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Use of two valuation functions in financial and actuarial transactions*†

by

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Abstract: Usually a financial transaction is being considered with a single valuation function, but sometimes, and depending on the economic context, it is necessary to use one function to value the set of incoming payments and another to assess the set of outgoing payments. This paper, therefore, sets out to demonstrate that those financial transactions which use two valuation (capitalization or discount) functions can be treated in the same way as those valued with a single valuation function in which some incoming or outgoing payments are random. Our methodology is applied to determine the periodic amounts necessary to reach a target capital at a certain point in the future time in two situations, namely when the withdrawal of the capital depends on the investor's survival, and when the receipt of the capital (by the investor or his/her beneficiaries) is assured.

Keywords: financial transaction, balance, periodic amounts, survival, target capital

1. Introduction

The aim of this paper is to introduce those financial transactions whose instalments (paid or received) are valued with different capitalization or discount functions. A well-known precedent of this type of financial transaction can be found in the sinking fund method of loan repayment (Cruz and Valls, 2008). In effect, let us consider a loan of amount C when the full loan principal will be repaid at the end of the entire loan duration and the interests will be paid periodically. If the borrower makes constant deposits D into a separate account

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(the so-called sinking fund) at the end of the n periods with the aim of repaying C, we can consider that this financial transaction is equivalent to receiving the amount C at instant 0, valued with an interest rate i_1, and paying the instalments

$$(D,\, 1), (D,\, 2), \ldots, (D,\, n),$$

valued with another interest rate i_2, such that

$$C(1+i_1)^n = Ds_{\overline{n|}\, i_2}, \tag{1}$$

where $s_{\overline{n|}\, i} = \frac{(1+i)^n-1}{i}$ is the future value of an annuity-immediate of amount \$1 (Bodie et al., 2004). It is worth noting that i_2 may differ from i_1 (Brigham and Daves, 2007).

The following situation provides another case in which it is convenient to use two valuation functions in a financial transaction. Consider a company A which has to pay n money amounts

$$(D_1, t_1), (D_2, t_2), \ldots, (D_n, t_n),$$

due to another company B, at an interest rate i_1. Nevertheless, the two companies agree that the payments (D_j, t_j), $j = 1, 2, \ldots, n$, are not going to be made when due, but that the equivalent payment C will be effected at a time t later than the due dates $t_1, t_2, \ldots, t_n$:

$$C = \sum_{j=1}^{n} D_j (1+i_1)^{t-t_j}.$$

In some cases, it is usual for C to be funded by a loan which has to be paid back by means of the cash flows $(P_1, t'_1), (P_2, t'_2), \ldots, (P_m, t'_m)$ at a different interest rate i_2, where $t_1 < t_2 < \cdots < t_n < t < t'_1 < t'_2 < \cdots < t'_m$. Therefore,

$$C = \sum_{k=1}^{m} P_k (1+i_2)^{t-t'_k}.$$

In this case, it would be useful for company A to determine the real interest rate corresponding to the complete repayment, for which it is convenient to consider a financial transaction with two valuation functions, one for the initially due amounts (D_j, t_j), $j = 1, 2, \ldots, n$, and another one for the delayed payments (P_k, t'_k), $k = 1, 2, \ldots, m$:

$$\sum_{j=1}^{n} D_j (1+i_1)^{t-t_j} = \sum_{k=1}^{m} P_k (1+i_2)^{t-t'_k},$$

giving rise to the following equation:

$$\sum_{j=1}^{n} D_j (1+i)^{t-t_j} = \sum_{k=1}^{m} P_k (1+i)^{t-t'_k},$$

in which the unknown parameter is i (the so-called effective interest) (Brealey et al., 2006). Observe that, although i_1 and i_2 are the real interest rates for both sets of the given financial transaction, the real interest rate for the whole financial transaction does not conform to any of the usual averages (arithmetic, geometric or harmonic) of these rates. Nevertheless, this paper offers a method of unifying the interest rates used in the transaction, achieving this objective from a completely different perspective.

