5)). Now, $\operatorname{dom}(I_{[-1,1]}) = [-1, 1]$, and since it seems to be impossible to separate y away from ± 1 in this case, the condition mentioned above in Section 1.2 is violated, and thus the theory of Colli, Gilardi and Sprekels (2018c) does not apply directly. In such situations, the so-called "deep quench" approximation has proven to be a useful tool in a number of cases in the framework of Cahn–Hilliard systems (see, e.g., Colli, Farshbaf-Shaker and Sprekels, 2015; Colli et al., 2015a; Colli, Gilardi and Sprekels 2017, 2019b; Colli and Sprekels, 2018). Also in the recent paper, Colli, Gilardi and Sprekels (2018d), we have chosen to approach the problem via the deep quench approximation. The general idea behind this approach is the following: we choose a monotone increasing function $\varphi \in C(0, 1]$ satisfying

$$\varphi(\alpha) > 0 \text{ in } (0, 1] \quad \text{and} \quad \lim_{\alpha \searrow 0} \varphi(\alpha) = 0, \tag{1.18}$$

and define the logarithmic functions

$$h(r) := \begin{cases} (1+r)\ln(1+r) + (1-r)\ln(1-r) & \text{if } r \in (-1, 1) \\ 2\ln(2) \quad \text{if } r \in \{-1, 1\} \\ +\infty \quad \text{if } r \notin [-1, 1] \end{cases} \tag{1.19}$$

$$h^\alpha(r) := \varphi(\alpha) h(r) \quad \text{for } r \in \mathbb{R} \text{ and } \alpha \in (0, 1]. \tag{1.20}$$

In view of (1.18), it is easily seen that

$$\lim_{\alpha \searrow 0} h^\alpha(r) = I_{[-1,1]}(r) \quad \forall\, r \in \mathbb{R}. \tag{1.21}$$

Moreover, $h'(r) = \ln(\frac{1+r}{1-r})$ and $h''(r) = \frac{2}{1-r^2}$, and thus

$$\lim_{\alpha \searrow 0} (h^\alpha)'(r) = 0 \ \ \forall\, r \in (-1,1), \quad \lim_{\alpha \searrow 0} \Big(\lim_{r \searrow -1} (h^\alpha)'(r) \Big) = -\infty,$$
$$\lim_{\alpha \searrow 0} \Big(\lim_{r \nearrow 1} (h^\alpha)'(r) \Big) = +\infty. \tag{1.22}$$

Hence, we may regard the graphs of the single-valued functions $(h^\alpha)'$ over the interval $(-1,1)$ as approximations to the graph of the subdifferential $\partial I_{[-1,1]}$. Observe that this is an *interior* approximation defined in the interior of the domain of $\partial I_{[-1,1]}$, in contrast to the *exterior* approximation obtained via the Moreau-Yosida approach.

In view of (1.21)–(1.22), it comes to one's mind to expect that the control problem $\mathbf{(CP}_0\mathbf{)}$ is closely related to the control problem (which in the following will be denoted by $\mathbf{(CP}_\alpha\mathbf{)}$) that arises when we choose in (1.5) $f_1 = h^\alpha$ for $\alpha > 0$. Indeed, by virtue of Colli, Gilardi and Sprekels (2019a), Theorems 2.6 and 2.8, the system (1.4)–(1.6) enjoys for both $f_1 = I_{[-1,1]}$ and $f_1 = h^\alpha$ a solution pair (μ, y) and (μ^α, y^α), respectively, and it turns out (see Colli, Gilardi and Sprekels, 2018d, Section 3) that (μ^α, y^α) converges in a suitable topology to (μ, y) as $\alpha \searrow 0$. Moreover, the optimal control problem $\mathbf{(CP}_\alpha\mathbf{)}$ belongs to the class of problems for which in Colli, Gilardi and Sprekels (2018c) the Fréchet differentiability of the control-to-state operator has already been proven and first-order necessary optimality conditions in terms of a variational inequality and the adjoint state system have been established. One can therefore hope to perform a passage to the limit as $\alpha \searrow 0$ in the state and the adjoint state variables in order to derive meaningful first-order necessary optimality conditions also for $\mathbf{(CP}_0\mathbf{)}$. We will show in Section 4.3 that this strategy actually succeeds, where, however, the problem $\mathbf{(CP}_\alpha\mathbf{)}$ will have to be replaced by a suitable *adapted* version.

1.4. Overview of related contributions

Let us add a few remarks on the existing literature. There exist numerous contributions on viscous/nonviscous, local/nonlocal, convective/nonconvective Cahn–Hilliard systems for the classical (non-fractional) case $A = B = -\Delta$, $2r = 2\sigma = 1$, or some nonlocal counterparts thereof, where various types of boundary conditions (e.g., Dirichlet, Neumann, dynamic) and different assumptions on the nonlinearity f were considered. We refer the interested reader to Colli, Gilardi and Sprekels (2018a) for a selection of associated references. Some papers also address the coupled Cahn–Hilliard/Navier–Stokes system (see, e.g., Frigeri, Gal and Grasselli, 2016; Frigeri et al., 2019, and the references given therein).

The literature on optimal control problems for non-fractional Cahn–Hilliard systems is still scarce. The case of Dirichlet and/or Neumann boundary conditions for various types of such systems were the subject of, e.g., the works by Colli et al. (2017), Colli, Gilardi and Sprekels (2016a, 2017), Duan and Zhao (2015), Wang and Nakagiri (2000), Zheng (2015), Zheng and Wang (2015), while the case of dynamic boundary conditions was studied in Colli et al. (2015a,b); Colli, Farshbaf-Shaker and Sprekels (2015); Colli, Gilardi and Sprekels (2015, 2016b, 2018b, 2019b); Colli and Sprekels (2018); Fukao and Yamazaki (2018), and Gilardi and Sprekels (2019). The optimal control of convective Cahn–Hilliard systems was addressed in Rocca and Sprekels (2015) and Zhao and Liu (2013, 2014), while the papers by Biswas, Dharmatti and Mohan (2018a,b,c); Frigeri, Grasselli and Sprekels (2018); Frigeri, Rocca and Sprekels (2016); Hintermüller et al. (2018); Hintermüller, Keil and Wegner (2018); Hintermüller and Wegner (2012, 2014, 2017), as well as by Tachim Medjo (2015) were concerned with coupled Cahn–Hilliard/Navier–Stokes systems.

While there are numerous papers on the general properties of fractional operators (for an extensive account of the existing literature, we refer the reader to the recent paper by Bonito et al., 2018), there are only few contributions to the theory of Cahn–Hilliard systems involving fractional operators. Regarding the connection of well-posedness and regularity results, see Ainsworth and Mao (2017), Akagi, Schimperna and Segatti (2016, 2019) for the case of the fractional negative Laplacian with zero Dirichlet boundary conditions, where the paper by Ainsworth and Mao (2017) also contributes to the numerical analysis; general operators other than the negative Laplacian have apparently only been studied in Colli, Gilardi and Sprekels (2019a); Gal (2017a,b; 2018) (however, we refer in this connection also to Colli and Gilardi, 2019, where a phase field system of Caginalp type was investigated). As of now, aspects of optimal control have been scarcely dealt with even for simpler linear evolutionary systems involving fractional operators; for such systems, some identification problems were addressed in Geldhauser and Valdinoci (2018), Sprekels and Valdinoci (2017) (see also Antil, Otárola and Salgado, 2018, for the stationary elliptic case), while for optimal control problems for such cases we refer to Antil, Khatri and Warma (2019), Antil and Otárola (2015, 2018), Antil, Otárola and Salgado (2016), Antil, Pfefferer and Rogovs (2018), Antil and Warma (2017, 2019). However, to the authors' best knowledge, optimal control problems for Cahn–Hilliard systems with general fractional order operators have so far only been addressed in Colli, Gilardi and Sprekels (2018c,d).

The paper is organized as follows: the subsequent Section 2 brings some auxiliary functional analytic material, while in Section 3 the relevant results concerning the state system (1.4)–(1.6) will be discussed. Section 4 then brings an analysis of the optimal control problem **(CP)**, where existence (in Section 4.1) and first-order necessary optimality conditions are derived (see Section 4.2 for the smooth case, and Section 4.3 for the double obstacle case).

Throughout the paper, for a general Banach space X we denote by $\|\cdot\|_X$ and X^* its norm and dual space, respectively. However, particular symbols are adopted for the spaces to be introduced in the next section.

2. Fractional powers and auxiliary results

In this section, we collect some auxiliary material concerning functional analytic notions. To this end, we generally assume that the conditions **(A1)** and **(A2)** are satisfied. Using the facts summarized in (1.10)–(1.12), we can define the powers of A and B for an arbitrary positive real exponent. For the first operator, we have

$$V_A^r := D(A^r) = \Big\{ v \in H : \sum_{j=1}^{\infty} |\lambda_j^r (v, e_j)|^2 < +\infty \Big\}, \tag{2.1}$$

$$A^r v = \sum_{j=1}^{\infty} \lambda_j^r (v, e_j) e_j \quad \text{for } v \in V_A^r, \tag{2.2}$$

the series being convergent in the strong topology of H, due to the properties (2.1) of the coefficients. In principle, we could endow V_A^r with the graph norm and inner product to obtain a Hilbert space; however, it is more convenient to work with the equivalent (see Colli, Gilardi and Sprekels, 2019a, Proposition 3.1) Hilbert norm

$$\|v\|_{A,r}^2 := \begin{cases} \|A^r v\|^2 = \displaystyle\sum_{j=1}^{\infty} |\lambda_j^r (v, e_j)|^2 & \text{if } \lambda_1 > 0, \\ |(v, e_1)|^2 + \|A^r v\|^2 = |(v, e_1)|^2 + \displaystyle\sum_{j=2}^{\infty} |\lambda_j^r (v, e_j)|^2 & \text{if } \lambda_1 = 0 \end{cases} \tag{2.3}$$

and the corresponding inner product

$$(v, w)_{A,r} = (A^r v, A^r w) \quad \text{or} \quad (v, w)_{A,r} = (v, e_1)(w, e_1) + (A^r v, A^r w),$$
$$\text{depending on whether } \lambda_1 > 0 \text{ or } \lambda_1 = 0, \quad \text{for } v, w \in V_A^r. \tag{2.4}$$

Remark 2 *Observe that in the case of $\lambda_1 = 0$ the constant value of e_1 equals one of the numbers $\pm|\Omega|^{-1/2}$, where $|\Omega|$ is the volume of Ω. It follows for every $v \in H$ that the first term $(v, e_1)e_1$ of the Fourier series of v is the constant function, whose value is the mean value of v, which is defined by*

$$\operatorname{mean}(v) := \frac{1}{|\Omega|} \int_\Omega v \,. \tag{2.5}$$

In the same way as for A, starting from (1.10)–(1.12) for B, we can define the power B^σ of B for every $\sigma > 0$, where for V_B^σ we choose the graph norm. We therefore set

$$V_B^\sigma := D(B^\sigma), \quad \text{with the norm } \|\cdot\|_{B,\sigma} \text{ associated to the inner product}$$
$$(v, w)_{B,\sigma} := (v, w) + (B^\sigma v, B^\sigma w) \quad \text{for } v, w \in V_B^\sigma. \tag{2.6}$$

Remark 3 *Let us briefly comment on the condition* **(A2)**(ii)*. We notice that the condition that* e_1 *be a constant belonging to* V_B^σ *holds true for many operators having a domain that involves Neumann boundary conditions. This is the case, for instance, if* B *is the negative Laplacian with domain* $D(-\Delta) = \{v \in H^2(\Omega) : \ \partial_{\mathbf{n}} v = 0 \ \text{ on } \Gamma\}$*. On the contrary, if* $B = -\Delta$ *with domain* $D(-\Delta) := H^2(\Omega) \cap H_0^1(\Omega)$*, then* $D(B)$ *does not contain any nonzero constant functions. However,* V_B^σ *does contain every constant function provided that* $\sigma \in (0, \frac{1}{4})$*, since* V_B^σ *coincides with the usual Sobolev-Slobodeckij space* $H^{2\sigma}(\Omega)$ *in this case.*

To resume our preparations, we observe that if r_i and σ_i are arbitrary positive exponents, then it is easily seen that we have the "Green type" formulas

$$(A^{r_1+r_2} v, w) = (A^{r_1} v, A^{r_2} w) \quad \text{for every } v \in V_A^{r_1+r_2} \text{ and } w \in V_A^{r_2}, \tag{2.7}$$

$$(B^{\sigma_1+\sigma_2} v, w) = (B^{\sigma_1} v, B^{\sigma_2} w) \quad \text{for every } v \in V_B^{\sigma_1+\sigma_2} \text{ and } w \in V_B^{\sigma_2}. \tag{2.8}$$

The next step is the introduction of some spaces with negative exponents. We set

$$V_A^{-r} := (V_A^r)^* \quad \text{for } r > 0, \tag{2.9}$$

and endow V_A^{-r} with the dual norm $\|\cdot\|_{A,-r}$ of $\|\cdot\|_{A,r}$. We use the symbol $\langle \cdot\,, \cdot \rangle_{A,r}$ for the duality pairing between V_A^{-r} and V_A^r and identify H with a subspace of V_A^{-r} in the usual sense, i.e., in order that $\langle z, v\rangle_{A,r} = (z,v)$ for every $z \in H$ and $v \in V_A^r$. Similarly, we set

$$V_B^{-\sigma} := (V_B^\sigma)^* \quad \text{for } \sigma > 0. \tag{2.10}$$

As V_B^σ is dense in H, we have the analogous embedding

$$H \subset V_B^{-\sigma}. \tag{2.11}$$

Observe that the following embedding results are valid:

$$\text{The embeddings } V_A^{r_2} \subset V_A^{r_1} \subset H \text{ are dense and compact for } 0 < r_1 < r_2. \tag{2.12}$$

$$\text{The embeddings } H \subset V_A^{-r_1} \subset V_A^{-r_2} \text{ are dense and compact for } 0 < r_1 < r_2. \tag{2.13}$$

$$\text{The embeddings } V_B^{\sigma_2} \subset V_B^{\sigma_1} \subset H \text{ are dense and compact for } 0 < \sigma_1 < \sigma_2. \tag{2.14}$$

We comment only on (2.12), noting that (2.14) follows similarly and (2.13) is a consequence of (2.12). Clearly, the embeddings are dense. For the compactness, we just notice that $\lim_{j\to\infty} \lambda_j^{r_1-r_2} = 0$, so that the mapping that to each sequence $\{c_j\} \in \ell^2$ associates the sequence $\{\lambda_j^{r_1-r_2} c_j\}$ is compact from ℓ^2 into itself.

We also note the validity of the Poincaré type inequality (see Colli, Gilardi and Sprekels, 2019a, Eq. (3.5))

$$\|v\| \le \widehat{c}\,\|A^r v\| \quad \text{for every } v \in V_A^r \text{ with } \operatorname{mean}(v) = 0, \tag{2.15}$$

with a constant $\widehat{c} > 0$ depending on r.

