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On the calculus of variations: opening simpler, direct routes*[1]

by

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Abstract: We first offer a direct, analytic way of deriving the Euler equation. The method we suggest avoids the difficulties met by Lagrange, which become serious in the case of complex functionals. It also leads to the new Hamiltonian, defined by Robert Dorfman; in turn, this Hamiltonian can be extended to high order functionals or n–fold integrals, thus providing, almost immediately, the corresponding Euler equations.

Keywords: Euler equations, optimal control, new Hamiltonians

1. Introduction and motivation

At the top of their fascinating essay on the birth of optimal control theory, Hans-Joseph Pesch and Michael Plail very appropriately chose a subtitle that said it all: "A History of Ingenious Ideas and Missed Opportunities" (Pesch and Plail, 2009). In this paper we would like to first show how this applies in exactly the same way to the birth of the calculus of variations as pioneered analytically by Lagrange, and, quite rightfully, enthusiastically endorsed by Euler. We will show that the corner stones of the calculus of variations can be made simpler if we do not miss a few crucial steps.[1]

Indeed, it turns out that inasmuch as the Hestenes approach to optimal control had been captured – and therefore restrained – by the calculus of variations as Pesch and Plail in their very precise way relate, young Giuseppe Ludovico de Lagrange–Tournier (he was 19 years old) was guided by the powerful hand

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[1]Some of these ideas found an expression in our recent paper (La Grandville, 2024); but here we offer an even more direct route to obtain the Euler equations, particularly in the integrated form.

of differential calculus when he sent his famous 1755 letter to Euler. Brilliant as his method and result were, they would come at a cost, as we will see.

In the usual, modern notation the problem addressed by Lagrange was to find a function $y(x)$ that would constitute an extremal of the integral

$$I\left[y\right] = \int_a^b F\left(x, y, y'\right) dx$$

subject to the conditions

$$y(a) = A, \quad y(b) = B$$

where $y(x)$ is continuously differentiable.

His first stroke of genius was to transform a functional such as $I\left[y\right] = \int_a^b F\left(x, y, y'\right) dx$ into a simple function of a sole variable by giving to the solution $y(x)$ a variation defined as a fixed function $\eta(x)$ multiplying a variable α. The functional $I\left[y(x) + \alpha\eta(x)\right]$ thus just turned into a function $I(\alpha)$, and $I'(0) = 0$ would lead to the Euler equation, a necessary condition for y to be an extremal.

Despite his extraordinary creativity, Euler had not been able to derive his equation from an analytic argument, and had to rely on geometrical reasoning. As Herman Goldstine relates in his splendid *History of the Calculus of Variations* (Goldstine, 1980), Euler himself tells us how he had hopelessly struggled to find such a method, and that all the merit of achieving this feat belonged to the 19 year-old. Here are Euler's own words (quoted by Goldstine, p. 110): "Even though the author of this [Euler] had meditated a long time and had revealed to friends his desire, yet the glory of first discovery was reserved to the very penetrating geometer from Torino La Grange who, having used analysis alone, has clearly attained the very same solution which the author had deduced from geometrical considerations"[2].

[2] Perhaps Euler had been too harsh on himself, for two reasons: first his geometric reasoning was vindicated two centuries later in the "direct methods" of the calculus of variations (see Gelfand and Fomin, 1963); secondly, his approach led him to the extremal of functionals depending on m-order derivatives such as

$$I\left[y(x)\right] = \int_a^b F\left(x, y, y', ..., y^{(m)}\right) dx,$$

given by the solution of the $2m$ order differential equation

$$\frac{\partial F}{\partial y}\left(x, y, y', ..., y^{(m)}\right) + \sum_{j=1}^{m} (-1)^j \frac{d^j}{dx^j} \frac{\partial F}{\partial y^{(j)}}\left(x, y, y', ..., y^{(m)}\right) = 0,$$

later called the Euler-Poisson equation. Furthermore, in the same way he was able to tackle functionals, defined by double integrals, such as

$$I\left[u(x, y)\right] = \int\int_R F\left(x, y, u, u_x, u_y\right) dxdy,$$

Lagrange, however, was bound to meet two difficulties. First, after deriving $I'(0)$ as

$$I'(0) = \int_a^b \left[F_y \eta(x) + F_{y'} \eta'(x)\right] dx$$

he needed to dispense with the arbitrariness of $\eta(x)$; but now sprung another beautiful idea: upon integrating by parts $\int_a^b F_{y'} \eta'(x) dx$ and using initial conditions $\eta(a) = \eta(b) = 0$, he came up with the necessary condition