The use of two valuation functions in a financial transaction appears when considering the multiple factors affecting the set of either received or paid amounts present in the transaction, but with the singularity that the factors affecting each set are different. One of these factors, that of inflation, was analyzed in depth by Cruz and González (2013), giving rise to an adjustment of the final payments by the lender or the borrower.

The organization of this paper is as follows. After a justification of the convenience of introducing the financial transactions with two valuation functions, Section 2 formalizes this concept and relates it to the financial transactions valued with a single valuation function in which some received or paid amounts are random. Section 3 presents the mathematical expression of the balance by using three methods: retrospective (backward), prospective (forward) and recursive. The recursive method allows us to construct the table of balance evolution. In Section 4 our findings are applied to the problem of calculating the final received amount, namely to determine the periodic random amounts which should be invested in order to reach a certain target capital at a predetermined future time (Dhaene et al., 2012). This final amount will be considered in two contexts: deterministic and random. Finally, Section 5 summarizes and concludes.

2. Financial transactions with two valuation functions

In order to facilitate the understanding of this section, it is important to offer some preliminary definitions of certain terms, which will be employed.

DEFINITION 1 *In the context of financial transactions, a valuation function* $F(t,p)$ *is a continuous real-valued positive function, defined within a subset of* $\Re \times \Re$*, satisfying the following conditions:*

1. $F(p,p) = 1$.
2. *Given a focal date* p, $F_p(t) := F(t,p)$ *is strictly decreasing.*

When $t \leqslant p$ *(which implies* $F(t,p) \geqslant 1$*), we shall say that* $F(t,p)$ *is a capitalization function, whilst if* $t \geqslant p$ *(which implies* $F(t,p) \leqslant 1$*), we shall say that* $F(t,p)$ *is a discount function.* $F(t,p)$ *represents the value at* p *of \$1 available at time* t *(Cruz and Valls, 2002).*

DEFINITION 2 *A financial transaction is a couple of sets of instalments (with specification of their respective maturities):*

$$\mathbf{P} = \{(C_1, t_1), (C_2, t_2), \ldots, (C_m, t_m)\}$$

and

$$\mathbf{N} = \{(C'_1, t'_1), (C'_2, t'_2), \ldots, (C'_n, t'_n)\},$$

where $t_1 < t_2 < \cdots < t_n$, $t'_1 < t'_2 < \cdots < t'_m$ *and* $t_1 < t'_1$. ***P** represents a series of amounts paid by a person, called the lender or investor, and received by another person, called the borrower or recipient, and **N** denotes the series of amounts repaid by the borrower so that the final balance is 0. Usually, the equilibrium between these two sets of payments is achieved by choosing a focal date (Ayres, 1963),* p*, and a valuation (capitalization or discount) function* $F(t,p)$ *which projects all the involved dates up to the time* p*, such that*

$$\sum_{i=1}^{m} C_i \cdot F(t_i, p) = \sum_{j=1}^{n} C'_j \cdot F(t'_j, p) \tag{2}$$

which is the so-called equation of financial equivalence at p *(De Pablo, 2000).*

On the other hand, on some occasions, it seems logical to assess the set **P** with a valuation function different from that used to value the set **N**. This can be due, for instance, to the following reasons:

1. There may be a significant variation in interest rates (Ferruz, 1994).
2. The period in which the lender/borrower pays/receives the amounts involves more risk.
3. The lender or the borrower is subject to an important risk of insolvency.
4. It is necessary to take into account the effect of inflation if payments are delayed.
5. The deliveries of the respective amounts take place in different countries (country risk) (López and Sebastián, 2008).