At this point, we introduce the Riesz isomorphism $\mathcal{R}_r : V_A^r \to V_A^{-r}$, associated with the inner product (2.4), which is given by

$$\langle \mathcal{R}_r v, w \rangle_{A,r} = (v, w)_{A,r} \quad \text{for every } v, w \in V_A^r. \tag{2.16}$$

Moreover, we set

$$V_0^r := V_A^r \quad \text{and} \quad V_0^{-r} := V_A^{-r} \quad \text{if } \lambda_1 > 0, \tag{2.17}$$

$$\begin{aligned} &V_0^r := \{v \in V_A^r :\ \operatorname{mean}(v) = 0\} \ \text{ and} \\ &V_0^{-r} := \{v \in V_A^{-r} :\ \langle v, 1 \rangle_{A,r} = 0\} \quad \text{if } \lambda_1 = 0\,. \end{aligned} \tag{2.18}$$

According to Colli, Gilardi and Sprekels (2019a), Proposition 3.2, $\mathcal{R}_r$ maps V_0^r onto V_0^{-r} and extends to V_0^r the restriction of A^{2r} to V_0^{2r}. In view of this result, it is reasonable to use a proper notation for the restrictions of $\mathcal{R}_r$ and $\mathcal{R}_r^{-1}$ to the subspaces V_0^r and V_0^{-r}, respectively. We set

$$A_0^{2r} := (\mathcal{R}_r)_{|V_0^r} \quad \text{and} \quad A_0^{-2r} := (\mathcal{R}_r^{-1})_{|V_0^{-r}}\,, \tag{2.19}$$

where the index 0 has no meaning if $\lambda_1 > 0$ (since then $V_0^{\pm r} = V_A^{\pm r}$), while it reflects the zero mean value condition in the case of $\lambda_1 = 0$. We thus have

$$A_0^{2r} \in \mathcal{L}(V_0^r, V_0^{-r}), \quad A_0^{-2r} \in \mathcal{L}(V_0^{-r}, V_0^r) \quad \text{and} \quad A_0^{-2r} = (A_0^{2r})^{-1}\,, \tag{2.20}$$

$$\langle A_0^{2r} v, w \rangle_{A,r} = (v, w)_{A,r} = (A^r v, A^r w) \quad \text{for every } v \in V_0^r \text{ and } w \in V_A^r\,, \tag{2.21}$$

$$\langle f, A_0^{-2r} f \rangle_{A,r} = \|A_0^{-2r} f\|_{A,r}^2 = \|f\|_{A,-r}^2 \quad \text{for every } f \in V_0^{-r}. \tag{2.22}$$

Notice that (2.22) implies that

$$\langle f', A_0^{-2r} f \rangle_{A,r} = \frac{1}{2}\,\frac{d}{dt}\,\|f\|_{A,-r}^2 \quad \text{a.e. in } (0,T), \ \text{for every } f \in H^1(0,T;V_0^{-r}). \tag{2.23}$$

Moreover, by virtue of Colli, Gilardi and Sprekels (2019a), Proposition 3.3, we have

$$\bigl(A^r A_0^{-2r} f, A^r v\bigr) = \langle f, v \rangle_{A,r} \quad \text{for every } f \in V_0^{-r} \text{ and } v \in V_A^r. \tag{2.24}$$

In addition (see Colli, Gilardi and Sprekels, 2019a, Proposition 3.4), the operator $A^{2r} \in \mathcal{L}(V_A^{2r}, H)$ can be extended in a unique way to a continuous linear operator, still termed A^{2r}, from V_A^r into V_0^{-r}, and we have

$$\|A^{2r} v\|_{A,-r} \le \|A^r v\| \quad \text{for every } v \in V_A^r. \tag{2.25}$$

3. General assumptions and the state system

3.1. General assumptions and well-posedness

In this section, we state our general assumptions and discuss the properties of the state system (1.4)–(1.6). Besides **(A1)** and **(A2)**, we generally assume the following for the data of the state system:

(A3) $r > 0$, $\sigma > 0$, and $\tau \geq 0$ are fixed real numbers.

(A4) $f = f_1 + f_2$, where f_1, f_2 and f satisfy:

$f_1 : \mathbb{R} \to [0, +\infty]$ is convex, proper, and l.s.c., with $f_1(0) = 0$.

$f_2 \in C^1(\mathbb{R})$, and f_2' is Lipschitz on $\mathbb{R}$ with Lipschitz constant $L > 0$.

It holds that $\liminf_{|s| \nearrow +\infty} \frac{f(s)}{s^2} > 0$.

(A5) $y_0 \in V_B^\sigma$, $f_1(y_0) \in L^1(\Omega)$, and, if $\lambda_1 = 0$, then $m_0 := \mathrm{mean}(y_0) \in \mathrm{int}(\mathrm{dom}(\partial f_1))$.

(A6) $u \in H^1(0, T; H)$.

Notice that **(A4)** holds true for all of the potentials (1.13)–(1.15). Moreover, the subdifferential ∂f_1 is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$; in this connection, we denote by $\mathrm{dom}(\partial f_1) := \{r \in \mathbb{R} : \partial f_1(r) \neq \emptyset\}$ the effective domain of ∂f_1, and, for $r \in \mathrm{dom}(\partial f_1)$, by $\partial f_1^o(r)$ the element of $\partial f_1(r)$ having minimal modulus. We also note that no condition on the mean value m_0 is required if $\lambda_1 > 0$.

Remark 4 *It is worth noting that in the case of $\lambda_1 = 0$ it follows from* **(A2)**(ii) *and* (2.2) *that $A^r \mathbf{1} = 0$, where, here and in the following, we denote by $\mathbf{1}$ the function that is identically equal to unity on either Ω or Q. We then conclude from* (1.4) *that*

$$\frac{d}{dt} \int_\Omega y(t) = 0 \text{ for a.e. } t \in (0, T), \text{ i.e.,}$$
$$\mathrm{mean}(y(t)) = m_0 \text{ for all } t \in [0, T]. \tag{3.1}$$

Moreover, owing to the third condition in **(A5)**, *there is some $\delta_0 > 0$ such that*

$$[m_0 - \delta_0, m_0 + \delta_0] \subset \mathrm{int}(\mathrm{dom}(\partial f_1)). \tag{3.2}$$

We have the following well-posedness result (see Colli, Gilardi and Sprekels, 2019a, Theorem 2.6).

Theorem 3.1 *Suppose that the general assumptions* **(A1)**–**(A6)** *are fulfilled. Then, the system* (1.4)–(1.6) *has at least one solution* (μ, y) *satisfying* (1.1)–(1.3). *Moreover, there are constants $K_1 > 0$ and $K_2 > 0$, which continuously depend only on the data of the state system (in particular, on $\|u\|_{H^1(0,T;H)}$ and, if $\lambda_1 = 0$, the constant δ_0 from* (3.2)*), such that*

$$\|\mu\|_{L^2(0,T;V_A^r)} + \|y\|_{H^1(0,T;V_A^{-r}) \cap L^\infty(0,T;V_B^\sigma)} + \|f_1(y)\|_{L^1(Q)} + \|\tau^{1/2} \partial_t y\|_{L^2(0,T;H)} \leq K_1, \tag{3.3}$$

$$\|y_1 - y_2\|_{L^\infty(0,T;V_A^{-r}) \cap L^2(0,T;V_B^\sigma)} + \|\tau^{1/2}(y_1 - y_2)\|_{L^\infty(0,T;H)} \leq K_2 \|u_1 - u_2\|_{L^2(0,T;H)}, \tag{3.4}$$

whenever $u_i \in H^1(0,T;H)$, $i=1,2$, *are given*
and (μ_i, y_i), $i=1,2$, *are corresponding solutions.*

In particular, the second solution component y *is uniquely determined. Moreover, the expression* $A^r\mu$ *is uniquely determined as well, and, if* $\lambda_1 > 0$, *then also the first solution component* μ *is uniquely determined.*

Proof We do not present a detailed proof here and have to refer to Colli, Gilardi and Sprekels (2019a) for full details. However, in order to give the reader a flavor of the proof, we sketch the argument, which consists of a number of steps.

Step 1: Approximation.
In this section, we construct an approximation to the state system. We first introduce the Moreau–Yosida regularizations f_1^λ and ∂f_1^λ of f_1 and ∂f_1, respectively, at the level $\lambda > 0$ (see, e.g., Brezis, 1973, pp. 28 and 39). Then (see **(A4)**), with suitable constants $\widehat{c}_1 > 0$, $\widehat{c}_2 > 0$,

$$f_1^\lambda(s) = \int_0^s \partial f_1^\lambda(r)\,dr \quad \text{and} \quad 0 \le f_1^\lambda(s) \le f_1(s) \quad \text{for every } s \in \mathbb{R}, \tag{3.5}$$

$$f_1^\lambda(s) + f_2(s) \ge \widehat{c}_1\, s^2 - \widehat{c}_2 \quad \text{for every } s \in \mathbb{R} \text{ and } \lambda > 0 \text{ small enough.} \tag{3.6}$$

Moreover, we recall that ∂f_1^λ is Lipschitz continuous, so that f_1^λ grows at most quadratically, and that the following properties hold true:

$$f_1^{\lambda'}(s) \ge f_1^{\lambda''}(s) \quad \text{if } \lambda' \le \lambda'' \quad \text{and} \quad \lim_{\lambda \searrow 0} f_1^\lambda(s) = f_1(s), \quad \text{for every } s \in \mathbb{R}, \tag{3.7}$$

$$|\partial f_1^\lambda(s)| \le |\partial f_1^\circ(s)| \quad \text{for every } s \in \operatorname{dom}(\partial f_1). \tag{3.8}$$

By replacing f_1 in (1.5) by f_1^λ, we obtain the following system:

$$\langle \partial_t y^\lambda(t), v\rangle_{A,r} + (A^r\mu^\lambda(t), A^r v) = 0 \qquad \text{for every } v \in V_A^r \text{ and a.e. } t \in (0,T), \tag{3.9}$$

$$\begin{aligned} &\big(\tau\partial_t y^\lambda(t), y^\lambda(t) - v\big) + \big(B^\sigma y^\lambda(t), B^\sigma(y^\lambda(t) - v)\big) + \int_\Omega f_1^\lambda(y^\lambda(t)) \\ &\quad + \big(f_2'(y^\lambda(t)) - u(t), y^\lambda(t) - v\big) \;\le\; \big(\mu^\lambda(t), y^\lambda(t) - v\big) + \int_\Omega f_1^\lambda(v) \\ &\qquad \text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0,T), \end{aligned} \tag{3.10}$$

$$y^\lambda(0) = y_0. \tag{3.11}$$

Now, recall that f_1^λ is differentiable on $\mathbb{R}$ with globally Lipschitz continuous derivative ∂f_1^λ. Therefore, it is not difficult to show that (3.10) is, in fact, equivalent to the pointwise variational equation

$$
\begin{aligned}
&(\tau\partial_t y^\lambda(t),v) + \big(B^\sigma y^\lambda(t), B^\sigma v\big) + \big(\partial f_1^\lambda(y^\lambda(t)) + f_2'(y^\lambda(t)) - u(t), v\big) \\
&= \big(\mu^\lambda(t),v\big) \quad \text{for every } v \in V_B^\sigma \text{ and for a.a. } t\in(0,T).
\end{aligned} \tag{3.12}
$$

The problem (3.9)–(3.11) for the Moreau–Yosida approximations is solved by time discretization. To this end, we fix an integer $N>1$ and set $h_N := T/N$, $t_N^n := nh_N$ for $0\le n\le N$, and $I_N^n := [t_N^n, t_N^{n+1}]$ for $0\le n\le N-1$. Then, the discrete problem consists in finding two $(N+1)$-tuples $(y_N^0,\dots,y_N^N)$ and $(\mu_N^0,\dots,\mu_N^N)$ satisfying

$$
\begin{aligned}
&y_N^0 = y_0,\ \mu_N^0 = 0,\ (y_N^1,\dots,y_N^N)\in (V_B^{2\sigma})^N, \\
&\qquad\qquad \text{and} \quad (\mu_N^1,\dots,\mu_N^N)\in (V_A^{2r})^N,
\end{aligned} \tag{3.13}
$$

and solving

$$
\frac{y_N^{n+1}-y_N^n}{h_N} + \mu_N^{n+1} + A^{2r}\mu_N^{n+1} \;=\; \mu_N^n, \tag{3.14}
$$

$$
\begin{aligned}
&\tau\,\frac{y_N^{n+1}-y_N^n}{h_N} + \big((L+1)I + B^{2\sigma} + \partial f_1^\lambda + f_2'\big)(y_N^{n+1}) \\
&= (L+1)y_N^n + \mu_N^{n+1} + u_N^{n+1},
\end{aligned} \tag{3.15}
$$

for $n=0,1,\dots,N-1$, where $I: H\to H$ is the identity and

$$
u_N^n := u(nh_N), \quad \text{for } n = 0,1,\dots,N, \tag{3.16}
$$

is well defined, since $u\in H^1(0,T;H)$. This problem can be solved uniquely inductively for $n=0,\dots,N-1$ by employing maximal monotonicity arguments (see the proof of Theorem 2.6 in Colli, Gilardi and Sprekels, 2019a).

At this point, we introduce the standard piecewise constant and piecewise linear interpolates with respect to time, which for $t\in I_N^n$, $0\le n\le N-1$, are defined by

$$
\begin{aligned}
&\underline{\mu}_{h_N}(\cdot,t) := \mu_N^n, \quad \overline{\mu}_{h_N}(\cdot,t) := \mu_N^{n+1}, \quad \underline{y}_{h_N}(\cdot,t) := y_N^n, \quad \overline{y}_{h_N}(\cdot,t) := y_N^{n+1}, \\
&\widehat{y}_{h_N}(\cdot,t) := \tfrac{t_N^{n+1}-t}{h_N}\, y_N^n + \tfrac{t-t_N^n}{h_N}\, y_N^{n+1}, \quad \overline{u}_{h_N}(\cdot,t) := u_N^{n+1}.
\end{aligned} \tag{3.17}
$$

In terms of these interpolates, the discrete system can be rewritten in the form

$$
\partial_t\widehat{y}_{h_N} + \overline{\mu}_{h_N} + A^{2r}\overline{\mu}_{h_N} = \underline{\mu}_{h_N}, \tag{3.18}
$$

$$
\begin{aligned}
&\tau\partial_t\widehat{y}_{h_N} + \big((L+1)\,I + B^{2\sigma} + \partial f_1^\lambda + f_2'\big)(\overline{y}_{h_N}) \\
&\qquad = (L+1)\,\underline{y}_{h_N} + \overline{\mu}_{h_N} + \overline{u}_{h_N},
\end{aligned} \tag{3.19}
$$

$$
\widehat{y}_{h_N}(0) = y_0. \tag{3.20}
$$

Step 2: A priori estimates.
We now derive some uniform (in $N \in \mathbb{N}$) a priori estimates. In the following, $C_i > 0$, $i \in \mathbb{N}$, will always denote constants that may depend on the data of the system, but not on $n, N \in \mathbb{N}$. We test (3.9) and (3.10) (by taking the scalar product in H) by $h_N \mu_N^{n+1}$ and $y_N^{n+1} - y_N^n$, respectively, and add the resulting identities. Noting an obvious cancellation, we obtain the equation

$$\begin{aligned} & h_N(\mu_N^{n+1} - \mu_N^n, \mu_N^{n+1}) + h_N(A^{2r}\mu_N^{n+1}, \mu_N^{n+1}) + \frac{\tau}{h_N}\|y_N^{n+1} - y_N^n\|^2 \\ & + (B^{2\sigma} y_N^{n+1}, y_N^{n+1} - y_N^n) + \big(((L+1)I + \partial f_1^\lambda + f_2')(y_N^{n+1}), y_N^{n+1} - y_N^n\big) \\ & = (L+1)(y_N^n, y_N^{n+1} - y_N^n) + (u_N^{n+1}, y_N^{n+1} - y_N^n). \end{aligned}$$

Now, we observe that the function $r \mapsto \frac{L+1}{2} r^2 + f_1^\lambda(r) + f_2(r)$ is convex on $\mathbb{R}$, since f_1^λ is convex and $|f_2''| \le L$ a.e. on $\mathbb{R}$. Thus,

$$\begin{aligned} & \big(((L+1)I + \partial f_1^\lambda + f_2')(y_N^{n+1}), y_N^{n+1} - y_N^n\big) \\ & \ge \frac{L+1}{2}\|y_N^{n+1}\|^2 + \int_\Omega \big(f_1^\lambda(y_N^{n+1}) + f_2(y_N^{n+1})\big) - \frac{L+1}{2}\|y_N^n\|^2 \\ & \quad - \int_\Omega \big(f_1^\lambda(y_N^n) + f_2(y_N^n)\big). \end{aligned}$$