$$I'(0) = \int_a^b \eta(x)(F_y - \frac{d}{dx} F_{y'}) dx \equiv \int_a^b \eta(x) g(x) dx = 0,$$

where $g(x)$ stands for $F_y - \frac{d}{dx} F_{y'}$. At this point, surely few would fail to conclude, as Giuseppe did, that since $\eta(x)$ was completely arbitrary, $g(x) \equiv F_y - \frac{d}{dx} F_{y'}$ must be equal to zero. Not entirely convinced, Euler asked his correspondent to please prove this. It turns out neither Euler nor Lagrange ever came up with a rigorous proof. It is more than one century later, after much debate, that Paul Du Bois-Raymond proved this proposition, which became known as the fundamental lemma of the calculus of variations (Du Bois-Reymond, 1879a,b).

If the Lagrange approach is relatively straightforward to apply in the most basic problem of the calculus of variations, described above, it is not the case any more when considering more complex functionals. To suppress the arbitrariness of the perturbation given to the solution in the case of a double integral $I\left[z(y,x)\right] = \iint_R F\left(x, y, z, z_x, z_y\right) dxdy$, we already need a very subtle appeal to Green's theorem. And the case of an n–fold integral

$$I\left[z(x_1, x_2, ..., x_n)\right] = \iint ... \int_R F(x_1, x_2, ...x_n, z, z_{x_1}, z_{x_2}, ...z_{x_n}) dx_1 ... dx_n$$

necessitates the extension of Green's theorem to $n-$space. Already in the case of this first order multiple integral, the issue is highly complex; to the best of our knowledge, it has been taken up in two textbooks only: see Gelfand and Fomin (1963) pp. 153-154, and Troutman (1983) pp. 179-181.

All these difficulties disappear if we proceed very differently. It was quite natural for Lagrange to give a variation to the assumed solution; his idea extended what had been done in differential calculus when searching for an extremum of a function. But he could have taken another route. He could have focused on the

leading to the second order partial differential equation

$$\frac{\partial F}{\partial u}\left(x, y, u, u_x, u_y\right) - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_x}\left(x, y, u, u_x, u_y\right) - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_y}\left(x, y, u, u_x, u_y\right) = 0,$$

which would later have so much importance in physics.

role played by the *derivative* y' with regard to the optimal value of the integral $I\left[y(x)\right] = \int_a^b F\left(x, y, y'\right) dx$ at any point of the interval $[a, b)$. By doing so, he would have entirely escaped the hurdles he met, which become so fateful in more complicated cases. And there would have been even more in store for him. In addition to obtaining the Euler equation in a few elementary calculations, he would have reaped two important benefits:

1. He would have been led to define a new Hamiltonian, whose structure is akin to the "modified Hamiltonian" set up by Robert Dorfman in 1969 in the *American Economic Review* (Dorfman, 1969). In our opinion, the discovery of this new Hamiltonian is a perfect example of a brilliant idea and, at the same time, a missed opportunity. That it was – and still is – entirely neglected by mathematicians can most probably be attributed to the fact that it appeared in a journal little familiar to them[3]. In order to pay tribute to Robert Dorfman, and to honor his memory, we called this new Hamiltonian a $Dorfmanian$ and denoted it D in La Grandville (2017, 2018, 2024).

2. In turn, Lagrange could have generalized this Dorfmanian to deal with more complex functionals, leading in a few lines to the corresponding Euler equations, without any of the difficulties encountered in the classical approach.

We are now set to demonstrate this.

2. Focusing on the crucial role played by y'

Supposing, as both Euler and Lagrange did, that $y(x)$ is optimal, we can ask: what should be the properties of the *optimal value of the slope* $y'(x)$ *at any point* $(x, y\left(x\right))$? As in the classical calculus of variations we assume that the solution $y(x)$ is a continuously differentiable function.

The answer will hinge upon a fundamental property of differentiability: any curve corresponding to a differentiable function $y(x)$ becomes a straight line on any interval Δx that tends to zero. Let us then focus on the role, both direct and indirect, played with respect to the functional $\int_x^b F\left(t, y, y'\right) dt$ by $y'(x)$ within a very short interval Δx.