Hereinafter, we shall suppose that $p \leqslant t_1$, whereby $F(t,p)$ will be a discount function. If $F(t,p)$ strictly represents the money value of time, and $R_1(t,p) \leqslant 1$ (respectively $R_2(t,p) \leqslant 1$) denotes the amount that the lender (respectively the borrower) is willing to pay and consequently the amount that the borrower (respectively the lender) is willing to receive at time p with the aim of avoiding all risks affecting \$1 at time t, the new functions to value **P** and **N** (all risks included) are

$$F_1(t,p) := R_1(t,p)F(t,p)$$

and

$$F_2(t,p) := R_2(t,p)F(t,p),$$

respectively. Therefore, the equation of financial equivalence at p (2) remains:

$$\sum_{i=1}^{m} C_i \cdot F_1(t_i, p) = \sum_{j=1}^{n} C'_j \cdot F_2(t'_j, p), \tag{3}$$

or, equivalently,

$$\sum_{i=1}^{m} \hat{C}_i \cdot F(t_i, p) = \sum_{j=1}^{n} \hat{C}'_j \cdot F(t'_j, p), \tag{4}$$

where $\hat{C}_i = C_i R_1(t_i, p)$ and $\hat{C}_j = C'_j R_2(t'_j, p)$.

In what follows, such financial transactions will be labelled as *double-valued financial transactions*.

Observe that, in general, if p is different from all dates of $\mathbf{P}$ and $\mathbf{N}$, then the double-valued financial transaction (3) (or, equivalently, the financial transaction (4)) is equivalent to a new single-valued financial transaction, whose set of (random) payments by the lender is:

$$\mathbf{P}' = \{(\xi_1, t_1), (\xi_2, t_2), \ldots, (\xi_m, t_m)\},$$

where ξ_i $(i = 1, 2, \ldots, m)$ is the dichotomous random variable defined by:

$$\xi_i = \begin{cases} C_i, \text{ with probability } & R_1(t_i, p) \\ 0, \quad \text{with probability } & 1 - R_1(t_i, p) \end{cases}$$

and whose set of payments by the borrower is:

$$\mathbf{N}' = \{(\xi'_1, t'_1), (\xi'_2, t'_2), \ldots, (\xi'_n, t'_n)\},$$

where ξ'_j $(j = 1, 2, \ldots, n)$ is the dichotomous random variable defined by:

$$\xi'_j = \begin{cases} C'_j, \text{ with probability } & R_2(t'_j, p) \\ 0, \quad \text{with probability } & 1 - R_2(t'_j, p) \end{cases}.$$

Observe that this statement is in accordance with the fact that both $1 - R_1(t, p)$ and $1 - R_2(t, p)$ are the distribution functions of certain random variables T_1 and T_2 (Cruz and Muñoz, 2005).

From a different point of view, if F_1 and F_2 $(F_1 > F_2)$ are two discount functions, a financial transaction, in which $\mathbf{P}$ is valued with F_1 and $\mathbf{N}$ is valued with F_2, is equivalent to another financial transaction, valued only with a common valuation function F $(F \geqslant F_1 > F_2)$, whose set of (random) payments by the lender is:

$$\mathbf{P}' = \{(\xi_1, t_1), (\xi_2, t_2), \ldots, (\xi_m, t_m)\},$$

where ξ_i $(i = 1, 2, \ldots, m)$ is the dichotomous random variable defined by:

$$\xi_i = \begin{cases} C_i, \text{ with probability } & \frac{F_1(t_i, p)}{F(t_i, p)} \\ 0, \quad \text{with probability } & 1 - \frac{F_1(t_i, p)}{F(t_i, p)} \end{cases}$$

and whose set of payments by the borrower is:

$$\mathbf{N}' = \{(\xi'_1, t'_1), (\xi'_2, t'_2), \ldots, (\xi'_n, t'_n)\},$$

Table 1. Randomness of **P** and **N** depending on the relative positions of F_1 and F_2

