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Stochastic optimization problems under incomplete information on distribution functions[1]

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Abstract: In this study we discuss a class of stochastic optimization problems for the cases where parameter of optimization is a probability distribution. The knowledge about this distribution is reduced to fixed values of some of its moments, or quantiles. Examples from reliability maintenance, inventory, and queueing theory illustrate the specific nature of the problems that goes beyond the known frames of the general theory.

The article is in the form of a survey of papers relevant to the stated scope, related mainly to some authors and less known results. Relationships with general theory, and discission on prospective other considerations arc also given.

Keywords: inventory models, moment restrictions, optimal control, queueing, reliability, technical maintenance, unreliable service, warranty.

1. Introduction and summary

Most problems of stochastic inventory control, reliability maintenance, etc. consider optimization of functionals that are expected value of some expression involving one or more random variables. This makes these functionals ultimately dependent on some probability distribution function $F(x)$. In practice this function is not completely known. Usually the information about participating random variables, and therefore about $F(x)$, is restricted to the knowledge of some of its initial moments, or some quantiles only. To specify the problem, we introduce the following definition and notations.

Definition 1 *We say that the cumulative probability distribution function (cdf) $F(x)$ belongs to the class $D(m_1, \ldots, m_n)$ iff it satisfies the equations*

$$1 = \int_{-\infty}^{\infty} dF(x), \quad \text{and } m_k = \int_{-\infty}^{\infty} x^k dF(x); \; k = 1, \ldots, n. \tag{1}$$

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Here $m_1, \ldots, m_n$ is a given seq11,ence of real numbers satisfying required conditions to represent the moments of some probability· distribv,tion.

The common problem is: to estimate the efficiency

$$\mathrm{Eff}(T,a,F^*) = extremum_{F\in D(m_1,\ldots m_n)} \mathrm{Eff}(T,a,F) \tag{2}$$

of the use of corresponding systems during a time interval [0, T], while the function $F(x)$ is incompletely known (i.e. it belongs to a certain class of cdf's, say, in the sense of Definition 1). In (2) a is some controllable parameter which can be the subject of other optimization.

To solve problems of type (2) usually the methods known from Markov-Tchebysheff moment problem (see e.g. Carlin and Studden, 1968, or Krein and Nudelman, 1973) are used. However, the specific conditions of reliability, warranty, inventory control, queuing, and other related problems generate some new aspects for application of these methods. In the present work we discuss several questions relevant to these fields in order to highlight what is similar and what is different in the theoretical setups and the reality of practice. The work has the form of a brief survey of some known (to the authors) results related to the optimal control problems in reliability maintenance under incomplete information. The discussed specifics are illustrated also on examples from warranties with restricted reliability data, as well as on examples from inventory control with incomplete demand information, and optimal control of an unreliable process. This survey has no pretension to completeness, it rather aims to present some peculiarities in the study. At the end we state some prospective development of problem (2), which appear to be of practical importance due to the necessity to introduce extended restrictions instead of (1). In this case the class of probability distributions O is determined by inequalities of the type $c_k \leq m_k \leq d_k$, $k = 1, 2, \ldots, n$. Such type of problems offer new challenge to mathematicians. In this context we review some resent results of Danielian and Tatalian (1987, 1987a, 1988, 1994).

2. Optimization problems in technical maintenance

In a series of works Barzilovich and Kashtanov (1971, 1975, 1983), and Kashtanov (1981, 1987) considered maintenance problems with emphasis on the specific probability information about the systems, available in practice. It generates a class of new optimization problems, requiring some specific methods for its solution. As an illustration we consider three kinds of problems in the next subsections.

2.1. Strategies for reliability maintenance

A system with preventive inspections, latent failures and renewals. Let the life time X of a system have cdf $F_X(x) = P(X < x)$. Suppose a failure during the system operating time is not discovered immediately but after a random time interval Z with cdf $F_Z(x) = P(Z < x)$. This is a latent failure. The random variables (r.v.) X and Z can be dependent and have common cdf $F_{X,Z}(x, z)$ with marginals $F_X(x)$ and $F_Z(z)$. The system starts to operate at $t = 0$, and a preventive inspection is assigned after a random time Y with cdf $F_Y(x) = P(Y < x)$. If in the random time interval $[0, Y]$ the system has no failure, then an inspection starts, and has a random duration g_1, whose cdf is $G_1(x)$. If the system has failed but the failure was not detected, the inspection would have a duration g_2 with cdf $G_2(x)$. On the other hand, if the failure is discovered at the time $X + Z$ before Y expires, then the inspection is also conducted, and is followed by a renewal of the system. This is started immediately after a latent failure turns into explicit one, and the residual inspection and repair will have duration g_3 with a cdf $G_3(x)$. During any inspection all the necessary repairs are made (their times are included in the inspection) and afterwards the system starts to operate as properly as a new system. The following inspections are assigned in the same way as for a new system. Thus the maintenance process is regenerative with points of regeneration at the end of any inspection. Note that the r.v.'s within the times between two inspections can be arbitrarily dependent.