The first, direct impact can be measured by $F\left(x, y, y'\right) \Delta x + o(\Delta x)$. Secondly, since $y'(x)$ will determine $y(x + \Delta x)$ through $y(x + \Delta x) = y(x) + y'(x)$ $\Delta x+$ $o(\Delta x)$, it will definitely have a bearing on the rest of the functional

[3]A referee very rightly noted that the Dorfman paper had been cited over 750 times. However, all the references that I saw dealt with the economic interpretation of the Pontryagin equations, given in the paper. None of them mentioned the new Hamiltonian introduced by Dorfman, and, of course, there was no indication of its potential extensions.

$\int_{x+\Delta x}^{b} F\left(x, y, y'\right) dx$. Therefore, if $y'(x)$ is to be optimal, any change in $y'(x)$ must be such that the sum of these two effects should be equal to zero.

What must be evaluated is the dual effect of any change in $y'(x)$ on the functional $\int_{x}^{b} F\left(t, y, y'\right) dt$, denoted I_x^b and equal to

$$\int_{x}^{b} F\left(t, y, y'\right) dt \equiv I_x^b \equiv F\left(x, y, y'\right) \Delta x \ + o(\Delta x) + \int_{x+\Delta x}^{b} F\left(t, y, y'\right) dt.$$

Consider in the righthand side the integral $\int_{x+\Delta x}^{b} F\left(t, y, y'\right) dt$: its optimal value crucially depends on the initial value $y(x + \Delta x)$ that has been optimally set by $y'(x)$. To underline this dependency, we write this optimal value as the single variable function $I_{x+\Delta x}^b \left[y(x + \Delta x)\right]$.

Fundamentally, since y and y' are supposed to be optimal over the entire interval of integration, our functional on the interval $[x, b]$ is now transformed into a simple composite function of y'.

Taking to zero the derivative of $\int_{x}^{b} F\left(t, y, y'\right) dt \equiv I_x^b$ with respect to y' and noting that $y(x + \Delta x) = y(x) + y'(x)\Delta x + o(\Delta x)$ gives

$$\frac{dI_x^b}{dy'} = \frac{\partial F}{\partial y'}\Delta x + \frac{dI_{x+\Delta x}^b}{dy(x + \Delta x)} \frac{dy(x + \Delta x)}{dy'} = 0.$$

We can now denote the rate of increase of the optimal value of the functional $I_{x+\Delta x}^b$with respect to $y(x + \Delta x)$, $\frac{dI_{x+\Delta x}^b}{dy(x+\Delta x)}$, as $p(x + \Delta x)$; this is the familiar concept of the co-state variable, or shadow price of y at the point $x + \Delta x$[4]. On the other hand, $dy(x + \Delta x)/\ dy'$ is simply Δx. Our necessary condition for y' to be optimal is thus

$$\frac{\partial F}{\partial y'}\Delta x + p(x + \Delta x)\Delta x = 0.$$

Simplifying and taking Δx to zero gives our first fundamental equation

$$\frac{\partial F}{\partial y'}\left(x, y, y'\right) + p(x) = 0. \tag{1}$$

Equation (1) embodies the very role that y' must fulfil if it is to maintain y in an extremal status: the sum of the consequences, direct and indirect, on the

[4]Very clear interpretations of the co-state variable as the shadow price have been given, for instance, in Kamien and Schwartz (2012) or in Chiang (2002).

optimal value of the functional due to a change in y' must be nil. The direct consequence is measured by the rate $\partial F/\partial y'$; the indirect one is the following: y' imparts a change in y, which, in turn, alters the optimal value of the functional over its remaining part; this is measured by the rate $p(x)$.

We will now use the identity to derive our second central equation; together with equation (1), it will constitute the Euler equation in parametric form. Consider the identity

$$F\left(x,y,y'\right)=-\frac{\partial}{\partial x}\int_x^b F\left(t,y,y'\right)dt;$$

differentiating it with respect to y, and applying Young's theorem gives

$$\frac{\partial F}{\partial y}=-\frac{\partial}{\partial y}\frac{\partial}{\partial x}\int_x^b F\left(t,y,y'\right)dt=-\frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_x^b F\left(t,y,y'\right)dt=-\frac{\partial}{\partial x}p(x)\equiv -p'(x)$$

and our second central equation

$$\frac{\partial F}{\partial y}\left(x,y,y'\right)+p'(x)=0. \tag{2}$$

Equations (1) and (2) immediately yield, explicitely, the Euler equation *both* in integral and differential form. For the integral form, we integrate (2) and use (1) for $p(x)$, getting

$$\int_a^x \frac{\partial F}{\partial y}\left(t,y,y'\right)dt-\frac{\partial F}{\partial y'}\left(x,y,y'\right)+C=0. \tag{3}$$