Relative positions	$F_1 > F_2$	$F_1 < F_2$	$F_1 > F_2$	$F_1 < F_2$	$\cdots$
Intervals for t	(p, t_1)	(t_1, t_2)	(t_2, t_3)	(t_3, t_4)	$\ldots$
Applied function	F_1	F_2	F_1	F_2	$\ldots$
Set of sure payments	**P**	**N**	**P**	**N**	$\ldots$
Set of random payments	$\mathbf{N}'$	$\mathbf{P}'$	$\mathbf{N}'$	$\mathbf{P}'$	$\ldots$

where ξ'_j $(j = 1, 2, \ldots, n)$ is the dichotomous random variable defined by:

$$\xi'_j = \begin{cases} C'_j, \text{ with probability } & \frac{F_2(t'_j,p)}{F(t'_j,p)} \\ 0, \quad \text{with probability } & 1 - \frac{F_2(t'_j,p)}{F(t'_j,p)} \end{cases}.$$

Observe that the expressions $\frac{F_1(t,p)}{F(t,p)}$ and $\frac{F_2(t,p)}{F(t,p)}$ denote two conditional probabilities, since the random variable T, whose distribution function is $1 - F$, has a first-order stochastic dominance over T_1 and T_2 (Bawa, 1975).

Observe also that there exists an infinite number of random single-valued financial transactions equivalent to a given deterministic double-valued one. Nevertheless, among all of them, we shall consider the *canonical* equivalent financial transaction, valued with the greater function (F_1), whose set of payments by the lender continues being **P** and whose set of payments by the borrower is:

$$\mathbf{N}' = \{(\xi'_1, t'_1), (\xi'_2, t'_2), \ldots, (\xi'_n, t'_n)\},$$

where ξ'_j $(j = 1, 2, \ldots, n)$ is the dichotomous random variable defined by:

$$\xi'_j = \begin{cases} C'_j, \text{ with probability } & \frac{F_2(t'_j,p)}{F_1(t'_j,p)} \\ 0, \quad \text{with probability } & 1 - \frac{F_2(t'_j,p)}{F_1(t'_j,p)} \end{cases}.$$

Observe again the conditional probabilities derived from the first-order stochastic dominance of $1 - F_1$ over $1 - F_2$.

An analogous conclusion holds if $F_1 < F_2$. Finally, if the relative positions of F_1 and F_2 change, the applied valuation function at each instant depends on the interval to which instant t belongs and the randomness of each set will also change.

Therefore, if $p < t_1 < t_2 < \cdots < t_k < t < t_{k+1}$, the resulting discount function to be applied to the canonical financial transaction is the following mixture of discount functions:

$$F(t,p) = \begin{cases} F_1(t,p), & \text{if} \quad p \leqslant t < t_1 \\ F_2(t,p), & \text{if} \quad t_1 \leqslant t < t_2 \\ \vdots & \qquad \vdots \end{cases},$$

Finally, the content of this section can be summarized in the following result.

THEOREM 1 *Every double-valued financial transaction is equivalent to an infinite number of single-valued random financial transactions, where the random amounts are dichotomous random variables, whose probability measure is determined by the original valuation functions being different for the two sets of cash flows. The random financial transaction of this class, valued with the minimum discount function, will be called the canonical financial transaction. Reciprocally, every single-valued random financial transaction is equivalent to a double-valued financial transaction.*

3. Mathematical expression of the balance

Let V_τ denote the balance at time τ. This parameter can be understood either as the balance just before or after a payment by the lender or the borrower (Gil and Gil, 1987; Gil, 1992). Of course, this distinction will only make sense when a payment matures at τ. In the first case (the balance is calculated just before a payment), V_τ will be represented as V_τ^-. In the second case (the balance is considered just after a payment), V_τ will be represented as V_τ^+. Both values can be determined by using two methods: the retrospective (backward) and the prospective (forward).