Hence, using this inequality and the formulas (2.7)–(2.8), and applying the elementary identity $b(b-a) = \frac{1}{2}b^2 - \frac{1}{2}a^2 + \frac{1}{2}(b-a)^2$, for $a, b \in \mathbb{R}$, to two terms on the left-hand side and to the first one on the right-hand side, we easily deduce from summation from $n = 0$ to $n = k-1$ with $k \le N$ that

$$\begin{aligned} & \frac{h_N}{2}\|\mu_N^k\|^2 + \sum_{n=0}^{k-1} \frac{h_N}{2}\|\mu_N^{n+1} - \mu_N^n\|^2 + \sum_{n=0}^{k-1} h_N \|A^r \mu_N^{n+1}\|^2 + \frac{1}{2}\|B^\sigma y_N^k\|^2 \\ & + \sum_{n=0}^{k-1} \frac{1}{2}\|B^\sigma(y_N^{n+1} - y_N^n)\|^2 - \frac{1}{2}\|B^\sigma y_0\|^2 + \tau \sum_{n=0}^{k-1} h_N \left\| \frac{y_N^{n+1} - y_N^n}{h_N} \right\|^2 \\ & + \int_\Omega \big(f_1^\lambda(y_N^k) + f_2(y_N^k)\big) - \int_\Omega \big(f_1(y_0) + f_2(y_0)\big) + \frac{L+1}{2}\sum_{n=0}^{k-1}\|y_N^{n+1} - y_N^n\|^2 \\ & \le (u_N^k, y_N^k) - (u_N^1, y_0) - \sum_{n=1}^{k-1}(u_N^{n+1} - u_N^n, y_N^n), \end{aligned} \tag{3.21}$$

where the expression on the right-hand side results from a summation by parts. Now, we observe that (3.6) implies that

$$\int_\Omega \big(f_1^\lambda(y_N^k) + f_2(y_N^k)\big) \ge \frac{1}{2}\int_\Omega \big(f_1^\lambda(y_N^k) + f_2(y_N^k)\big) + \frac{\widehat{c}_1}{2}\|y_N^k\|^2 - \frac{\widehat{c}_2}{2},$$

for sufficiently small $\lambda > 0$. In particular, the above integral is bounded from below. We treat the right-hand side of (3.21) by using the Young and Schwarz inequalities for finite sums and the fact that $u \in H^1(0,T;H)$ to obtain

$$(u_N^k, y_N^k) - (u_N^1, y_0) - \sum_{n=1}^{k-1} (u_N^{n+1} - u_N^n, y_N^n) \le \frac{\widehat{c}_1}{4} \|y_N^k\|^2 + \sum_{n=1}^{k-1} h_N \|y_N^n\|^2 + C_1.$$

By combining the above estimates, we can infer from the discrete Gronwall lemma that

$$\begin{aligned} & h_N \, \|\mu_N^k\|^2 + \sum_{n=0}^{k-1} h_N \, \|\mu_N^{n+1} - \mu_N^n\|^2 + \sum_{n=0}^{k-1} h_N \|A^r \mu_N^{n+1}\|^2 \\ & + \tau \sum_{n=0}^{k-1} h_N \Big\| \frac{y_N^{n+1} - y_N^n}{h_N} \Big\|^2 + \|y_N^k\|_{B,\sigma}^2 + \int_\Omega \big(f_1^\lambda(y_N^k) + f_2(y_N^k) \big) \\ & + \sum_{n=0}^{k-1} \|B^\sigma (y_N^{n+1} - y_N^n)\|^2 + \sum_{n=0}^{k-1} \|y_N^{n+1} - y_N^n\|^2 \le C_2 \\ & \qquad \text{for } k = 0, \dots, N. \end{aligned} \tag{3.22}$$

In terms of the interpolates defined in (3.17), by neglecting the first contribution and recalling that $\mu_N^0 = 0$, we have that

$$\begin{aligned} & \|\overline{\mu}_{h_N} - \underline{\mu}_{h_N}\|_{L^2(0,T;H)} + \|A^r \overline{\mu}_{h_N}\|_{L^2(0,T;H)} + \|A^r \underline{\mu}_{h_N}\|_{L^2(0,T;H)} \\ & \quad + \|\underline{y}_{h_N}\|_{L^\infty(0,T;V_B^\sigma)} + \|\overline{y}_{h_N}\|_{L^\infty(0,T;V_B^\sigma)} + \|\widehat{y}_{h_N}\|_{L^\infty(0,T;V_B^\sigma)} \\ & \quad + \tau^{1/2} \|\partial_t \widehat{y}_{h_N}\|_{L^2(0,T;H)} + \|f_1^\lambda(\overline{y}_{h_N}) + f_2(\overline{y}_{h_N})\|_{L^\infty(0,T;L^1(\Omega))} \\ & \quad + h_N^{-1/2} \|B^\sigma (\overline{y}_{h_N} - \underline{y}_{h_N})\|_{L^2(0,T;H)} + h_N^{-1/2} \|\overline{y}_{h_N} - \underline{y}_{h_N}\|_{L^2(0,T;H)} \le C_3 \,. \end{aligned} \tag{3.23}$$

Due to the Lipschitz continuity of f_2' and (3.22), we easily infer that

$$\|f_2(\overline{y}_{h_N})\|_{L^\infty(0,T;L^1(\Omega))} \le C_4, \quad \text{whence} \quad \|f_1^\lambda(\overline{y}_{h_N})\|_{L^\infty(0,T;L^1(\Omega))} \le C_5 \,. \tag{3.24}$$

Moreover, we conclude from (3.22) by comparison in (3.18) that

$$\|\partial_t \widehat{y}_{h_N}\|_{L^2(0,T;V_A^{-r})} \le C_6. \tag{3.25}$$

At this point, we claim that the estimate for $A^r \overline{\mu}_{h_N}$ in (3.23) can be improved to

$$\|\overline{\mu}_{h_N}\|_{L^2(0,T;V_A^r)} + \|\underline{\mu}_{h_N}\|_{L^2(0,T;V_A^r)} \le C_7 \,. \tag{3.26}$$

By recalling (3.23) and (2.3), we see that there is nothing to prove if $\lambda_1 > 0$. Suppose now that $\lambda_1 = 0$. We then have to estimate the mean value of $\overline{\mu}_{h_N}$. To this end, we first derive an estimate for $\partial f_1^\lambda(\overline{y}_{h_N})$. Now recall that $m_0 \mathbf{1} \in V_B^\sigma$

by **(A2)**(ii) and that by **(A5)** we have $m_0 \in \operatorname{int}(\operatorname{dom}(\partial f_1))$. We thus can make use of an inequality due to Miranville and Zelik (2004) (see also Gilardi, Miranville and Schimperna, 2009, p. 908 for a detailed proof); namely, there is some $C_0 > 0$ such that

$$\partial f_1^\lambda(s)(s-m_0) \geq \delta_0 |\partial f_1^\lambda(s)| - C_0 \quad \text{for all } s \in \mathbb{R} \text{ and } \lambda \in (0,1), \tag{3.27}$$

where δ_0 is the constant from (3.2). Then, we test (3.15) by $y_N^{n+1} - m_0\mathbf{1}$ and use (3.23), (3.27), and the fact that $u \in H^1(0,T;H)$, to find that

$$\begin{aligned}
&\int_\Omega \big(\delta_0\, |\partial f_1^\lambda(y_N^{n+1})| - C_0\big) \;\leq\; \int_\Omega \partial f_1^\lambda(y_N^{n+1})(y_N^{n+1} - m_0\mathbf{1}) \\
&= -\tau\Big(\frac{y_N^{n+1} - y_N^n}{h_N}, y_N^{n+1} - m_0\mathbf{1}\Big) - (L+1)(y_N^{n+1} - y_N^n, y_N^{n+1} - m_0\mathbf{1}) \\
&\quad - \big(B^{2\sigma} y_N^{n+1}, y_N^{n+1} - m_0\mathbf{1}\big) - \big(f_2'(y_N^{n+1}), y_N^{n+1} - m_0\mathbf{1}\big) \\
&\quad + (\mu_N^{n+1} + u_N^{n+1}, y_N^{n+1} - m_0\mathbf{1}) \\
&\leq\; C_8\, \tau \Big\| \frac{y_N^{n+1} - y_N^n}{h_N} \Big\| \, (\|y_N^{n+1}\| + 1) + C_9\, (\|y_N^{n+1}\|^2 + \|y_N^n\|^2 + 1) \\
&\quad + C_{10}\, \|u_N^{n+1}\|\, (\|y_N^{n+1}\| + 1) + \big|\big(B^\sigma y_N^{n+1}, B^\sigma(y_N^{n+1} - m_0\mathbf{1})\big)\big| \\
&\quad + (\mu_N^{n+1}, y_N^{n+1} - m_0\mathbf{1}) \\
&\leq\; C_8\, \tau \,\Big\| \frac{y_N^{n+1} - y_N^n}{h_N} \Big\| + C_{11} + (\mu_N^{n+1}, y_N^{n+1} - m_0\mathbf{1})\,.
\end{aligned} \tag{3.28}$$

Now observe that (3.14) implies that, for $n = 0, \ldots, N-1$,

$$\operatorname{mean}(y_N^{n+1} + h\mu_N^{n+1}) - \operatorname{mean}(y_N^n + h\mu_N^n) = \; -\frac{h}{|\Omega|}\, (A^r \mu_N^{n+1}, A^r(\mathbf{1})) = 0,$$

so that $\operatorname{mean}(y_N^{n+1} + h\mu_N^{n+1}) = m_0$ for every n, since $\mu_N^0 = 0$. Hence, by taking advantage of Poincaré's inequality (2.15), we obtain the estimate

$$\begin{aligned}
&(\mu_N^{n+1}, y_N^{n+1} - m_0\mathbf{1}) \\
&= (\mu_N^{n+1} - \operatorname{mean}(\mu_N^{n+1}), y_N^{n+1} - m_0\mathbf{1}) + (\operatorname{mean}(\mu_N^{n+1}), y_N^{n+1} - m_0\mathbf{1}) \\
&\leq\; \widehat{c}\, \|A^r \mu_N^{n+1}\|\, \|y_N^{n+1} - m_0\mathbf{1}\| + (\operatorname{mean}(\mu_N^{n+1}), -h\mu_N^{n+1}) \\
&\leq\; C_{12}\, \|A^r \mu_N^{n+1}\| - |\Omega|\, h\, (\operatorname{mean}(\mu_N^{n+1}))^2 \;\leq\; C_{12}\, \|A^r \mu_N^{n+1}\|\,.
\end{aligned}$$

Therefore, (3.28) becomes

$$\|\partial f_1^\lambda(y_N^{n+1})\|_{L^1(\Omega)} \;\leq\; C_{13}\, \Big(\tau\, \Big\| \frac{y_N^{n+1} - y_N^n}{h_N} \Big\| + \|A^r \mu_N^{n+1}\| + 1\Big). \tag{3.29}$$

From this, we readily conclude that

$$\begin{aligned}
\|\partial f_1^\lambda(\overline{y}_{h_N})\|^2_{L^2(0,T;L^1(\Omega))} &= \sum_{n=0}^{k-1} h_N \|\partial f_1^\lambda(y_N^{n+1})\|^2_{L^1(\Omega)} \\
\leq C_{14} \Big(\tau^2 \sum_{n=0}^{k-1} h_N \Big\| \frac{y_N^{n+1} - y_N^n}{h_N} \Big\|^2 &+ \sum_{n=0}^{k-1} h_N \|A^r \mu_N^{n+1}\|^2 + 1\Big) \leq C_{15} \,.
\end{aligned} \tag{3.30}$$

At this point, we simply integrate (3.1) over Ω to have, a.e. in $(0,T)$, that

$$\begin{aligned}
|\Omega| \operatorname{mean}(\overline{\mu}_{h_N}) &= \tau \int_\Omega \partial_t \widehat{y}_{h_N} + (L+1) \int_\Omega (\overline{y}_{h_N} - \underline{y}_{h_N}) + \big(B^\sigma \overline{y}_{h_N}, B^\sigma \mathbf{1}\big) \\
&\quad + \int_\Omega \partial f_1^\lambda(\overline{y}_{h_N}) + \int_\Omega f_2'(\overline{y}_{h_N}) - \int_\Omega \overline{u}_{h_N} \,.
\end{aligned}$$

Thus, $\operatorname{mean}(\overline{\mu}_{h_N})$ is bounded in $L^2(0,T)$, owing to (3.22) and (3.30), which proves (3.26) for $\overline{\mu}_{h_N}$. Since $A^r \mu^0 = A^r 0 = 0$, and as $\overline{\mu}_{h_N} - \underline{\mu}_{h_N}$ is bounded in $L^2(0,T;H)$ by virtue of (3.22), the same estimate holds for $\underline{\mu}_{h_N}$. Hence, (3.26) holds true also in the case of $\lambda_1 = 0$.

Step 3: Existence.

Upon collecting the estimates (3.23)–(3.26), and using standard weak and weak-star compactness results, we see that there are a subsequence, which is again indexed by $N \in \mathbb{N}$, and functions y^λ and μ^λ such that, as $N \to \infty$,

$$\overline{y}_{h_N} \to y^\lambda, \quad \underline{y}_{h_N} \to y^\lambda, \quad \widehat{y}_{h_N} \to y^\lambda, \quad \text{all weakly-star in } L^\infty(0,T;V_B^\sigma), \tag{3.31}$$

$$\partial_t \widehat{y}_{h_N} \to \partial_t y^\lambda \quad \text{weakly in } L^2(0,T;V_A^{-r}), \tag{3.32}$$

$$\partial_t \widehat{y}_{h_N} \to \partial_t y^\lambda \quad \text{weakly in } L^2(0,T;H) \quad \text{if } \tau > 0, \tag{3.33}$$

$$\overline{\mu}_{h_N} \to \mu^\lambda \quad \text{weakly in } L^2(0,T;V_A^r), \tag{3.34}$$

provided that $\lambda > 0$ is small enough. By letting $N \to \infty$ in (3.20), we see that y^λ satisfies (3.11). Moreover, it is not hard to verify (see the proof of Theorem 2.6 in Colli, Gilardi and Sprekels, 2019a) that

$$\underline{\mu}_{h_N} \to \mu^\lambda \quad \text{weakly in } L^2(0,T;V_A^r). \tag{3.35}$$

Therefore, the pair (y^λ, μ^λ) solves (3.9). In order to deal with the nonlinear terms of (3.1), we recall the compact embedding $V_B^\sigma \subset H$ (see (2.14)) and invoke well-known strong compactness results (see, e.g., Simon, 1987, Section 8, Corollary 4). In fact, we can infer from (3.31)–(3.32) that

$$\widehat{y}_{h_N} \to y^\lambda \quad \text{strongly in } L^\infty(0,T;H). \tag{3.36}$$

Since it is easily checked that $\|\overline{y}_{h_N} - \widehat{y}_{h_N}\|_{L^2(0,T;H)} \to 0$ as $N \to \infty$, we conclude that

$$\overline{y}_{h_N} \to y^\lambda, \quad \underline{y}_{h_N} \to y^\lambda, \quad \text{both strongly in } L^2(0,T;H), \tag{3.37}$$

where the latter convergence result follows from (3.23) (see the last term on the left-hand side of this estimate). By Lipschitz continuity, it turns out that

$$(\partial f_1^\lambda + f_2')(\overline{y}_{h_N}) \to (\partial f_1^\lambda + f_2')(y^\lambda) \quad \text{strongly in } L^2(0,T;H).$$

Moreover, as we can assume that $\overline{y}_{h_N}$ converges to y^λ pointwise a.e. in Q and it is known that f_1^λ grows at most quadratically, we can also apply (3.25) and Fatou's lemma to deduce that

$$\int_\Omega f_1^\lambda(y^\lambda(t)) \,\le\, \liminf_{N\to\infty} \int_\Omega f_1^\lambda(\overline{y}_{h_N}(t)) \,\le\, C_{16} \quad \text{for a.a. } t \in (0,T), \tag{3.38}$$

whence

$$\|f_1^\lambda(y^\lambda)\|_{L^\infty(0,T;L^1(\Omega))} \le C_{16}\,. \tag{3.39}$$

Therefore, we can pass to the limit in the time-integrated version of (3.1) (written with time-dependent test functions) and deduce that the pair (y^λ, μ^λ) also solves the time-integrated version of (3.12), which is equivalent to (3.10).