For the differential form, we can either directly differentiate (3), or differentiate (1) and use (2), obtaining the Euler equation

$$\frac{\partial F}{\partial y}\left(x,y,y'\right)-\frac{d}{dx}\frac{\partial F}{\partial y'}\left(x,y,y'\right)=0. \tag{4}$$

This very simple derivation of the Euler equation was obtained through straight differential calculus. It did not require introducing any arbitrary perturbation to the solution, nor did we have to later dispense with this arbitrariness. Finally, we did not need to resort to the Du Bois-Reymond theorem, a necessary step to obtain an integral form, whose proof definitely required a lot of ingenuity from its author[5].

[5] After demonstrating the fundamental lemma of the calculus of variations, Du Bois-Reymond took one step further (Du Bois-Reymond, 1879a,b). In order to show that the Euler differential equation was not necessarily of second order, he had the ingenious idea of integrating by parts the *first* part of $I'(0)$, which led him, thanks to a new theorem bearing his name, to the Euler equation in integrated form. In contrast to the lemma, his theorem leaves little space to intuition; it is the following: if $\int_a^b M(x)h'(x)dx=0$ for arbitrary $h(x)$, it is necessary that $M(x)$ be constant. For a proof, see for instance Mark Kot (2014) or Hans Sagan (2012).

3. The symmetry of the central equations (1) and (2): some important, welcome consequences

Equations (1) and (2) offer a striking symmetry, reflecting the dual, complementary roles played by y and y' if y is an extremum. No doubt that if Lagrange had taken the route described above, obtaining the Euler equation in two lines, his next step would have been to define a function of x, y, y' and p such that its gradient with respect to y and y', taken to 0, would have yielded (1) and (2).

Such a function could have been

$$D = F(x, y, y') + p(x)y' + p'(x)y = F(x, y, y') + \frac{d}{dx}[p(x)y(x)], \tag{5}$$

which has an immediate interpretation: at any point x, it is the rate at which both y and y' contribute to the value of the functional. It is made of two components. The first is their direct contribution, measured by the integrand of the functional $\int_a^b F(x, y, y')\,dx$. The second, all important component, reflects the rate of change in the value of the extremal, which has itself two causes: a change in the *amount* of y, equal to $p(x)y'$; and a change in the *price* of y, expressed as $p'(x)y$.

It turns out that this expression has the same structure as the "modified Hamiltonian", introduced and designated as such by Robert Dorfman in his remarkable essay *An Economic Interpretation of Optimal Control Theory* (1969). Robert Dorfman's first objective was to make intuitive sense of the seemingly mysterious equations at the heart of the Pontryagin maximum principle. To achieve this in a simple, dynamic setting, Dorfman considered the following problem. Defining a state variable as $k(t)$ and a control variable as $v(t)$[6], the objective was to find an optimal path of $v(t)$, maximizing the functional

$$I = \int_0^T U(k, v, t)\,dt \tag{6}$$

subject to the constraint

$$\dot{k} = f(k, v, t). \tag{7}$$

Denoting the co-state variable as $\lambda(t)$, the traditional Hamiltonian reads as

$$H = U(k, v, t) + \lambda(t) f(k, v, t), \tag{8}$$

and the fundamental equations of the Pontryagin maximum principle are

$$\frac{\partial H}{\partial v} = \frac{\partial U}{\partial v} + \lambda(t)\frac{\partial f}{\partial v} = 0 \tag{9}$$

[6]Dorfman's notation for the control variable was $x(t)$; in order to avoid any confusion with our integration variable x, we have taken the liberty of denoting Dorfman's control variable as $v(t)$.

and

$$\frac{\partial H}{\partial k}=\frac{\partial U}{\partial k}+\lambda\left(t\right)\frac{\partial f}{\partial k}=-\dot{\lambda}\left(t\right). \tag{10}$$

Admittedly, it is difficult to give immediate, intuitive sense to these equations. Dorfman's first merit was to give a beautiful derivation of both equations from economic reasoning. But Dorfman did not stop there. He realized that both equations could be directly obtained from a "modified Hamiltonian", denoted by him H^*, and equal to

$$H^*=U\left(k,v,t\right)+\lambda\dot{k}+\dot{\lambda}k=U\left(k,v,t\right)+\frac{d}{dt}[\lambda k]. \tag{11}$$

H^* is thus the traditional Hamiltonian augmented by the product of the derivative of the co-state variable and the state variable, $\dot{\lambda}\left(t\right)k\left(t\right)$. Its significance is the following: at any time t, H^* is the value directly imparted to the functional, at a rate equal to the integrand $U\left(k,v,t\right)$, and indirectly by the rate of increase in the value of the state variable $\frac{d}{dt}[\lambda k]$.