For the sake of simplicity, we denote the ratio $\frac{F_j(\tau,p)}{F_j(\upsilon,p)}$ by $f_j(\tau,\upsilon)$, $j=1$ and 2, which, for additive discount functions, is equal to $F_j(\tau,\upsilon)$.

Table 2. Balance in a double-valued financial transaction according to two methods

Method	V_τ^-	V_τ^+
Retrospective (backward)	$\sum_{t_i<\tau} C_i f_1(t_1,\tau) - \sum_{t'_j<\tau} C'_j f_2(t'_j,\tau)$	$\sum_{t_i\leqslant\tau} C_i f_1(t_1,\tau) - \sum_{t'_j\leqslant\tau} C'_j f_2(t'_j,\tau)$
Prospective (forward)	$\sum_{t'_j\geqslant\tau} C'_j f_2(t'_j,\tau) - \sum_{t_i\geqslant\tau} C_i f_1(t_1,\tau)$	$\sum_{t'_j>\tau} C'_j f_2(t'_j,\tau) - \sum_{t_i>\tau} C_i f_1(t_1,\tau)$

Hereinafter, we shall use the following notation:

$$VP_\tau^- := \sum_{t_i<\tau} C_i f_1(t_1,\tau), \quad VP_\tau^+ := \sum_{t_i\leqslant\tau} C_i f_1(t_1,\tau),$$

and

$$VN_\tau^- := \sum_{t'_j<\tau} C'_j f_2(t'_j,\tau) \text{ and } VN_\tau^+ := \sum_{t'_j\leqslant\tau} C'_j f_2(t'_j,\tau).$$

On the other hand, starting from the initial value $V_0^-=0$, and taking into account that either $V_\tau^+ = V_\tau^- + C_\tau$ or $V_\tau^+ = V_\tau^- - C'_\tau$, the balance available at each time is given by the recursive relations:

$$VP_{\upsilon}^{-} = f_1(\tau, \upsilon)VP_{\tau}^{+} \text{ and } VN_{\upsilon}^{-} = f_2(\tau, \upsilon)VN_{\tau}^{+},$$

where τ and υ denote two consecutive dates such that $\tau<\upsilon$. The results are shown in Table 3 (for the sake of simplicity, we shall assume that the dates of **P** and **N** are alternate).

Table 3. Partial and global evolution of the balance in a double-valued financial transaction

$\tau,\ \upsilon$	$f_1(\tau, \upsilon)$	$f_2(\tau, \upsilon)$	VP_{τ}^{-}	VN_{τ}^{-}	V_{τ}^{-}	C_{τ}	C'_{τ}	VP_{τ}^{+}	VN_{τ}^{+}	V_{τ}^{+}
t_1	-	-	0	-	0	C_1	-	C_1	-	C_1
t'_1	$f_1(t_1, t'_1)$	-	$VP_{t'_1}^{-}$	-	$VP_{t'_1}^{-}$	-	C'_1	$VP_{t'_1}^{-}$	C'_1	$V_{t'_1}^{+}$
t_2	$f_1(t'_1, t_2)$	$f_2(t'_1, t_2)$	$VP_{t_2}^{-}$	$VN_{t_2}^{-}$	$V_{t_2}^{-}$	C_2	-	$VP_{t_2}^{+}$	$VN_{t_2}^{-}$	$V_{t_2}^{+}$
t'_2	$f_1(t_2, t'_2)$	$f_2(t_2, t'_2)$	$VP_{t'_2}^{-}$	$VN_{t'_2}^{-}$	$V_{t'_2}^{-}$	-	C'_2	$VP_{t'_2}^{-}$	$VN_{t'_2}^{+}$	$V_{t'_2}^{+}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t_m	$f_1(\cdot, t_m)$	$f_2(\cdot, t_m)$	$VP_{t_m}^{-}$	$VN_{t_m}^{-}$	$V_{t_m}^{-}$	C_m	-	$VP_{t_m}^{+}$	$VN_{t_m}^{-}$	$V_{t_m}^{+}$
t'_n	$f_1(t_m, t'_n)$	$f_2(t_m, t'_n)$	$VP_{t'_n}^{-}$	$VN_{t'_n}^{-}$	$V_{t'_n}^{-}$	-	C'_n	$VP_{t'_n}^{-}$	$VN_{t'_n}^{+}$	0