We have thus shown that the system (3.9)–(3.11), governed by the Moreau–Yosida approximations, has for sufficiently small $\lambda > 0$ a solution (μ^λ, y^λ) having the expected regularity. More precisely, it follows from the above estimates and the semicontinuity properties of norms that

$$\begin{aligned}&\|y^\lambda\|_{H^1(0,T;V_A^{-r})\cap L^\infty(0,T;V_B^\sigma)} + \|\mu^\lambda\|_{L^2(0,T;V_A^r)} + \tau^{1/2}\|\partial_t y^\lambda\|_{L^2(0,T;H)} \\ &\quad + \|f_1^\lambda(y^\lambda)\|_{L^\infty(0,T;L^1(\Omega))} \,\le\, C_{17},\end{aligned} \tag{3.40}$$

for $\lambda > 0$ small enough. Hence, there exist a strictly decreasing sequence $\lambda_n \searrow 0$ and a pair (μ, y) satisfying, as $n \to \infty$,

$$y^{\lambda_n} \to y \quad \text{weakly star in } H^1(0,T;V_A^{-r}) \cap L^\infty(0,T;V_B^\sigma), \tag{3.41}$$

$$\mu^{\lambda_n} \to \mu \quad \text{weakly in } L^2(0,T;V_A^r), \tag{3.42}$$

$$\partial_t y^{\lambda_n} \to \partial_t y \quad \text{weakly in } L^2(0,T;H) \quad \text{ if } \tau > 0. \tag{3.43}$$

Then, it is immediately seen that (μ, y) solves (1.4) and that y satisfies the initial condition (1.6). Moreover, invoking the compact embedding $V_B^\sigma \subset H$ (see (2.14) and, e.g., Simon, 1987, Section 8, Corollary 4), we may without loss of generality assume that

$$y^{\lambda_n} \to y \quad \text{strongly in } L^\infty(0,T;H) \quad \text{and} \quad \text{a.e. in } Q, \tag{3.44}$$

which implies that $f_2'(y^{\lambda_n})$ converges to $f_2'(y)$ in the same space, by Lipschitz continuity.

It remains to prove that the variational inequality (1.5) holds true, as well, and that

$$\int_Q f_1(y) \le \liminf_{n\to\infty} \int_Q f_1^{\lambda_n}(y^{\lambda_n}) < +\infty. \tag{3.45}$$

We notice that the right-hand side of (3.45) is finite due to (3.39). In particular, the requirement $f_1(y) \in L^1(Q)$ (see (1.3)) will be fulfilled once the first inequality of (3.45) is shown. To prove (3.45), we take arbitrary indices m and n with $n > m$. Then $\lambda_n < \lambda_m$, and, owing to (3.7),

$$f_1^{\lambda_m}(y^{\lambda_n}) \le f_1^{\lambda_n}(y^{\lambda_n}) \quad \text{a.e. in } Q\text{, for every } n > m,$$

whence also (since $f_1^{\lambda_m}$ is continuous)

$$f_1^{\lambda_m}(y) = \lim_{n\to\infty} f_1^{\lambda_m}(y^{\lambda_n}) = \liminf_{n\to\infty} f_1^{\lambda_m}(y^{\lambda_n}) \le \liminf_{n\to\infty} f_1^{\lambda_n}(y^{\lambda_n}) \quad \text{a.e. in } Q.$$

Thus, by virtue of the second property stated in (3.7),

$$f_1(y) = \lim_{m\to\infty} f_1^{\lambda_m}(y) \;\le\; \liminf_{n\to\infty} f_1^{\lambda_n}(y^{\lambda_n}) \quad \text{a.e. in } Q, \tag{3.46}$$

and (3.45) follows from Fatou's lemma. From this point, it is not too difficult (see the proof of Theorem 2.6 in Colli, Gilardi and Sprekels, 2019a) to show that also (1.5) is satisfied. With this, it is shown that (μ, y) is a solution to the system (1.4)–(1.6) having the properties (1.1)–(1.3).

STEP 4: CONTINUOUS DEPENDENCE AND UNIQUENESS.

We pick two data u_i, $i = 1, 2$, and corresponding solutions (μ_i, y_i), and set, for convenience, $u := u_1 - u_2$, $y := y_1 - y_2$, and $\mu := \mu_1 - \mu_2$. Now, we write equation (1.4) at the time s for these solutions, take the difference, and test it by $v = A_0^{-2r} y(s)$, where we observe that $y(s) \in V_0^{-r}$, since $y \in L^2(0,T;H)$ and $\text{mean}(y(s)) = 0$ if $\lambda_1 = 0$ by the conservation property (3.1), so that v is a well-defined element of V_A^r. Moreover, $A_0^{-2r} y \in L^\infty(0,T;V_A^r)$, since $y \in L^\infty(0,T;V_A^{-r})$ by (1.1). By integrating over $(0,t)$ with respect to s, where $t \in (0,T)$ is arbitrary, we obtain the identity

$$\int_0^t \langle \partial_t y(s), A_0^{-2r} y(s) \rangle_{A,r}\, ds + \int_0^t \big(A^r \mu(s), A^r A_0^{-2r} y(s) \big)\, ds = 0\,. \tag{3.47}$$

Now, apply (2.23) and (2.24), noting that $\mu \in L^2(0,T;V_A^r)$. Then (3.47) becomes

$$\frac{1}{2}\, \|y(t)\|^2_{A,-r} + \int_0^t (y(s), \mu(s))\, ds = 0. \tag{3.48}$$

Next, write (1.5) for u_i and (μ_i, y_i), $i = 1, 2$, test them by y_2 and y_1, respectively, add the resulting inequalities, integrate over $(0,t)$ as before, and rearrange.

Then,

$$\frac{\tau}{2}\,\|y(t)\|^2 + \int_0^t \|B^\sigma y(s)\|^2\,ds - \int_0^t \bigl(\mu(s),y(s)\bigr)\,ds$$
$$\le \int_0^t \bigl(u(s),y(s)\bigr)\,ds \,-\, \int_0^t \bigl(f_2'(y_1(s)) - f_2'(y_2(s)),y(s)\bigr)\,ds\,.$$

By adding this to (3.48), and accounting for the Lipschitz continuity of f_2' and the Cauchy–Schwarz and Young inequalities, we deduce that

$$\frac{1}{2}\,\|y(t)\|^2_{A,-r} + \frac{\tau}{2}\,\|y(t)\|^2 + \int_0^t \|B^\sigma y(s)\|^2\,ds$$
$$\le \frac{1}{4}\int_0^t \|u(s)\|^2\,ds + (L+1)\int_0^t \|y(s)\|^2\,ds. \tag{3.49}$$

At this point, recall that the embeddings $H \subset V_A^{-r}$ and $V_B^\sigma \subset H$ are compact, by (2.13) and (2.14). Therefore, we can infer from the compactness inequality (Ehrling's lemma) that there is some constant $c>0$ such that

$$(L+1)\int_0^t \|y(s)\|^2\,ds \,\le\, \frac{1}{2}\int_0^t \|B^\sigma y(s)\|^2\,ds + c\int_0^t \|y(s)\|^2_{A,-r}\,ds.$$

By combining this with (3.49) and applying Gronwall's lemma, we conclude that (3.4) holds true with a constant K_2 as in the statement. In particular, if $u_1 = u_2$ and (μ_1,y_1), (μ_2,y_2) are corresponding solutions, then $y_1 = y_2$, i.e., the second solution component is uniquely determined. Moreover, it follows from (1.4) that almost everywhere in $(0,T)$ we have

$$0 \,=\, \langle \partial_t(y_1-y_2), \mu_1-\mu_2\rangle_{A,r} \,+\, \|A^r(\mu_1-\mu_2)\|^2 \,=\, \|A^r(\mu_1-\mu_2)\|^2\,,$$

which implies that $A^r\mu_1 = A^r\mu_2$. By (2.3), then also $\mu_1=\mu_2$ if $\lambda_1>0$. This finally concludes the proof of the assertion. $\square$

3.2. Higher regularity

Under additional assumptions on the data, we have stronger regularity results in both the viscous and nonviscous cases. Recall now that, for every $r\in\mathbb{R}$, $\partial f_1^\circ(r)$ denotes the element of the set $\partial f_1(r)$ having minimal modulus. We then also assume that either $\tau>0$ and

$$y_0 \in V_B^{2\sigma} \quad \text{and} \quad \partial f_1^\circ(y_0)\in H, \tag{3.50}$$

or that $\tau=0$ and, for some $M_0>0$ and every sufficiently small $\lambda>0$ and $t>0$,

$$y_0 \in V_B^{2\sigma} \quad \text{and} \quad \|\mu_0^\lambda(t)\|_{A,r} \le M_0, \quad \text{where} \tag{3.51}$$
$$\mu_0^\lambda(t) := B^{2\sigma}y_0 + (\partial f_1^\lambda + f_2')(y_0) - u(t). \tag{3.52}$$

More precisely, it is assumed that the element $\mu_0^\lambda(t)$ (which is well defined by (3.52) due to the first assumption on y_0) belongs to V_A^r and satisfies the above estimate. This very restrictive assumption is, for example, satisfied if each of the four contributions to the right-hand side of (3.52) satisfies bounds like (3.51), separately. For instance, if A^{2r} is the Laplace operator with zero Dirichlet boundary conditions and ∂f_1 is single-valued and of class C^2 in the interior of its domain, then one can assume that $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and that $\min y_0 > \inf(\mathrm{dom}(\partial f_1))$ and $\max y_0 < \sup(\mathrm{dom}(\partial f_1))$. These assumptions keep $\partial f_1^\lambda(y_0)$ bounded in $H^2(\Omega)$, indeed.

Theorem 3.2 *In addition to the assumptions of Theorem 3.1, suppose that either $\tau > 0$ and* (3.50) *or $\tau = 0$ and* (3.51)–(3.52) *are fulfilled. Then, any solution (μ, y) in the sense of Theorem 3.2 also has the regularity properties*

$$\partial_t y \in L^\infty(0,T;V_A^{-r}) \cap L^2(0,T;V_B^\sigma) \ \text{ and } \ \mu \in L^\infty(0,T;V_A^r) \ \text{ if } \tau \geq 0, \tag{3.53}$$

$$\partial_t y \in L^\infty(0,T;H) \ \text{ and } \ \mu \in L^\infty(0,T;V_A^{2r}) \ \text{ if } \tau > 0, \tag{3.54}$$

and the estimate

$$\begin{aligned} &\|\partial_t y\|_{L^\infty(0,T;V_A^{-r}) \cap L^2(0,T;V_B^\sigma)} + \|\mu\|_{L^\infty(0,T;V_A^r)} + \|\tau^{1/2}\partial_t y\|_{L^\infty(0,T;H)} \\ &+ \|\tau^{1/2}\mu\|_{L^\infty(0,T;V_A^{2r})} \leq K_3, \end{aligned} \tag{3.55}$$

holds with a constant K_3 that depends only on the structure of the system, the norms of the data, the width δ_0 satisfying (3.2) *if $\lambda_1 = 0$, the constant M_0 satisfying* (3.51) *if $\tau = 0$, and T.*

Proof Also this result is established by showing a corresponding a priori estimate on the level of the time-discrete system (3.14)–(3.15), see the proof of Theorem 2.8 in Colli, Gilardi and Sprekels (2019a). □

In the following, we derive a strict separation result for a special case, in which the nonlinearity f_1' exhibits the same singular behavior as in the case of the logarithmic potential (1.14).

Theorem 3.3 *In addition to the assumptions of Theorem 3.1, suppose that the following conditions are fulfilled:*

$$B = -\Delta \ \text{ with } \ D(-\Delta) = \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0 \ \text{ on } \Gamma\} \quad \text{and} \quad 2\sigma = 1. \tag{3.56}$$

$$V_A^{2r} \subset L^\infty(\Omega). \tag{3.57}$$

$$\text{The constant } \tau \text{ is positive.} \tag{3.58}$$

$$y_0 \in H^2(\Omega) \quad \text{and} \quad -1 < \min_{y\in\overline{\Omega}} y_0(x) \leq \max_{y\in\overline{\Omega}} y_0(x) < 1. \tag{3.59}$$

$$\begin{aligned} &f_1 \in C^0[-1,1] \cap C^1(-1,1) \quad \text{satisfies} \quad f_1(r) = +\infty \ \text{ for } \ r \notin [-1,1], \\ &\lim_{r\searrow -1} f_1'(r) = -\infty, \quad \text{and} \quad \lim_{r\nearrow 1} f_1'(r) = +\infty. \end{aligned} \tag{3.60}$$

Moreover, let (μ, y) *be any solution in the sense of Theorem 3.1. Then* $\|\mu\|_{L^\infty(Q)}$ *is bounded by a global constant, and there are constants* $-1 < r_* \le \min_{x\in\overline{\Omega}} y_0(x) \le \max_{x\in\overline{\Omega}} y_0(x) < r^* < 1$ *such that*

$$r_* \le y \le r^* \quad \text{a.e. in } Q. \tag{3.61}$$

Proof We only sketch the argument. For full details, we refer to Colli, Gilardi and Sprekels (2018c), Section 3, Example 1. At first, notice that (3.59) and (3.60) imply that $\partial f_1^\circ(y_0) = f_1'(y_0) \in H$, so that (3.50) holds true. Therefore, by virtue of Theorem 3.2 and (3.57), $\|\mu\|_{L^\infty(Q)}$ is bounded by a global constant. Moreover, since $f_1(y) \in L^1(Q)$ by (1.3), we must have $y \in [-1,1]$ a.e. in Q, so that $\|f_2'(y)\|_{L^\infty(Q)}$ is finite. Hence, there is some global constant $M > 0$ such that $\|\mu + u - f_2'(y)\|_{L^\infty(Q)} \le M$. By virtue of assumption (3.60), there are constants $r_*, r^* \in (-1,1)$ such that $r_* \le y_0 \le r^*$ in $\overline{\Omega}$ and

$$f_1'(r) + M \le 0 \quad \forall\, r \in (-1, r_*) \qquad \text{and} \qquad f_1'(r) - M \ge 0 \quad \forall\, r \in (r^*, 1).$$

Now, observe that $V_B^{1/2} = H^1(\Omega)$ if (3.56) is valid. We thus may take $v = y(t) - (y(t)-r^*)^+ \in H^1(\Omega)$ in the variational inequality (1.5), where $(y(t)-r^*)^+$ is the positive part of $y(t) - r^*$. We then find for almost every $t \in (0,T)$ the inequality

$$\begin{aligned} &\frac{\tau}{2}\frac{d}{dt}\|(y(t)-r^*)^+\|^2 + \int_\Omega |\nabla (y(t)-r^*)^+|^2 \\ &\le \int_\Omega \big[f_1(y(t)-(y(t)-r^*)^+) - f_1(y(t)) + (\mu(t) + u(t) - f_2'(y(t)))(y(t)-r^*)^+\big]. \end{aligned} \tag{3.62}$$

At this point, using the differentiability and convexity of f_1, it is not difficult to show that (see Colli, Gilardi and Sprekels, 2018c, Section 3, Example 1) the integrand of the integral on the right-hand side is nonpositive, i.e., the expression on the right-hand side of (3.62) is nonpositive. Now, we integrate (3.62) over $(0,t)$, where $t \in (0,T]$ is arbitrary. Since $(y_0 - r^*)^+ = 0$ by assumption, we have $(y - r^*)^+ = 0$ a.e. in Q, which implies that $y \le r^*$ a.e. in Q. Similarly, we obtain that $y \ge r_*$ a.e. in Q. □

Remark 5 *Apparently, the convex part of the logarithmic potential* (1.14) *(and thus also the convex part of any of the functions* h^α *for* $\alpha > 0$*, see* (1.20)*) satisfies the condition* (3.60)*. We also note that the constant* $M > 0$*, occurring in the above proof, depends in a monotone increasing way on* $\|u\|_{L^\infty(Q)}$*; it thus follows from the above argument that the constants* r_*, r^* *can be chosen independently of* u *if* u *belongs to a bounded subset of* $L^\infty(Q)$*.*

Remark 6 *The condition* (3.57) *is satisfied, for instance, if* $A = -\Delta$ *with zero Dirichlet or Neumann boundary condition and* $r > \frac{3}{8}$*. Indeed, we then have* $V_A^{2r} \subset H^{4r}(\Omega)$ *and* $4r > \frac{3}{2}$*.*

4. The optimal control problem

4.1. Existence of optimal controls

In this section, we begin to study the control problem **(CP)**, where we closely follow the lines of Colli, Gilardi and Sprekels (2018c,d). We generally assume that **(A1)**–**(A5)** and the following assumptions are also satisfied:

(A7) $\beta_i \geq 0,\ \ i = 1, 2, 3, \quad y_\Omega \in H,\ \ y_Q \in L^2(Q),$ and $\rho_i > 0,\ i = 1, 2.$

We denote the control space by

$$\mathcal{X} := H^1(0, T; H) \cap L^\infty(Q). \tag{4.1}$$

Moreover, we generally assume that $\mathcal{U}_{\text{ad}}$ is given by (1.17), and we fix once and for all some $R > 0$ such that

$$\mathcal{U}_{\text{ad}} \subset \mathcal{U}_R := \{u \in \mathcal{X} : \|u\|_{\mathcal{X}} < R\}. \tag{4.2}$$

Since for $u \in \mathcal{X}$ the assumption **(A6)** is automatically fulfilled, it then follows from Theorem 3.1 that the "partial" control-to-state operator

$$\mathcal{S}^2 : u \mapsto \mathcal{S}^2(u) := y, \tag{4.3}$$

where (μ, y) denotes a solution to (1.4)–(1.6) in the sense of Theorem 3.1 (which means, in particular, that (1.1)–(1.3) are valid), is well defined as a mapping from $\mathcal{X}$ into the Banach space $H^1(0, T; V_A^{-r}) \cap L^\infty(0, T; V_B^\sigma)$, if $\tau = 0$, or $H^1(0, T; H) \cap L^\infty(0, T; V_B^\sigma)$, if $\tau > 0$.