As we mentioned in our introduction, to pay tribute to Robert Dorfman and to honor his memory, we call this new Hamiltonian a Dorfmanian and denote it as D. Observe now that if we set

$$\dot{k}=f(k,v,t)=v, \tag{12}$$

i.e. if the derivative $\dot{k}$ is the control variable, the Dorfmanian becomes

$$D=U\left(k,\dot{k},t\right)+\lambda\dot{k}+\dot{\lambda}k=U\left(k,\dot{k},t\right)+\frac{d}{dt}[\lambda k] \tag{13}$$

and has exactly the structure of the expression Lagrange could have written (see equ. (5)) to obtain the Euler equation in a few lines.

True, in this simple case, the Pontryagin maximum principle would also have led to the Euler equations in two lines. But as soon as we want to optimize more elaborate functionals, such as n–fold integrals, we will meet a difficulty even more challenging than extending Green's theorem, and which, to the best of our knowledge, has never been overcome: extending optimal control theory to n–space. In contrast, the Dorfmanian, beyond offering a highly intuitive approach, has the considerable advantage, by its possible generalisation, to allow the easy and practically immediate handling of such functionals.

We have indicated in La Grandville (2017, 2018, 2024) a number of these extensions, yielding in a few elementary steps the corresponding Euler equations. Here we will present a case that offers a particularly important simplification.

4. Deriving the Euler-Ostrogradski equation in a few lines

Consider the problem of extremizing the functional

$$I\left[z(x_1, x_2, ..., x_n)\right] = \iint ... \int_R F(x_1, x_2, ..., x_n,\ z,\ z_{x_1}, z_{x_2}, ...z_{x_n})dx_1...dx_n. \tag{14}$$

where $F(.)$ and the domain of integration R have the same properties as those adopted by Gelfand and Fomin (19630) pp. 153-154, and Troutman (1983) pp. 179-18İ: $F(.)$ has continuous first and second order derivatives with respect to all its arguments, and the domain R has sufficient regularity.

We first suppose that the solution $z(x_1, x_2, ..., x_n)$ is known. We then define, and denote as $p(x_1, x_2, ..., x_n)$ a function equal to the rate of increase of the optimal value of the functional, with respect to one additional unit of the optimum z at point $(x_1, x_2, ..., x_n)$. As before, $p(x_1, x_2, ..., x_n)$ can be conceived as the shadow price of $z(x_1, x_2, ..., x_n)$.

We then build a Dorfmanian that will take into account all direct and indirect effects of z and its n partial derivatives $z_{x_1}, z_{x_2}, ..., z_{x_n}$ on the functional. The direct effect is the integrand of the functional, $F(x_1, x_2, ..., x_n, z, z_{x_1}, z_{x_2}, ...z_{x_n})$. The indirect effect is measured by the sum of the derivatives of the product $pz = p(x_1, x_2, ..., x_n)\ z(x_1, x_2, ..., x_n)$ with respect to each variable x_i, $i = 1, ..., n$.

Omitting for simplicity the arguments of F, z, z_{x_i}, p, we can extend the Dorfmanian to the following expression:

$$D = F + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(pz) = F + p\sum_{i=1}^{n} z_{x_i} + z\sum_{i=1}^{n} p_{x_i}. \tag{15}$$

We now take to zero the gradient of D with respect to z and z_{x_i}, $i = 1, ..., n$. This results in the $n+1$ following equations:

$$\frac{\partial D}{\partial z} = \frac{\partial F}{\partial z} + \sum_{i=1}^{n} p_{x_i} = 0 \tag{16}$$

$$\frac{\partial D}{\partial z_{x_i}} = \frac{\partial F}{\partial z_{x_i}} + p(x_1, ..., x_i, ..., x_n) = 0, \quad i = 1, ..., n. \tag{17}$$