4. An application to random savings transactions

In this section, we are going to apply the former results to financial savings transactions. Let us consider the financial transaction, in which an x-year-old investor places periodic amounts a_s at time $s-1$ ($s = 1, 2, \ldots, n$) in order to receive at time n the saving fund C_n, provided that he/she lives up to this time. In this case and following our notation, the set of payments by the investor (Suárez, 1991) is

$$\mathbf{P} = \{(a_1,\ 0), (a_2,\ 1), \ldots, (a_n,\ n-1)\},$$

whilst the set of payments by the recipient is

$$\mathbf{N} = \{(C_n,\ n)\}.$$

Under this initial assumption (i.e., survival of the investor at time n), the equation of financial equivalence at instant 0, taking into account a variable interest rate, is:

$$C_n \cdot {}_nE_x = \sum_{s=1}^{n} a_s \cdot {}_{s-1}E_x, \tag{5}$$

where ${}_sE_x = {}_sp_x \prod_{h=1}^{s} (1+i_h)^{-1}$ is the actuarial discount factor (Valls and Cruz, 2013), and ${}_0p_x$, ${}_1p_x, \ldots$, ${}_np_x$ are the survival probabilities of an x-year-old person at moments 0, 1, ..., n, respectively, and, therefore, the probabilities associated with the amounts in **P** and **N**. It is well known that

$$ {}_sp_x = \frac{l_{x+s}}{l_x}, $$

where l_z is the number of persons who reach the age z. Obviously, Equation (5) can be written down as:

$$ C_n \cdot {}_np_x \prod_{h=1}^{n} (1+i_h)^{-1} = \sum_{s=1}^{n} a_s \cdot {}_{s-1}E_x, $$

which leads to

$$ C_n \prod_{h=1}^{n} (1+i_h)^{-1} = \sum_{s=1}^{n} \frac{a_s}{{}_np_x} {}_{s-1}E_x. \tag{6} $$

Finally, Equation (6) can be written down as:

$$ C_n \prod_{h=1}^{n} (1+i_h)^{-1} = \sum_{s=1}^{n} \frac{a_s}{{}_np_x} {}_{s-1}p_x \prod_{h=1}^{s-1} (1+i_h)^{-1}. \tag{7} $$

Some comments:

1. The left-hand side of Equation (5) indicates that the investor is expecting to receive (in case of survival) C_n at moment n, whilst equations (6) and (7) point out that the investor (in case of survival) or his/her beneficiaries (in case of investor's death) will be sure to receive C_n at the moment n.
2. The periodic amounts, which will accumulate to C_n at time n are:
   (a) a_s, in the case of the investor's likely survival (which does not guarantee the future payment of all amounts in **P** and **N**).
   (b)
   $$ \frac{a_s}{{}_np_x} := a'_s, $$
   in the case where the withdrawal of C_n is guaranteed but not the payment of the periodic amounts (which depends on the investor's survival).
   (c)
   $$ \frac{a_s}{{}_np_x} {}_{s-1}p_x := a''_s, $$
   in the case where both the withdrawal of C_n and the payment of the periodic amounts are guaranteed.

3. Logically, the following inequality holds:
$$a_s < a''_s < a'_s,$$
since a'_s and a''_s have to assure a certain final amount and, moreover, a'_s has to compensate for the randomness of the periodic amounts.
4. In other words, if the periodic amounts in paragraphs 1 and 2 above are equal, the corresponding saving funds would be C_n and C'_n, satisfying the following equality:
$$C'_n = {}_np_x \cdot C_n.$$

We can summarize these comments in Table 4 (The case dealing only with the randomness of **N** will not be considered in this paper.)