Remark 7 *As stated in Theorem 3.1, the constants K_1 and K_2 depend continuously on $\|u\|_{H^1(0,T;H)}$. We can therefore choose these constants in such a way that the estimates* (3.3) *and* (3.4) *are valid whenever the respective controls u belong to $\mathcal{U}_R$.*

We have the following general existence result.

Theorem 4.1 *Suppose that* **(A1)**–**(A5)**, **(A7)**, (1.17), *and* (4.2) *are fulfilled. Then the optimal control problem* **(CP)** *has a solution.*

Proof Let $\{(y_n, u_n)\}$ be a minimizing sequence for **(CP)**. Then, in particular, $\{u_n\} \subset \mathcal{U}_{\text{ad}}$ and $y_n = \mathcal{S}^2(u_n)$ for some solution (μ_n, y_n) to (1.4)–(1.6) in the sense of Theorem 3.1, for $n \in \mathbb{N}$. According to Remark 7, we may then assume that there are $u \in \mathcal{U}_{\text{ad}}$ and functions (μ, y) such that

$$u_n \to u \quad \text{weakly-star in } \mathcal{X}, \tag{4.4}$$

$$\mu_n \to \mu \quad \text{weakly in } L^2(0, T; V_A^r), \tag{4.5}$$

$$y_n \to y \quad \text{weakly-star in } H^1(0, T; V_A^{-r}) \cap L^\infty(0, T; V_B^\sigma), \tag{4.6}$$

$$\tau \partial_t y_n \to \tau \partial_t y \quad \text{weakly in } L^2(0, T; H). \tag{4.7}$$

By virtue of Simon (1987), Section 8, Corollary 4, we may without loss of generality also assume that

$$y_n \to y \quad \text{strongly in } C^0([0,T];H) \text{ and pointwise a.e. in } Q. \tag{4.8}$$

In particular, we have that $y(0) = y_0$ and, by the Lipschitz continuity of f_2', $f_2'(y_n) \to f_2'(y)$ strongly in $C^0([0,T];H)$. Moreover, taking the limit as $n \to \infty$ in the time-integrated version of (1.4), written for (μ_n, y_n) with test functions $v \in L^2(0,T;V_A^r)$, we readily see that (μ, y) solves (1.4). Moreover, by the lower semicontinuity of f_1 and (4.8), $f_1(y) \le \liminf_{n\to\infty} f_1(y_n)$ a.e. in Q, and it follows from (3.3) and Fatou's lemma that

$$\int_Q f_1(y) \,\le\, \liminf_{n\to\infty} \int_Q f_1(y_n) \,\le\, K_1, \tag{4.9}$$

which shows that $f_1(y) \in L^1(Q)$, in particular. With this, it is now an easy task to take the limit as $n \to \infty$ in the time-integrated version of (1.5), written for (μ_n, y_n) and u_n with test functions $v \in L^2(0,T;V_B^\sigma)$, to infer that (μ, y) satisfies the variational inequality (1.5). In conclusion, we have $y = \mathcal{S}^2(u)$, so that (y,u) is admissible for **(CP)**. It then follows from the semicontinuity properties of the cost functional (1.16) that (y,u) is an optimal pair. □

4.2. The differentiable case

In this section, we derive first-order necessary optimality conditions for the case of smooth nonlinearities, where our analysis follows the lines of Colli, Gilardi and Sprekels (2018c). In addition to **(A1)**–**(A5)**, **(A7)**, (1.17) and (4.2), we make the following assumptions:

(A8) $\tau > 0$, and (3.50) is satisfied.

(A9) $f_2 \in C^3(\mathbb{R})$, and f_1 satisfies the conditions $f_1 \in C^3(\mathrm{int}(\mathrm{dom}(f_1)))$ and $f_1'' \ge 0$ in $\mathrm{int}(\mathrm{dom}(f_1))$.

Notice that **(A9)** is trivially satisfied for the classical regular potential (1.13), but it also holds true for the logarithmic and double obstacle potentials, where in the latter two cases $\mathrm{int}(\mathrm{dom}(f_1)) = (-1,1)$.

Remark 8 *We also observe that with the above assumptions all of the conditions to apply Theorem 3.2 are fulfilled with* $\tau > 0$. *Therefore, the solutions* (μ, y) *in the sense of Theorem 3.1 associated with* $u \in \mathcal{U}_R$ *enjoy the regularity* (3.54)*, and for all of these solutions the estimate* (3.55) *holds.*

The following global boundedness condition was crucial for the analysis of the control problem, carried out in Colli, Gilardi and Sprekels (2018c).

(GB) There is a closed interval $[a,b] \subset \mathrm{int}(\mathrm{dom}(f_1))$ such that the following holds true: whenever (μ, y) is a solution to the state system in the sense of Theorem 3.1, associated with some $u \in \mathcal{U}_R$, then $y \in [a,b]$ almost everywhere in Q.

Remark 9 *The condition* **(GB)** *is very restrictive and has to be verified from case to case. By virtue of Theorem 3.3, it is fulfilled if also the conditions* (3.56)–(3.60) *are valid. Recall that* (3.60) *holds true for the logarithmic potential. Hence, the following analysis applies to the logarithmic case if, in addition to the general assumptions of this section, the conditions* (3.56)–(3.59) *are fulfilled. We also remark that in Colli, Gilardi and Sprekels (2018c) two further nontrivial examples for the validity of* **(GB)** *have been presented.*

Remark 10 *If* **(GB)** *is fulfilled, then it follows from Colli, Gilardi and Sprekels (2019a), Remark 4.1, that the whole solution pair* (μ, y) *to the state system* (1.4)–(1.6) *is uniquely determined. Hence, the "total" control-to-state operator (compare the definition in* (4.3)*)*

$$\mathcal{S} : \mathcal{X} \ni u \mapsto \mathcal{S}(u) = (\mathcal{S}^1(u), \mathcal{S}^2(u)) := (\mu, y) \tag{4.10}$$

is well defined. In addition, there is a constant $K_4 > 0$*, which depends only on the data of the state system and* R*, such that the following holds true: whenever* $(\mu, y) = \mathcal{S}(u)$ *for some* $u \in \mathcal{U}_R$*, then*

$$\max_{i=0,1,2,3} \|f_1^{(i)}(y)\|_{L^\infty(Q)} \le K_4. \tag{4.11}$$

Remark 11 *Let us assume that in addition to* **(GB)** *and the general assumptions of this section the following condition is satisfied:*

(A10) $V_B^\sigma \cap L^\infty(\Omega)$ *is dense in* V_B^σ.

Then, it is easily seen (see Colli, Gilardi and Sprekels, 2018c, Remark 3.5) that the variational inequality (1.5) *is equivalent to the variational equation*

$$\begin{aligned} &(\tau \partial_t y(t), v) + (B^\sigma y(t), B^\sigma(v)) + (f_1'(y(t)), v) + (f_2'(y(t)), v) \\ &= (\mu(t) + u(t), v) \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V_B^\sigma. \end{aligned} \tag{4.12}$$

A fortiori, by virtue of the bounds (3.55) *and a comparison in equation* (4.12)*, we have* $B^{2\sigma} y = \mu + u - \tau\, \partial_t y - f'(y) \in L^\infty(0,T;H)$*, whence we infer the additional regularity*

$$y \in L^\infty(0,T; V_B^{2\sigma}). \tag{4.13}$$

Hence, the solution (μ, y) *is strong, and* (1.8) *is valid almost everywhere in* Q.

In the following, we will always assume that the conditions **(GB)** and **(A10)** are satisfied and account for Remark 11. We now improve the stability estimate (3.4) established in Theorem 3.1.

Theorem 4.2 *Suppose that* **(A1)**–**(A5)**, **(A7)**–**(A10)**, (1.17), (4.2), *and* **(GB)** *are satisfied. Then there is a constant* $K_5 > 0$*, which depends only on the data of the state system and* R*, such that the following holds true: whenever*

$u_i \in \mathcal{U}_R$, $i = 1, 2$, are given and (μ_i, y_i), $i = 1, 2$, are the associated solutions to the state system (1.7)–(1.9)*, then, for every $t \in (0, T]$, it holds that*

$$\|\mu_1 - \mu_2\|_{L^2(0,t;V_A^{2r})} + \|y_1 - y_2\|_{H^1(0,t;H)\cap L^\infty(0,t;V_B^\sigma)} \leq K_5 \|u_1 - u_2\|_{L^2(0,t;H)} . \tag{4.14}$$

Proof Here, we refer the reader to the proof of Theorem 3.6 in Colli, Gilardi and Sprekels (2018c). □

Next, we aim at showing the Fréchet differentiability of the control-to-state mapping $\mathcal{S}$. To this end, we fix some $\bar{u} \in \mathcal{U}_R$ and set $(\bar{\mu}, \bar{y}) = \mathcal{S}(\bar{u})$. We then consider, for an arbitrary $k \in \mathcal{X}$ the linearized system

$$\partial_t \xi + A^{2r}\eta = 0 \quad \text{in } Q, \tag{4.15}$$

$$\tau\,\partial_t \xi + B^{2\sigma}\xi + f''(\bar{y})\xi = \eta + k \quad \text{in } Q, \tag{4.16}$$

$$\xi(0) = 0 \quad \text{in } \Omega. \tag{4.17}$$

More precisely, we consider the following weak version of the system (4.15)–(4.17):

$$(\partial_t \xi(t), v) + (A^r \eta(t), A^r v) = 0 \quad \text{for a.e. } t \in (0, T) \text{ and all } v \in V_A^r, \tag{4.18}$$

$$(\tau\,\partial_t \xi(t), v) + (B^\sigma \xi(t), B^\sigma v) + (f''(\bar{y}(t))\xi(t), v) = (\eta(t) + k(t), v) \quad \text{for a.e. } t \in (0, T) \text{ and all } v \in V_B^\sigma, \tag{4.19}$$

$$\xi(0) = 0. \tag{4.20}$$

For this system, we have the following result.

Theorem 4.3 *Under the assumptions of Theorem 4.2, the linearized system* (4.18)–(4.20) *admits for every $\bar{u} \in \mathcal{U}_{\rm ad}$ and every $k \in \mathcal{X}$ a unique solution (η, ξ) such that*

$$\eta \in L^2(0, T; V_A^r), \quad \xi \in H^1(0, T; H) \cap L^\infty(0, T; V_B^\sigma). \tag{4.21}$$

Moreover, there is a constant $K_6 > 0$, which depends only on the data of the state system and $R > 0$, such that

$$\|\eta\|_{L^2(0,T;V_A^r)} + \|\xi\|_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)} \leq K_6 \|k\|_{L^\infty(Q)} . \tag{4.22}$$

Proof We have to skip the somewhat technical and lengthy argument, which proceeds along similar lines as the proof of Theorem 3.1, and refer the interested reader to the proof of Theorem 4.1 from Colli, Gilardi and Sprekels (2018c). □

After these preparations, the road is paved for establishing the Fréchet differentiability of the control-to-state operator $\mathcal{S}$. We need, however, yet another assumption.

(A11) V_B^σ is continuously embedded in $L^4(\Omega)$.

REMARK 12 *Using standard embedding results for Sobolev-Slobodeckij spaces, it was shown in Colli, Gilardi and Sprekels (2018c) that* **(A11)** *is fulfilled if, e.g.,* $B = -\Delta$ *with zero Dirichlet or Neumann boundary conditions and* $\sigma \geq 3/8$.

Recalling Theorem 4.3, we have the following result, see Colli, Gilardi and Sprekels (2018c), Theorem 4.2.