Differentiating the last n equations with respect to x_i leads to

$$\frac{\partial}{\partial x_i}\frac{\partial F}{\partial z_{x_i}} + p_{x_i} = 0, \quad i = 1, ..., n. \tag{18}$$

Summing the equations of system (18) gives

$$\sum_{i=1}^{n} p_{x_i} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial z_{x_i}}; \tag{19}$$

substituting this sum into (16) gives us the second order partial differential Euler-Ostrogradski equation

$$\frac{\partial F}{\partial z} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial z_{x_i}} = 0. \tag{20}$$

Provided that the second order partial derivatives of z exist, its explicit form is

$$\begin{aligned}
&\frac{\partial F}{\partial z} - [\frac{\partial^2 F}{\partial z_{x_1} \partial x_1} + \frac{\partial^2 F}{\partial z_{x_1} \partial z} z_{x_1} + \left(\frac{\partial^2 F}{\partial z_{x_1} \partial z_{x_1}} z_{x_1 x_1} + \ldots + \frac{\partial^2 F}{\partial z_{x_1} \partial z_{x_n}} z_{x_1 x_n} \right) \\
&+ \ldots + \\
&\frac{\partial^2 F}{\partial z_{x_n} \partial x_n} + \frac{\partial^2 F}{\partial z_{x_n} \partial z} z_{x_n} + \left(\frac{\partial^2 F}{\partial z_{x_n} \partial z_{x_1}} z_{x_n x_1} + \ldots + \frac{\partial^2 F}{\partial z_{x_n} \partial z_{x_n}} z_{x_n x_n} \right)] = 0,
\end{aligned}$$

an equation with $1 + n(n + 2)$ terms. Complex as this equation is, it was very simply obtained thanks to the extended Dorfmanian, with no need to generalize neither the fundamental lemma of the calculus of variations, nor Green's theorem to $n-$space.

A conjecture

We finally take the liberty of indulging into a conjecture, in the form of a question. Recall how, even in simple cases, it is difficult to define sufficient conditions for an extremum to furnish a maximum or a minimum. Would the simplicity provided by the Dorfmanian to obtain necessary conditions for optimizing functionals carry over to sufficient conditions?

References

Chiang, A. (2002) *Elements of Dynamic Optimization.* McGraw Hill.

du Bois-Reymond, P. (1879a) Erläuterungen zu den Anfangsgründen Variationsrechnung. *Mathematische Annalen*, **15**, 2, June 1879, 283-314.

du Bois-Reymond, P. (1879b) Fortsetzung der Erläuterungen zu den Anfangsgründen Variationsrechnung. *Mathematische Annalen*, **15**, 3-4, September 1879, 564-576.

Dorfman, R. (1969) An Economic Interpretation of Optimal Control Theory. *American Economic Review*, **59**, 5, 817-31.

GELFAND, I. AND FOMIN, S. (1963) *Calculus of Variations*. Prentice Hall, Englewood Cliffs, NJ.

GOLDSTINE, H. (1980) *A History of the Calculus of Variations from the 17th Century to the 19th Century*. Springer-Verlag, New York.

KAMIEN, M. I. AND SCHWARTZ, N. L. (2012) *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management* (second edition). Dover Publications Inc., Mineola, NY.

KOT, M. (2014) *A First Course in the Calculus of Variations*. American Mathematical Society, Providence, R.I.

LA GRANDVILLE, OLIVIER DE (2017) *Economic Growth: a Unified Approach*, with a foreword and two contributions by Robert M. Solow, Second edition. Cambridge University Press, Cambridge, U.K.

LA GRANDVILLE, OLIVIER DE (2018) Introducing the Dorfmanian, a Powerful Tool for the Calculus of Variations. *The Australian Journal of Mathematical Analysis with Applications*, **15**, 1, article 2, 1-12.

LA GRANDVILLE, OLIVIER DE (2024) A New Look at the Equations of the Calculus of Variations. *The Australian Journal of Mathematical Analysis with Applications*, **21**, 1, article 14.

PESCH, H. J. AND PLAIL, M. (2009) The Maximum Principle of Optimal Control: A History of Ingenious Ideas and Missed Opportunities' *Control and Cybernetics* **38**, 4A, 973-995.

SAGAN, H. (2012) *Introduction to the Calculus of Variations*. Dover Publications.

TROUTMAN, J. (1983) *Variational Calculus with Elementary Convexity*. Springer Verlag, New York.