Table 4. The cases studied according to randomness of **P** and **N**

	Randomness		
	Neither **P** nor **N**	**P** and not **N**	Both **P** and **N**
Periodic amounts	a''_s	a'_s	a_s
Savings fund	C_n	C_n	C_n
Exponential discounting	both **P** and **N**	**N** and not **P**	neither **P** nor **N**
Actuarial discounting	neither **P** nor **N**	**P** and not **N**	both **P** and **N**

Let us analyze these random savings transactions in the following subsections.

4.1. The withdrawal of the savings fund is assured

In this case, compared with the accompanying sure transaction, the investor should make higher periodic payments to compensate for his/her risk of death before time n. This difference, denoted by r'_s, will be called the *risk premium*, since it compensates the recipient (investment holder or savings guarantor) for the assumed risk. Thus,

$$r'_s := a'_s - a''_s.$$

In this case,

$$r'_s = a'_s(1 - {}_{s-1}p_x) > 0.$$

The difference between the periodic amounts and the risk premium, is the *savings premium*, denoted by $\bar{r}_s$,

$$\bar{r}_s := a'_s - r'_s = a'_s \cdot {}_{s-1}p_x.$$

Analogous interpretations in dividing the total premium into a savings premium and a risk premium can be found in Gerber (1997).

The balance (*saved amount*) at time τ, which is denoted here by V'_τ, is defined by the following equation by using the backward method:

$$\prod_{h=1}^{\tau}(1+i_h)^{-1}V'_\tau=\sum_{s=1}^{\tau}a'_s\cdot{}_{s-1}E_x. \tag{8}$$

In order to determine the equation corresponding to the recursive method, we have to consider the Equation (8) for the previous period. Thus, by subtracting both equalities, one has:

$$\prod_{h=1}^{\tau-1}(1+i_h)^{-1}\left[(1+i_\tau)^{-1}V'_\tau-V'_{\tau-1}\right]=a'_\tau\cdot{}_{\tau-1}E_x,$$

from where the recursive relationship arises:

$$V'_\tau=(V'_{\tau-1}+a'_\tau\cdot{}_{\tau-1}p_x)(1+i_\tau). \tag{9}$$

From Equation (9), we can deduce the expression for the increase of the saved amount $\Delta'_\tau:=V'_\tau-V'_{\tau-1}$ (observe that V'_τ includes $V'_{\tau-1}$, $\bar{r}_\tau$ and the interests of both parameters during the period $]\tau-1,\tau]$, which takes the time value of money into account):

$$\Delta'{}_\tau=(V'_{\tau-1}+a'_\tau\cdot{}_{\tau-1}p_x)i_\tau+a'_\tau\cdot{}_{\tau-1}p_x. \tag{10}$$

Finally, taking into account that $\Delta'_\tau:=I'_\tau+\bar{r}_\tau$, we can deduce the expression of the *earned interest*:

$$I'_{\tau^-}=(V'_{\tau-1}+a'_\tau\cdot{}_{\tau-1}p_x)i_\tau. \tag{11}$$

Some comments:

1. Observe that in all previous expressions the balance is affected only by the interest rate, whilst the payments by the investor appear multiplied by the actuarial discount factor. This is logical, taking into account the different functions valuing savings and withdrawals.
2. In this kind of transaction, if the investor survives beyond (or dies before) the completion date, the target capital would be greater (or less) than the obtained in an assured savings transaction, so that the bank earnings would be higher (or lower).