THEOREM 4.4 *Suppose that the assumptions of Theorem 4.2 and* **(A11)** *are fulfilled. Then the control-to-state operator* $\mathcal{S} : u \mapsto \mathcal{S}(u) = (\mu, y)$ *is Fréchet differentiable in* $\mathcal{U}_R$ *when viewed as a mapping between the spaces* $\mathcal{X} = H^1(0,T;H) \cap L^\infty(Q)$ *and* $\mathcal{Y} := L^2(0,T;V_A^r) \times \left(H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma)\right)$. *Moreover, whenever* $\bar{u} \in \mathcal{U}_R$ *with* $(\bar{\mu}, \bar{y}) = \mathcal{S}(\bar{u})$ *is given, then the Fréchet derivative* $D\mathcal{S}(\bar{u}) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *of* $\mathcal{S}$ *at* $\bar{u}$ *is specified by the identity* $D\mathcal{S}(\bar{u})(k) = (\eta, \xi)$, *where* (η, ξ) *is the unique solution to the weak formulation* (4.18)–(4.20) *of the linearized system.*

PROOF For the reader's convenience, and in order to motivate the assumption **(A11)**, we sketch the proof. Since $\mathcal{U}_R$ is open, there is some $\Lambda > 0$ such that $\bar{u} + k \in \mathcal{U}_R$ whenever $k \in \mathcal{X}$ and $\|k\|_{\mathcal{X}} \leq \Lambda$. In the following, we consider only such perturbations k, for which we define the quantities

$$(\mu^k, y^k) := \mathcal{S}(\bar{u} + k), \quad \rho^k := \mu^k - \bar{\mu} - \eta^k, \quad z^k := y^k - \bar{y} - \xi^k,$$

where (η^k, ξ^k) denotes the unique solution (η, ξ) to the system (4.18)–(4.20). Obviously, we have $\rho^k \in L^2(0,T;V_A^r)$, $z^k \in H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma)$, as well as

$$(\partial_t z^k(t), v) + \left(A^r \rho^k(t), A^r v\right) = 0 \quad \text{for a.e. } t \in (0,T) \text{ and all } v \in V_A^r, \tag{4.23}$$

$$\tau\left(\partial_t z^k(t), v\right) + \left(B^\sigma z^k(t), B^\sigma v\right) + \left(f'(y^k(t)) - f'(\bar{y}(t)) - f''(\bar{y}(t))\xi^k(t), v\right)$$

$$= (\rho^k(t), v) \quad \text{for a.e. } t \in (0,T) \text{ and all } v \in V_B^\sigma, \tag{4.24}$$

$$z^k(0) = 0. \tag{4.25}$$

In addition, by Taylor's theorem and (4.11), we have almost everywhere in Q that

$$\left|f'(y^k) - f'(\bar{y}) - f''(\bar{y})\xi^k\right| \leq C_1\left(|z^k| + |y^k - \bar{y}|^2\right), \tag{4.26}$$

where, here and in the remainder of the proof, the constants $C_i > 0$, $i \in \mathbb{N}$, depend only on the data of the problem and R, but not on the special choice of $k \in \mathcal{X}$ with $\|k\|_{\mathcal{X}} \leq \Lambda$. Using (4.2) in Theorem 4.2 and **(A11)**, we infer that, for any $t \in (0,T]$,

$$\|y^k - \bar{y}\|_{L^\infty(0,t;L^4(\Omega))} \leq C_2 \|k\|_{L^2(0,t;H)}. \tag{4.27}$$

Now recall that by (4.22) the mapping $k \mapsto (\eta^k, \xi^k)$ is continuous from $\mathcal{X}$ into $\mathcal{Y}$. According to the notion of Fréchet differentiability, it therefore suffices to construct an increasing function $Z : (0, \Lambda) \to (0, +\infty)$, such that $\lim_{\lambda \searrow 0} \frac{Z(\lambda)}{\lambda^2} = 0$ and

$$\|\rho^k\|^2_{L^2(0,T;V_A^r)} + \|z^k\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)} \leq Z\left(\|k\|_{L^2(0,T;H)}\right) . \tag{4.28}$$

At this point, we take $v = \rho^k(t)$ in (4.23) and formally test (4.24) by $\partial_t z^k(t)$, then add the resulting equations and integrate over Q_t, where $t \in (0, T]$. In addition, we add the term $\int_0^t\!\int_\Omega z^k\, \partial_t z^k$ to both sides of the result. Invoking (4.26), we then obtain the inequality

$$\begin{aligned} &\frac{1}{2}\left(\|z^k(t)\|^2 + \|B^\sigma z^k(t)\|^2\right) + \tau \int_0^t\!\!\int_\Omega |\partial_t z^k|^2 + \int_0^t \|A^r \rho^k(s)\|^2\, ds \\ &\leq C_3 \int_0^t\!\!\int_\Omega |z^k|\, |\partial_t z^k| + C_4 \int_0^t\!\!\int_\Omega |\partial_t z^k| \left|y^k - \bar{y}\right|^2 =: I_1 + I_2, \end{aligned} \tag{4.29}$$

with obvious notation. Now, by Young's inequality, we have that

$$I_1 \leq \frac{\tau}{4} \int_0^t\!\!\int_\Omega |\partial_t z^k|^2 + C_5 \int_0^t\!\!\int_\Omega |z^k|^2 ,$$

while, by also using Hölder's inequality and (4.27),

$$\begin{aligned} I_2 &\leq C_4 \int_0^t \|\partial_t z^k(s)\|\, \|y^k(s) - \bar{y}(s)\|^2_{L^4(\Omega)}\, ds \\ &\leq \frac{\tau}{4} \int_0^t\!\!\int_\Omega |\partial_t z^k|^2 + C_6\, \|k\|^4_{L^2(0,T;H)} . \end{aligned}$$

Employing Gronwall's lemma, we thus conclude from (4.29) the estimate

$$\|z^k\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)} + \|A^r \rho^k\|^2_{L^2(0,T;H)} \leq C_7\, \|k\|^4_{L^2(0,T;H)} . \tag{4.30}$$

At this point, we have to distinguish between two cases. At first, if $\lambda_1 > 0$, then we have $\|\rho^k\|_{L^2(0,T;V_A^r)} \leq C_8\, \|A^r \rho^k\|_{L^2(0,T;H)}$, and thus (4.28) follows from (4.30) with the choice $Z(\lambda) = (1 + C_8) C_7\, \lambda^4$.

In the case of $\lambda_1 = 0$, we need to estimate the mean value of ρ^k. For the sake of shortness, we skip this argument, here, and refer the reader to the proof of Theorem 4.2 in Colli, Gilardi and Sprekels (2018c). □

Using the above result and the fact that $\mathcal{U}_{\mathrm{ad}}$ is a closed and convex subset of $\mathcal{X}$, we infer from the chain rule the following first-order necessary optimality condition:

Theorem 4.5 *Let the assumptions of Theorem 4.4 be satisfied, and assume that* $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ *with* $(\bar{\mu}, \bar{y}) = \mathcal{S}(\bar{u})$ *is a solution to the optimal control problem* **(CP)**.

Then, for every $v \in \mathcal{U}_{\mathrm{ad}}$*, the following inequality holds:*

$$\beta_1 \int_\Omega (\bar{y}(T) - y_\Omega)\, \xi(T) \,+\, \beta_2 \int_0^T\!\!\int_\Omega (\bar{y} - y_Q)\, \xi \,+\, \beta_3 \int_0^T\!\!\int_\Omega \bar{u}\, (v - \bar{u}) \,\geq\, 0, \tag{4.31}$$

where (η, ξ) *is the unique solution to the system* (4.18)–(4.20) *associated with* $k = v - \bar{u}$.

Next, we aim at eliminating the quantities η and ξ from (4.31) by means of the adjoint state variables. To this end, we consider the adjoint state system, which formally reads

$$A^{2r} p - q \,= 0 \quad \text{in } Q, \tag{4.32}$$

$$-(\partial_t p + \tau\, \partial_t q) + B^{2\sigma} q + f''(\bar{y})\, q \,=\, \beta_2(\bar{y} - y_Q) \quad \text{in } Q, \tag{4.33}$$

$$p(T) + \tau\, q(T) \,=\, \beta_1(\bar{y}(T) - y_\Omega) \quad \text{in } \Omega. \tag{4.34}$$

We consider a variational formulation of the above formal problem. To this end, we recall the definition (2.10) of $V_B^{-\sigma}$ and the embedding $H \subset V_B^{-\sigma}$ (see (2.11)); let us use the simpler notation $\langle \cdot\,, \cdot \rangle$ without indices for the duality pairing between $V_B^{-\sigma}$ and V_B^{σ}. For the adjoint state (p, q), we require the following regularity conditions:

$$p \in L^2(0,T; V_A^{2r}), \tag{4.35}$$

$$q \in L^2(0,T; V_B^{\sigma}), \tag{4.36}$$

$$p + \tau q \in H^1(0,T; V_B^{-\sigma}). \tag{4.37}$$

The adjoint problem then reads as follows:

$$(A^r p(t), A^r v) - (q(t), v) = 0 \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V_A^r, \tag{4.38}$$

$$\begin{aligned} &- \langle \partial_t (p + \tau q)(t), v \rangle + (B^\sigma q(t), B^\sigma v) + (\psi(t) q(t), v) = (g_2(t), v) \\ &\quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V_B^{\sigma}, \end{aligned} \tag{4.39}$$

$$(p + \tau q)(T) = g_1, \tag{4.40}$$

where, for brevity, we have set

$$\psi := f''(\bar{y}), \quad g_1 := \beta_1(\bar{y}(T) - y_\Omega) \quad \text{and} \quad g_2 := \beta_2(\bar{y} - y_Q). \tag{4.41}$$

We have written for convenience the weak form (4.38), which still makes sense under the weaker regularity requirement $p \in L^2(0,T; V_A^r)$. However, it is immediately seen that such a regularity and (4.38) imply (4.35) and

$$q = A^{2r} p. \tag{4.42}$$

The solution of the problem (4.38)–(4.40) requires (in particular, in the case of $\lambda_1 = 0$) a considerable functional analytic effort and cannot be carried out in detail in this overview paper. We thus have to refer to Theorem 5.8 from Colli, Gilardi and Sprekels (2018c), which we state here without proof:

Theorem 4.6 *Suppose that the assumptions of Theorem 4.4 are fulfilled, assume that $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$, and let $(\bar{\mu}, \bar{y}) = \mathcal{S}(\bar{u})$ be the corresponding state. Then, the adjoint problem* (4.38)–(4.40) *has a unique solution (p, q) satisfying* (4.35)–(4.37).

We conclude this section with the first-order necessary condition for optimality, expressed in terms of the adjoint state variables. Its proof uses quite standard arguments. However, there is a technical point where the following integration-by-parts formula, proven in Colli, Gilardi and Sprekels (2018b), Lemma 4.5, is needed: let $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$ be a Hilbert triplet and $(\cdot, \cdot)_{\mathcal{H}}$ denote the inner product in $\mathcal{H}$. If

$$w \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \quad \text{and} \quad z \in H^1(0,T;\mathcal{V}^*) \cap L^2(0,T;\mathcal{H}),$$

then the function $t \mapsto (w(t), z(t))_{\mathcal{H}}$ is absolutely continuous, and for every $t_1, t_2 \in [0, T]$ we have that

$$\int_{t_1}^{t_2} \bigl\{ \bigl(\partial_t w(s), z(s)\bigr)_{\mathcal{H}} + \langle \partial_t z(s), w(s) \rangle_{\mathcal{V}} \bigr\} \, ds = \bigl(w(t_2), z(t_2)\bigr)_{\mathcal{H}} - \bigl(w(t_1), z(t_1)\bigr)_{\mathcal{H}} . \tag{4.43}$$

This formula is used in the sequel of the paper as well.

Theorem 4.7 *Let the assumptions of Theorem 4.4 be satisfied, and assume that $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ is a solution to the optimal control problem* **(CP)** *with associated state $(\bar{\mu}, \bar{y}) = \mathcal{S}(\bar{u})$. Moreover, let (p, q) be the unique solution to the corresponding adjoint problem. Then, the following variational inequality holds true:*

$$\int_0^T \int_\Omega (q + \beta_3 \bar{u})(v - \bar{u}) \ \geq \ 0 \quad \forall \, v \in \mathcal{U}_{\mathrm{ad}}. \tag{4.44}$$

In particular, if $\beta_3 \neq 0$, then the optimal control $\bar{u}$ coincides with the $L^2(0,T;H)$ - projection of $-\beta_3^{-1} q$ on $\mathcal{U}_{\mathrm{ad}}$.

Before going to the proof of the above result, we note that it remains a difficult task to exploit the variational inequality (4.44) for, e.g., numerical purposes, since the structure of the admissible set $\mathcal{U}_{\mathrm{ad}}$ is complicated.

Proof Fix any $v \in \mathcal{U}_{\mathrm{ad}}$, set $k := v - \bar{u}$, and consider the solutions (η, ξ) and (p, q) to the corresponding linearized system (4.18)–(4.20) and the adjoint system (4.38)–(4.40), respectively. We test (4.18) and (4.19) by $p(t)$ and $q(t)$, respectively. Then, we add the resulting equalities and integrate over $(0, T)$. With the notations (4.41), we have that

$$\begin{aligned}
&\int_0^T \bigl\{ \bigl(\partial_t \xi(t), p(t)\bigr) + \bigl(A^r \eta(t), A^r p(t)\bigr) \bigr\} \, dt \\
&\quad + \int_0^T \bigl\{ \bigl(\tau \, \partial_t \xi(t), q(t)\bigr) + \bigl(B^\sigma \xi(t), B^\sigma q(t)\bigr) + \bigl(\psi(t) \xi(t), q(t)\bigr) \bigr\} \, dt \\
&= \int_0^T \bigl(\eta(t) + k(t), q(t)\bigr) \, dt \, .
\end{aligned}$$

At the same time, we test (4.38) and (4.39) by $-\eta(t)$ and $-\xi(t)$, sum up, and integrate with respect to t to obtain

$$\begin{aligned}&\int_0^T \bigl\{-\bigl(A^r p(t), A^r \eta(t)\bigr) + \bigl(q(t), \eta(t)\bigr)\bigr\}\, dt \\ &\quad + \int_0^T \bigl\{\langle \partial_t (p+\tau q)(t), \xi(t)\rangle - \bigl(B^\sigma q(t), B^\sigma \xi(t)\bigr) - \bigl(\psi(t) q(t), \xi(t)\bigr)\bigr\}\, dt \\ &= -\int_0^T \bigl(g_2(t), \xi(t)\bigr)\, dt.\end{aligned}$$

Addition of the above equations yields that

$$\begin{aligned}&\int_0^T \bigl\{\bigl(\partial_t \xi(t), (p+\tau q)(t)\bigr) + \langle \partial_t (p+\tau q)(t), \xi(t)\rangle\bigr\}\, dt \\ &= \int_0^T \bigl(k(t), q(t)\bigr)\, dt - \int_0^T \bigl(g_2(t), \xi(t)\bigr)\, dt,\end{aligned} \tag{4.45}$$

and we infer from the integration-by-parts formula (4.43) and the Cauchy conditions (4.20) and (4.40) that

$$\bigl(g_1, \xi(T)\bigr) + \int_0^T \bigl(g_2(t), \xi(t)\bigr)\, dt = \int_0^T \bigl(q(t), k(t)\bigr)\, dt\,. \tag{4.46}$$

Insertion of this formula in (4.31) yields the validity of (4.44). □

4.3. The double obstacle case

Let the assumptions **(A1)**–**(A3)**, **(A7)**, and (3.56)–(3.59) be satisfied. We consider in this section the optimal control problem **(CP$_0$)**, which is given by (see (1.16) and (1.17)):

(CP$_0$) Minimize $\mathcal{J}(y,u)$ over $\mathcal{U}_{\rm ad}$ subject to the state system (1.4)–(1.6) with $f = f_{\rm obs}$, where $f_{\rm obs}$ is defined by (1.15).

Then, **(A4)**–**(A9)** and, since $V_B^\sigma = V_B^{1/2} = H^1(\Omega)$ in this case, also **(A10)** and **(A11)** are automatically satisfied. Moreover, we know from Theorem 3.1 that the corresponding state system (1.4)–(1.6) has for every $u \in \mathcal{U}_{\rm ad}$ a solution (μ, y), satisfying (1.1)–(1.3). More precisely, since $I_{[-1,1]}(y) \in L^1(Q)$ for our notion of solution, we must have $y \in [-1,1]$ a.e. in Q, so that $\int_\Omega I_{[-1,1]}(y(t)) = 0$ for a.e. $t \in (0,T)$. Therefore, the inequality (1.5) takes in this case the form

$$\begin{aligned}&(\tau \partial_t y(t), y(t) - v) + \int_\Omega \nabla y(t) \cdot \nabla (y(t) - v) \;+\; (f_2'(y(t)), y(t) - v) \\ &\le\; (\mu(t) + u(t), y(t) - v) + \int_\Omega I_{[-1,1]}(v) \\ &\qquad\qquad\qquad \text{for a.e. } t \in (0,T) \text{ and all } v \in H^1(\Omega).\end{aligned} \tag{4.47}$$

Observing that in this case only the second solution component y is uniquely determined, we introduce the corresponding "partial" control-to-state operator

$$\mathcal{S}_0^2 : \mathcal{X} \ni u \mapsto \mathcal{S}_0^2(u) := y. \tag{4.48}$$

As already mentioned in the introduction, the main difficulty, inherent in $(\mathbf{CP}_0)$, is the fact that it seems impossible to separate the values of y away from the critical points ± 1, so that the validity of the condition **(GB)** cannot be expected and the nondifferentiability of the nonlinearity $I_{[-1,1]}$ at ± 1 comes fully into play. Hence, an argumentation along the lines of the previous section is not possible. In order to circumvent this difficulty, we approach the problem via the "deep quench" approximation, described in Section 1.3. Thus, let the functions φ, h, and h^α, be defined as in (1.18)–(1.20). We then consider for $\alpha > 0$ the optimal control problem

$(\mathbf{CP}_\alpha)$ Minimize the cost functional (1.16) over $\mathcal{U}_{\rm ad}$ subject to the state system (1.4)–(1.6), where $f_1 = h^\alpha$.

As noted in Remark 9, it follows from Theorem 3.3 that the condition **(GB)** is satisfied for the $\alpha-$state system (1.4)–(1.6), which then, by Remark 10, has for every $u \in \mathcal{U}_R$ a unique solution (μ^α, y^α); we denote the corresponding "total" solution operator (compare (4.10)) by

$$\mathcal{S}_\alpha : \mathcal{X} \ni u \mapsto \mathcal{S}_\alpha(u) = (\mathcal{S}_\alpha^1(u), \mathcal{S}_\alpha^2(u)) := (\mu^\alpha, y^\alpha). \tag{4.49}$$

More precisely, by virtue of (3.59) and (3.61) in Theorem 3.3, there are constants $r_*(\alpha), r^*(\alpha) \in (-1,1)$, for every $\alpha \in (0,1]$, such that, for every $u \in \mathcal{U}_R$,

$$r_*(\alpha) \le y^\alpha \le r^*(\alpha) \quad \text{a.e. in } Q, \quad r_*(\alpha) \le y_0 \le r^*(\alpha) \quad \text{a.e. in } \Omega. \tag{4.50}$$

In addition, it follows from Remark 11 that the variational inequality (1.5) for $f_1 = h^\alpha$ is actually equivalent to the variational equation (4.12) with $f_1 = h^\alpha$. A fortiori, we have $y^\alpha \in L^\infty(0,T;V_B^{2\sigma})$ (see (4.13)), and (μ^α, y^α) is even a strong solution that satisfies (1.7) and (1.8) almost everywhere in Q with $f_1 = h^\alpha$.

We now derive some a priori bounds for the family $\{(\mu^\alpha, y^\alpha)\}$. To this end, we take a closer look at the proof of Theorem 3.1. It turns out that the a priori estimates, carried out there, are in fact independent of $\alpha \in (0,1]$ and $u \in \mathcal{U}_R$. A fortiori, the same holds true for the estimates leading to (3.55) (see Colli, Gilardi and Sprekels, 2019a). We therefore have, invoking (3.57), that

$$\begin{aligned}&\|y^\alpha\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V_B^\sigma)} + \|\mu^\alpha\|_{L^\infty(0,T;V_A^{2r})} + \|\mu^\alpha\|_{L^\infty(Q)} \le K_3\\ &\text{for all } \alpha \in (0,1] \text{ and } u \in \mathcal{U}_R.\end{aligned} \tag{4.51}$$

Notice that a global bound resembling (4.11) for $f_1 = h^\alpha$ cannot be expected to hold true, since it may well happen that $r_*(\alpha) \searrow -1$ and/or $r^*(\alpha) \nearrow +1$ as $\alpha \searrow 0$, so that $(h^\alpha)'(y^\alpha)$ and $(h^\alpha)''(y^\alpha)$ may become unbounded as $\alpha \searrow 0$.

The following approximation result, which is a special case of Theorem 3.2 from Colli, Gilardi and Sprekels (2018d), is rather intuitive.

THEOREM 4.8 *Suppose that the assumptions* **(A1)**–**(A3)**, **(A7)**, (1.18)–(1.20), *and* (3.56)–(3.59), *are fulfilled, and let sequences* $\{\alpha_n\} \subset (0,1]$ *and* $\{u_n\} \subset \mathcal{X}$ *be given such that* $\alpha_n \searrow 0$ *and* $u_n \to u$ *weakly-star in* $\mathcal{X}$ *as* $n \to \infty$ *for some* $u \in \mathcal{X}$. *Moreover, let* $(\mu^{\alpha_n}, y^{\alpha_n}) := \mathcal{S}_{\alpha_n}(u_n)$ *for all* $n \in \mathbb{N}$. *Then there is a solution* (μ, y) *to* (1.4)–(1.6) *in the sense of Theorem 3.2 for* $f_1 = I_{[-1,1]}$ *(i.e., we have* $y = \mathcal{S}_0^2(u)$*) and a subsequence* $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ *of* $\{\alpha_n\}_{n\in\mathbb{N}}$ *such that, as* $k \to \infty$,

$$\mu^{\alpha_{n_k}} \to \mu \quad \textit{weakly-star in } L^\infty(0,T;V_A^{2r}), \tag{4.52}$$

$$\begin{aligned} y^{\alpha_{n_k}} \to y \quad & \textit{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma) \\ & \textit{and strongly in } C^0([0,T];H). \end{aligned} \tag{4.53}$$

REMARK 13 *Since the second solution component* y *is by Theorem 3.1 uniquely determined, it follows that the convergence property* (4.53) *is in fact valid for the entire sequence* $\{\alpha_n\}_{n\in\mathbb{N}}$*; we also have* $A^r\mu^{\alpha_n} \to A^r\mu$ *weakly in* $L^2(0,T;H)$, *and thus also* (4.52) *holds for the entire sequence if* $\lambda_1 > 0$.

Theorem 4.8 has an important consequence for the approximation of optimal pairs for $(\mathbf{CP}_0)$. In this connection, we know from Theorem 4.1 that $(\mathbf{CP}_\alpha)$ has for every $\alpha > 0$ a minimizer $\bar{u}^\alpha \in \mathcal{U}_{\text{ad}}$ with associated state $(\bar{\mu}^\alpha, \bar{y}^\alpha) = \mathcal{S}_\alpha(\bar{u}^\alpha)$. The following result is not difficult to prove (see the proof of Corollary 4.3 in Colli, Gilardi and Sprekels, 2018d):

THEOREM 4.9 *Let the assumptions of Theorem 4.8 be satisfied, and suppose that* $\bar{u}^{\alpha_n} \in \mathcal{U}_{\text{ad}}$ *is a minimizer for* $(\mathbf{CP}_{\alpha_n})$ *with associated state* $(\bar{\mu}^{\alpha_n}, \bar{y}^{\alpha_n}) = \mathcal{S}_{\alpha_n}(\bar{u}^{\alpha_n})$, $n \in \mathbb{N}$, *for some sequence* $\alpha_n \searrow 0$. *Then, there are a subsequence* $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ *and functions* $\bar{u}$, $\bar{\mu}$, $\bar{y}$ *such that, as* $k \to \infty$,

$$\bar{u}^{\alpha_{n_k}} \to \bar{u} \quad \textit{weakly-star in } \mathcal{X}, \tag{4.54}$$

$$\bar{\mu}^{\alpha_{n_k}} \to \bar{\mu} \quad \textit{weakly-star in } L^\infty(0,T;V_A^{2r}), \tag{4.55}$$

$$\begin{aligned} \bar{y}^{\alpha_{n_k}} \to \bar{y} \quad & \textit{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma) \\ & \textit{and strongly in } C^0([0,T];H). \end{aligned} \tag{4.56}$$

Moreover, $\bar{u} \in \mathcal{U}_{\text{ad}}$ *is a minimizer of* $(\mathbf{CP}_0)$, *where* $(\bar{\mu}, \bar{y})$ *solves the system* (1.4)–(1.6) *with* $f_1 = I_{[-1,1]}$ *and* $u = \bar{u}$.

Theorems 4.8 and 4.9 indicate that optimal controls of $(\mathbf{CP}_\alpha)$ are "close" to optimal controls of $(\mathbf{CP}_0)$. However, they do not yield sufficient information on the minimizers of $(\mathbf{CP}_0)$. In order to find first-order necessary optimality conditions, we recall that in the previous section we have been able to derive such conditions for the problem $(\mathbf{CP}_\alpha)$ (see Remark 9). Thus, we can hope to establish corresponding results for $(\mathbf{CP}_0)$ through an approximation process by taking the limit as $\alpha \searrow 0$. It seems, however, that such an approach fails since the convergence property (4.54) is not sufficient to pass to the limit as $\alpha \searrow 0$ in the variational inequality (4.44) (written for $\bar{u}^\alpha$ and the corresponding adjoint state q^α). For this, we seem to need a strong convergence of $\{\bar{u}^\alpha\}$ in $L^2(Q)$.

To this end, we employ a well-known trick, introduced in Barbu (1981). Let us assume that $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ is any optimal control for $(\mathbf{CP}_0)$ with associated state $(\bar{\mu}, \bar{y})$, where $\bar{y} = \mathcal{S}_0^2(\bar{u})$. We associate with it the *adapted cost functional*

$$\widetilde{\mathcal{J}}(y,u) := \mathcal{J}(y,u) \; + \; \frac{1}{2} \, \|u - \bar{u}\|^2_{L^2(Q)} \tag{4.57}$$

and a corresponding *adapted optimal control problem*:

$(\widetilde{\mathbf{CP}}_\alpha)$ Minimize the cost functional (4.57) over $\mathcal{U}_{\mathrm{ad}}$, subject to the state system (1.4)–(1.6), where $f_1 = h^\alpha$.

With the same direct argument as in the proof of Theorem 4.1, we can show that $(\widetilde{\mathbf{CP}}_\alpha)$ has a solution. The following result (see Theorem 4.5 in Colli, Gilardi and Sprekels, 2018d) indicates why the adapted control problem suits better our intended approximation approach.

Theorem 4.10 *Let the assumptions* **(A1)**–**(A3)**, **(A7)**, (1.17), (1.18)–(1.20), *and* (3.56)–(3.59), *be fulfilled, suppose that* $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ *is an arbitrary optimal control of* $(\mathbf{CP}_0)$ *with associated state* $(\bar{\mu}, \bar{y})$, *where* $\bar{y} = \mathcal{S}_0^2(\bar{u})$, *and let* $\{\alpha_n\} \subset (0,1]$ *be any sequence such that* $\alpha_n \searrow 0$ *as* $n \to \infty$. *Then, there exist a subsequence* $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ *of* $\{\alpha_n\}$, *and, for every* $k \in \mathbb{N}$, *an optimal control* $\bar{u}^{\alpha_{n_k}} \in \mathcal{U}_{\mathrm{ad}}$ *of the adapted problem* $(\widetilde{\mathbf{CP}}_{\alpha_{n_k}})$ *with associated state* $(\bar{\mu}^{\alpha_{n_k}}, \bar{y}^{\alpha_{n_k}}) = \mathcal{S}_{\alpha_{n_k}}(\bar{u}^{\alpha_{n_k}})$, *such that, as* $k \to \infty$,

$$\bar{u}^{\alpha_{n_k}} \to \bar{u} \quad \textit{strongly in } L^2(Q), \tag{4.58}$$

and the properties (4.55) and (4.56) *are satisfied. Moreover, we have*

$$\lim_{k\to\infty} \widetilde{\mathcal{J}}(\bar{y}^{\alpha_{n_k}}, \bar{u}^{\alpha_{n_k}}) \; = \; \mathcal{J}(\bar{y}, \bar{u}) \, . \tag{4.59}$$

We now discuss the first-order necessary optimality conditions for $(\widetilde{\mathbf{CP}}_\alpha)$. Obviously, the adjoint system is the same as for $(\mathbf{CP}_\alpha)$, and Theorem 4.6 applies to this situation. More precisely, the adjoint state (p^α, q^α) solves the variational system (see (4.38)–(4.40))

$$(A^r p^\alpha(t), A^r v) - (q^\alpha(t), v) = 0 \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V_A^r, \tag{4.60}$$

$$\begin{aligned} & -\langle \partial_t (p^\alpha + \tau q^\alpha)(t), v \rangle + \int_\Omega \nabla q^\alpha(t) \cdot \nabla v + ((\psi_1^\alpha(t) + \psi_2^\alpha(t)) \, q^\alpha(t), v) \\ & = (g_2^\alpha(t), v) \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in H^1(\Omega), \end{aligned} \tag{4.61}$$

$$(p^\alpha + \tau q^\alpha)(T) = g_1^\alpha, \tag{4.62}$$

where, for $\alpha > 0$,

$$\begin{aligned} & \psi_1^\alpha := (h^\alpha)''(\bar{y}^\alpha), \quad \psi_2^\alpha := f_2''(\bar{y}^\alpha), \\ & g_1^\alpha := \beta_1(\bar{y}^\alpha(T) - y_\Omega), \quad g_2^\alpha := \beta_2(\bar{y}^\alpha - y_Q). \end{aligned} \tag{4.63}$$

We note that (4.62) has a proper meaning, since $p^\alpha + \tau q^\alpha \in C^0([0,T]; H^1(\Omega)^*)$, by virtue of (4.37). Besides, owing to the general bounds (4.50) and (4.51), we have that

$$\|\psi_2^\alpha\|_{L^\infty(Q)} + \|g_1^\alpha\|_{L^2(\Omega)} + \|g_2^\alpha\|_{L^2(Q)} \le C_1 \quad \forall\, \alpha \in (0,1], \tag{4.64}$$

where, here and in the following, $C_i > 0$, $i \in \mathbb{N}$, denote constants that may depend on the data of the system, but not on $\alpha \in (0,1]$. Observe that a corresponding bound for ψ_1^α cannot be expected.

On the other hand, the variational inequality, characterizing the optimal controls, is different (obtained, nevertheless, using the same arguments that led to (4.44) in Theorem 4.7). Namely, if $\bar{u}^\alpha \in \mathcal{U}_{\rm ad}$ is optimal for $(\widetilde{\mathbf{CP}}_\alpha)$ and (p^α, q^α) is the associated adjoint state, then we have that

$$\int_0^T\!\!\int_\Omega (q^\alpha + \beta_3 \bar{u}^\alpha + (\bar{u}^\alpha - \bar{u}))(v - \bar{u}^\alpha) \,\ge\, 0 \quad \forall\, v \in \mathcal{U}_{\rm ad}. \tag{4.65}$$

Our aim is to let α tend to zero in both the above inequality and the adjoint system. Thus, we have to derive some a priori estimates for the adjoint variables that are uniform with respect to $\alpha \in (0,1]$. To this end, we set

$$H_0 := \{v \in H : \mathrm{mean}(v) = 0\}, \quad H^{1,0}(\Omega) := H^1(\Omega) \cap H_0, \tag{4.66}$$

and recall the definition (2.18) of V_0^r. We then claim that, for all $\alpha \in (0,1]$,

$$\|p^\alpha\|_{L^\infty(0,T;V_A^{2r})} + \|q^\alpha\|_{L^\infty(0,T;H)\cap L^2(0,T;H^1(\Omega))} \,\le\, C_2 \quad \text{if } \lambda_1 > 0, \tag{4.67}$$

$$\|p^\alpha - \mathrm{mean}(p^\alpha)\mathbf{1}\|_{L^\infty(0,T;V_0^{2r})} + \|q^\alpha\|_{L^\infty(0,T;H_0)\cap L^2(0,T;H^{1,0}(\Omega))} \le C_2 \quad \text{if } \lambda_1 = 0. \tag{4.68}$$

Remark 14 *Let us briefly motivate why in the case $\lambda_1 = 0$ the stronger estimate (4.67) cannot be expected to hold. Indeed, in this case we have $A^r\mathbf{1} = 0$, and thus insertion of $v = \mathbf{1}$ in (4.60) shows that q^α has zero mean value almost everywhere in $(0,T)$. Now, we insert $v = \mathbf{1}$ in (4.61) and integrate with respect to time over $[t,T]$ for arbitrary $t \in [0,T)$. Using the integration-by-parts formula (4.43), we then find the identity*

$$\mathrm{mean}(p^\alpha(t) + \tau q^\alpha(t)) = \mathrm{mean}(g_1^\alpha) + \frac{1}{|\Omega|}\int_t^T\!\!\int_\Omega (g_2^\alpha - \psi_1^\alpha q^\alpha - \psi_2^\alpha q^\alpha)\,, \tag{4.69}$$

where the left-hand side equals $\mathrm{mean}(p^\alpha(t))$ for almost every $t \in (0,T)$. Now observe that the term $-\int_t^T\int_\Omega \psi_1^\alpha q^\alpha$ on the right-hand side cannot be controlled uniformly in α, and thus we cannot expect to find a bound for the $L^\infty(0,T)$–norm of $\mathrm{mean}(p^\alpha)\mathbf{1}$. In fact, (4.3) is the best we can hope for.