4.2. The withdrawal of the savings fund depends on the investor's survival

In this case, the saved amount at time τ, denoted further on here by V^-_τ, is defined by the following equation, as obtained by using the backward method:

$${}_\tau E_x V^-_\tau=\sum_{s=1}^{\tau}a_s\cdot{}_{s-1}E_x. \tag{12}$$

For determining the equation corresponding to the recursive method, we have to consider the expression (12) for the previous period. Thus, by subtracting both equalities, one has:

$$_{\tau-1}E_x \left[\frac{_\tau p_x}{_{\tau-1}p_x}(1+i_\tau)^{-1}V_\tau^- - V_{\tau-1}^- \right] = a_\tau \cdot {}_{\tau-1}E_x,$$

from where the recursive relationship arises:

$$V_\tau^- = (V_{\tau-1}^- + a_\tau)\frac{_{\tau-1}p_x}{_\tau p_x}(1+i_\tau). \tag{13}$$

From Equation (13), we can deduce the expression of $\Delta_\tau^- := V_\tau - V_{\tau-1}$ (observe that $\frac{_{\tau-1}p_x}{_\tau p_x}(1+i_\tau) - 1$ is the true interest for the τ-th period):

$$\Delta_\tau^- = V_{\tau-1}^- \left[\frac{_{\tau-1}p_x}{_\tau p_x}(1+i_\tau) - 1 \right] + a_\tau \frac{_{\tau-1}p_x}{_\tau p_x}(1+i_\tau), \tag{14}$$

which leads to

$$\Delta_{\tau^-} = (V_{\tau-1} + a_\tau)\,\frac{_{\tau-1}p_x}{_\tau p_x} i_\tau + V_{\tau-1}\left(\frac{_{\tau-1}p_x}{_\tau p_x} - 1 \right) + a_\tau \frac{_{\tau-1}p_x}{_\tau p_x},$$

where

$$I_\tau := \left(V_{\tau-1}^- + a_\tau\right) \frac{_{\tau-1}p_x}{_\tau p_x} i_\tau \tag{15}$$

and

$$\bar{\bar{r}}_\tau := V_{\tau-1}^- \left(\frac{_{\tau-1}p_x}{_\tau p_x} - 1 \right) + a_\tau \frac{_{\tau-1}p_x}{_\tau p_x}. \tag{16}$$

Considering that

$$r''_\tau = a_\tau - \bar{\bar{r}}_\tau,$$

equation (16) leads to

$$r''_\tau = \left(V_{\tau-1}^- + a_\tau\right)\left(1 - \frac{_{\tau-1}p_x}{_\tau p_x}\right). \tag{17}$$

Observe that, in this case, the risk premium is negative, which indicates that the lender obtains an earning (discount) due to the assumed risk. Moreover, the risk affects the periodic amounts and the saved amount, due to the possibility of losing it, if the lender dies.

Some comments:

1. It should be noted that in this subsection both the balance and the amounts have been affected by the actuarial discount factor.
2. In order to guarantee reaching the same target capital, it will be necessary to continue making payments for a longer period, or to increase the value of such payments.

5. Conclusion

Traditionally, the financial transactions have been assessed with a single valuation (capitalization or discount) function. Nevertheless, the different timing of incoming payments (savings) and outgoing payments (withdrawals) can recommend the use of different functions affecting the sets of payments by the lender and the borrower. This situation has been presented in this paper, which has considered mutual transactions involving random amounts. Apart from this novel approach to financial transactions, the main contribution of this paper is its application to actuarial operations, in which a problem of calculating the final received amount is presented. In effect, under the assumption of random periodic instalments, we discuss the cases in which the withdrawal of the target capital depends or not on the investor's survival. These different approaches will give rise to the mathematical expressions of the so-called risk and savings premium, balance, earned interest and increase of capital.

This paper allows for applying a clearer methodology when two sets of cash flows are affected by different risks and thus also by different discount functions. Consequently, it allows for unifying the treatment of risk and discount by only using a methodology based on financial tools. Thus, the control of risks affecting each set of cash flows is more accurate, since some mistakes can be avoided when writing the equation of financial equivalence within some uncertain contexts.

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