We only sketch the proof of the above claim for the simpler case $\lambda_1 > 0$, referring to Colli, Gilardi and Sprekels (2018d) for the more delicate case of

$\lambda_1 = 0$. So, let $\lambda_1 > 0$. Then the mapping A^{2r} is a topological isomorphism from V_A^{2r} onto H with the inverse $A^{-2r} = (A^{2r})^{-1} : H \to V^{2r}$, where

$$A^{-2r} v = \sum_{j=1}^{\infty} \lambda_j^{-2r} (v, e_j) e_j \quad \forall\, v \in H. \tag{4.70}$$

We can thus rewrite (4.60) as $p^\alpha = A^{-2r} q^\alpha$, and (4.61) and (4.62) as, respectively,

$$- \langle \partial_t (A^{-2r} + \tau I)\, q^\alpha(t), v\rangle + \int_\Omega \nabla q^\alpha(t) \cdot \nabla v + ((\psi_1^\alpha(t) + \psi_2^\alpha(t))\, q^\alpha(t), v)$$

$$= (g_2^\alpha(t), v) \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in H^1(\Omega), \tag{4.71}$$

$$(A^{-2r} + \tau I)\, q^\alpha(T) = g_1^\alpha. \tag{4.72}$$

Notice that also the linear operator $A^{-2r} + \tau I \in \mathcal{L}(H,H)$ is a topological isomorphism, so that (4.72) can be equivalently written as

$$q^\alpha(T) = (A^{-2r} + \tau I)^{-1} g_1^\alpha, \tag{4.73}$$

which gives $q^\alpha(T)$ a proper meaning as well.

At this point, we test (4.71) by $q^\alpha(t)$ and integrate with respect to time over (t,T), where $t \in [0,T)$. We then conclude the equation

$$\int_t^T \langle -\partial_t \big((A^{-2r} + \tau I) q^\alpha\big)(\rho), q^\alpha(\rho)\rangle \, d\rho + \int_t^T \|\nabla q^\alpha(\rho)\|^2 \, d\rho + \int_t^T \int_\Omega \psi_1^\alpha \, |q^\alpha|^2$$

$$= \int_t^T \int_\Omega \big(-\psi_2^\alpha q^\alpha + g_2^\alpha\big)\, q^\alpha \,, \tag{4.74}$$

where the last term on the left-hand side is nonnegative and, owing to (4.64), the right-hand side is bounded by an expression of the form

$$C_3 \,+\, C_4 \int_t^T \int_\Omega |q^\alpha|^2 \,. \tag{4.75}$$

Now observe that, by definition (2.2), and since $\lambda_1 > 0$, it holds for every $v \in H$ that

$$(A^{-2r} + \tau I)^{1/2} v \,=\, \sum_{j=1}^{\infty} \big(\lambda_j^{-2r} + \tau\big)^{1/2} (v, e_j) e_j, \tag{4.76}$$

and thus

$$\left\|(A^{-2r}+\tau I)^{1/2}v\right\|^2 \;=\; \sum_{j=1}^{\infty}\big(\lambda_j^{-2r}+\tau\big)\,|(v,e_j)|^2 \;\geq\; \tau\,\|v\|^2, \tag{4.77}$$

$$\left\|(A^{-2r}+\tau I)^{1/2}v\right\|^2 \;\leq\; \sum_{j=1}^{\infty}(\lambda_1^{-2r}+\tau)|(v,e_j)|^2 \;\leq\; (\lambda_1^{-2r}+\tau)\,\|v\|^2, \tag{4.78}$$

$$\left\|(A^{-2r}+\tau I)^{-1}v\right\|^2 \;=\; \sum_{j=1}^{\infty}(\lambda_j^{-2r}+\tau)^{-2}|(v,e_j)|^2 \;\leq\; \tau^{-2}\,\|v\|^2. \tag{4.79}$$

Now, it is easily verified that

$$-\,\langle\partial_t(A^{-2r}+\tau I)\,q^\alpha(t),q^\alpha(t)\rangle \;=\; -\,\frac{1}{2}\,\frac{d}{dt}\left\|(A^{-2r}+\tau I)^{1/2}q^\alpha(t)\right\|^2. \tag{4.80}$$

Therefore, on account of (4.73), the first term on the left-hand side of (4.74) is equal to

$$\frac{1}{2}\left\|(A^{-2r}+\tau I)^{1/2}\,q^\alpha(t)\right\|^2 - \frac{1}{2}\left\|(A^{-2r}+\tau I)^{1/2}(A^{-2r}+\tau I)^{-1}g_1^\alpha\right\|^2, \tag{4.81}$$

which, by (4.64) and (4.77)–(4.79), is bounded from below by $\frac{\tau}{2}\,\|q^\alpha(t)\|^2 - C_5$, with some global constant $C_5>0$. At this point, we invoke Gronwall's lemma, and the fact that $p^\alpha = A^{-2r}q^\alpha$, to conclude that (4.67) is valid.

With (4.67) and (4.68) demonstrated, it is now not too difficult to derive further estimates for the adjoint variables, where, from now on, we treat the cases $\lambda_1>0$ and $\lambda_1=0$ simultaneously. Here, it is understood that the spaces V_0^r and the operators A_0^r are defined as in (2.17) and (2.19), respectively. We now introduce the space

$$\mathcal{Z} := \begin{cases} \{v\in H^1(0,T;H)\cap L^2(0,T;H^1(\Omega)) : v(0)=0\} & \text{if } \lambda_1>0 \\ \{v\in H^1(0,T;H_0)\cap L^2(0,T;H^1(\Omega)) : v(0)=0\} & \text{if } \lambda_1=0 \end{cases}, \tag{4.82}$$

which is a Hilbert space when endowed with its natural inner product and norm. Moreover, setting

$$G=H \quad \text{for } \lambda_1>0 \quad \text{and} \quad G=H_0 \quad \text{for } \lambda_1=0, \tag{4.83}$$

we see that the embedding $\mathcal{Z}\subset C^0([0,T];G)$ is continuous. Furthermore, we also have the dense and continuous embedding $\mathcal{Z}\subset L^2(0,T;G)\subset\mathcal{Z}^*$, where it is understood that

$$\langle v,z\rangle_{\mathcal{Z}} = \int_0^T (v(t),z(t))\,dt \quad \text{for all } z\in\mathcal{Z} \text{ and } v\in L^2(0,T;G). \tag{4.84}$$

In order not to have to distinguish between the two cases, we use in the following the same notation $\langle\cdot,\cdot\rangle$ for the dual pairings $\langle\cdot,\cdot\rangle_{H^1(\Omega)}$ and $\langle\cdot,\cdot\rangle_{H^{1,0}(\Omega)}$, where the former corresponds to the case $\lambda_1>0$ and the latter to the case $\lambda_1=0$.

At this point, by invoking (4.35)–(4.37) for $\lambda_1 > 0$ and the corresponding regularity results, see Colli, Gilardi and Sprekels (2018d), for $\lambda_1 = 0$, we may employ the integration-by-parts formula (4.43) to conclude that for every $v \in \mathcal{Z}$ there holds

$$\begin{aligned}
\langle -\partial_t(p^\alpha + \tau q^\alpha), v\rangle_{\mathcal{Z}} \;&=\; -\int_0^T \langle \partial_t(A_0^{-2r} q^\alpha(t) + \tau q^\alpha(t)), v(t)\rangle \, dt \\
&= \int_0^T (\partial_t v(t), (A_0^{-2r} + \tau I) q^\alpha(t))\, dt \;-\; (g_1^\alpha, v(T)) \\
&\le\; \|\partial_t v\|_{L^2(0,T;H)}\, \|(A_0^{-2r} + \tau I) q^\alpha\|_{L^2(0,T;H)} \;+\; \|g_1^\alpha\|_H\, \|v(T)\|_H \\
&\le\; C_6\, \|v\|_{\mathcal{Z}} \quad \forall\, \alpha \in (0,1],
\end{aligned} \tag{4.85}$$

which implies that

$$\|\partial_t(p^\alpha + \tau q^\alpha)\|_{\mathcal{Z}^*} \;\le\; C_7 \quad \forall\, \alpha \in (0,1]. \tag{4.86}$$

Now, observe that for any $v \in \mathcal{Z}$ it holds that

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \nabla q^\alpha \cdot \nabla v + \int_0^T (\psi_2^\alpha(t) q^\alpha(t), v(t))\, dt \;-\; \int_0^T (g_2^\alpha(t), v(t))\, dt \\
&\le \|q^\alpha\|_{L^2(0,T;H^1(\Omega))} \|v\|_{\mathcal{Z}} + C_8\, \|q^\alpha\|_{L^2(0,T;H)} \|v\|_{L^2(0,T;H)} + C_9\, \|v\|_{L^2(0,T;H)} \\
&\le\; C_{10}\, \|v\|_{\mathcal{Z}} \quad \forall\, \alpha \in (0,1],
\end{aligned}$$

and it follows from the comparison in (4.61) that, with $\Lambda^\alpha := \psi_1^\alpha q^\alpha = \varphi(\alpha) h''(\bar y^\alpha) q^\alpha$,

$$\|\Lambda^\alpha\|_{\mathcal{Z}^*} \;\le\; C_{11} \quad \forall\, \alpha \in (0,1]. \tag{4.87}$$

At this point, we choose any sequence $\{\alpha_n\}$ such that $\alpha_n \searrow 0$. By Theorem 4.10, we may, without loss of generality, assume that

$$\bar u^{\alpha_n} \to \bar u \quad \text{strongly in } \; L^2(Q), \tag{4.88}$$

$$\bar y^{\alpha_n} \to \bar y \quad \text{weakly-star in } \; W^{1,\infty}(0,T;H) \cap H^1(0,T;H^1(\Omega)), \tag{4.89}$$

and, by compact embedding, that

$$\bar y^{\alpha_n} \to \bar y \quad \text{strongly in } \; C^0([0,T];L^p(\Omega)) \quad \forall\, p \in [1,6), \tag{4.90}$$

which entails, in particular, that

$$f_2''(\bar y^{\alpha_n}) \to f_2''(\bar y) \quad \text{strongly in } \; C^0([0,T];L^p(\Omega)) \quad \forall\, p \in [1,6), \tag{4.91}$$

$$g_1^{\alpha_n} \to \beta_1(\bar y(T) - y_\Omega) \quad \text{strongly in } \; H, \tag{4.92}$$

$$g_2^{\alpha_n} \to \beta_2(\bar y - y_Q) \quad \text{strongly in } \; L^2(Q). \tag{4.93}$$

Moreover, by virtue of the estimates (4.67), (4.3), (4.86), and (4.87), there are limits $\zeta, \bar{q}, \Lambda$ such that, at least for a subsequence, which is again indexed by n,

$$\partial_t(A_0^{-2r} + \tau I)q^{\alpha_n} \to \zeta \quad \text{weakly in } \mathcal{Z}^*, \tag{4.94}$$

$$q^{\alpha_n} \to \bar{q} \quad \text{weakly-star in } L^\infty(0,T;G) \cap L^2(0,T;H^1(\Omega)), \tag{4.95}$$

$$A_0^{-2r} q^{\alpha_n} \to A_0^{-2r}\bar{q} \quad \text{weakly-star in } L^\infty(0,T;V_0^{2r}), \tag{4.96}$$

$$\Lambda^{\alpha_n} \to \Lambda \quad \text{weakly in } \mathcal{Z}^*. \tag{4.97}$$

The limit $\zeta \in \mathcal{Z}^*$ is readily identified: by formula (4.43) we have, for every $v \in \mathcal{Z}$,

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \langle \partial_t(A_0^{-2r} + \tau I)q^{\alpha_n}(t), v(t)\rangle \, dt \\ &= \lim_{n\to\infty} \Big[- \int_0^T \big(\partial_t v(t), (A_0^{-2r} + \tau I)q^{\alpha_n}(t)\big)\, dt + (g_1^{\alpha_n}, v(T)) \Big] \\ &= - \int_0^T \big(\partial_t v(t), (A_0^{-2r} + \tau I)\bar{q}(t)\big)\, dt \, + \, \beta_1(\bar{y}(T) - y_\Omega, v(T)) \, =: \, \langle \zeta, v\rangle_{\mathcal{Z}}. \end{aligned} \tag{4.98}$$

Moreover, by combining the strong convergence (4.91) with (4.95), it is easily checked that

$$f_2''(\bar{y}^{\alpha_n})\, q^{\alpha_n} \to f_2''(\bar{y})\, \bar{q} \quad \text{weakly in } L^2(Q). \tag{4.99}$$

At this point, we recall that, for a.e. $t \in (0,T)$ and $v \in \mathcal{Z}$,

$$\langle -\partial_t(p^\alpha + \tau q^\alpha)(t), v(t)\rangle \, = \, \langle -\partial_t(A_0^{-2r} + \tau I)q^\alpha(t), v(t)\rangle.$$

We now write the adjoint system (4.60)–(4.62) for $\alpha = \alpha_n$, insert $v = v(t)$ for an arbitrary $v \in \mathcal{Z}$, integrate the resulting identity with respect to time over $[0,T]$, and pass to the limit as $n \to \infty$. The following equation then results:

$$\begin{aligned} \langle \Lambda, v\rangle_{\mathcal{Z}} \, &= \, - \int_0^T \big(\partial_t v(t), (A_0^{-2r} + \tau I)\bar{q}(t)\big)\, dt \, + \beta_1(\bar{y}(T) - y_\Omega, v(T)) \\ &\quad - \int_0^T\!\!\int_\Omega \nabla \bar{q} \cdot \nabla v \, + \int_0^T\!\!\int_\Omega \big(\beta_2(\bar{y} - y_Q) - f_2''(\bar{y})\bar{q}\big)\, v \quad \forall\, v \in \mathcal{Z}. \end{aligned} \tag{4.100}$$

Finally, taking the limit as $n \to \infty$ in (4.65), and using (4.88) and (4.91), we infer that

$$\int_0^T\!\!\int_\Omega (\bar{q} \, + \, \beta_3\, \bar{u})(v - \bar{u}) \, \geq \, 0 \quad \forall\, v \in \mathcal{U}_{\rm ad}. \tag{4.101}$$

From the above considerations, we can conclude the following first-order necessary optimality conditions for the optimal control problem $(\mathbf{CP}_0)$:

Theorem 4.11 *Suppose that the conditions of Theorem 4.10 are satisfied, and let* $\bar{u} \in \mathcal{U}_{\rm ad}$ *be an optimal control for* $(\mathbf{CP}_0)$ *with associated state* $(\bar{\mu}, \bar{y})$, *where* $\bar{y} = \mathcal{S}_0^2(\bar{u})$. *Then, there exist* $(\bar{q}, \Lambda)$ *such that the following statements hold true:*
(i) $\bar{q} \in L^\infty(0,T;H) \cap L^2(0,T;H^1(\Omega))$, $\Lambda \in \mathcal{Z}^*$.
(ii) *The adjoint equation* (4.100) *is fulfilled.*
(iii) *The necessary optimality condition* (4.101) *is satisfied.*

Remark 15 *From* (4.101) *we infer that, for* $\beta_3 \neq 0$, $\bar{u}$ *is nothing but the* $L^2(Q)$*-orthogonal projection of* $-\beta_3^{-1} q$ *onto* $\mathcal{U}_{\rm ad}$.

Remark 16 *Unfortunately, we are unable to derive any complementarity slackness conditions for the Lagrange multiplier* Λ. *Indeed, while it is easily seen that*

$$\liminf_{n\to\infty} \int_0^T \int_\Omega \Lambda^{\alpha_n} q^{\alpha_n} = \liminf_{n\to\infty} \int_0^T \int_\Omega \frac{2\varphi(\alpha_n)}{1-(\bar{y}^{\alpha_n})^2} |q^{\alpha_n}|^2 \geq 0 \quad \forall\, n \in \mathbb{N},$$

the convergence properties (4.89) *and* (4.95) *do not suffice to conclude that* $\langle \Lambda, \bar{q} \rangle_{\mathcal{Z}} \geq 0$.

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