As measures for quality of the maintenance the following profit functions are considered:

(i) The availability coefficient

$$R(z) = \lim_{t\to\infty} R(t, z),$$

where $R(t, z)$ is the probability that the system is correctly operating within the interval $[t, t + z)$;

(ii) The expected specific profit (or losses) per unit time

$$L_1 = \lim_{t\to\infty} \frac{g(t)}{t},$$

where $g(t)$ is the expected profit for the system operation on $(0, t)$.

Using the points of regeneration and the Smith's renewal theorem from renewal theory (see e.g. Gnedenko et al., 1983), the following expression for $R(z)$ can be derived:

$$R(z) = \frac{\int_0^\infty A(t, z)\,dF_Y(t)}{\int_0^\infty B(t)\,dF_Y(t)}. \tag{3}$$

Here the notations

$$A(t, z) = \int_t^\infty [1 - F(x + z)]\,dx, \tag{4}$$

and

$$B(t) = \int_0^t [1 - \psi(x)]\,dx + Eg_1 + (Eg_2 - Eg_1)F_X(t) + (Eg_3 - Eg_2)\psi(t), \tag{5}$$

with

$$\psi(t) = P(X + Z < t) = \int_0^t \int_0^{t-z} dF_{X,Z}(x, z) \tag{6}$$

are used. One can easily check that $B(t)$ is the conditional expected duration of a separate period of regeneration for the system under the condition that the next inspection will start at time $t > 0$. In an analogous manner one can derive $g(t)$ and L_1 in (ii), and will obtain:

$$L_1 = L_1(F_Y) = \frac{\int_0^\infty C(t)dF_Y(t)}{\int_0^\infty B(t)dF_Y(t)}, \tag{7}$$

where $B(t)$ is given by (5). The function

$$C(t) = [c_0(t) - c_1(t)][1 - F_X(t)] + \int_0^t c_0(v)dF_X(v)$$

$$- \int_0^t c_1(v)\left(\int_{t-v}^{\infty} f_{X,Z}(t-v, z)dz + \int_{t-v}^{t} f_{X,Z}(t, v)\right) dv,$$

$$- c_4\psi(t) - c_3(t)\int_0^t \int_{t-x}^{\infty} f_{X,Z}(x,z)dxdz \tag{8}$$

represents the expected profit in the time period $0, t)$; $\psi(t)$ is given by (6); $f_{X,Z}(x,z)$ is the common probability density function (p.d.f.) corresponding to the cdf $F_{X,Z}(x,z)$ of the random vector (X, Z); $c_i(x)$ (for $i = 1, 2, 3, 4$) are loss functions corresponding to time of duration x which the system spent in operation while being into the state i, defined as follows: *State 1* means work in presence of a latent failure ($i = 1$). *State 2* - inspections started before the latent failure occurred ($i = 2$). *State 3* - inspection started during a latent failure was in presence ($i = 3$). *State 4* - an inspection started after a latent failure had been detected, and then ($i = 4$). The only positive profit (function) is $c_0(x)$ while the system operates correctly for time of continuous duration x. Abbreviations used in (8) are

$$c_k = \int_0^\infty c_k(x)dG_{k-1}(x), \quad k = 2,3,4. \tag{9}$$

The problem is how to determine "the most favorable" cdf $F_Y(y)$ of preventive inspection period Y, in order to achieve the best quality of maintenance.

In cases of complete information it is supposed that all the cdf's $F_X(x)$; $F_Z(x)$, $G_i(x)$, $i = 1, 2, 3$ are known. Then one has to find the optimal cdf $F_Y(x)$ which gives the extreme value of the profit functions, given by either (3), or (7), and no additional restrictions apply. The solutions to this kind of optimization problems with cost function given as a fraction of linear functionals of type like (3), and (7) can be significantly simplified. We recommend the use of a Barzilovich and Kashtanov's result (1971, pp. 26-28). We formulate it and give a new simplified proof.

Theorem 1 *Consider the fv,nctional*

$$\mathrm{J}\,(\mathrm{G1}, \ldots\, G_N)\,\bar{\cdot}\; \frac{J_{\!\mathit{IJI}} NA(x1, \ldots X_N)dG1\,(\mathrm{x1}) \ldots dG_N\,(X_N)}{J_{\!\mathit{IJI}} NB\,(x1,\text{---}X_N)dG1\,(\mathrm{x1}\,), \ldots dG_N\,(x_N)}, \tag{10}$$

where the functions $A(x_1, \ldots, x_N)$, $B\,(x_1, \ldots, x_N)$ *are known, and* $B\,(x)$ *is either positive, or negative for all* $x = (x_1, \ldots, x_N)$ E at^N. *Let* $\mathbf{J}(G)$ *be bounded over the set* D = {G1(x), ..., $G_N\,(x)$; *cdf's on* [O,oo)}. *Then the extreme problem: To .find*

$$\mathit{extremum0}_{E^{\mathit{rl}}}\ \mathrm{J}\,(\mathrm{G1}, \ldots\, G_N) = \mathrm{J}\,(Gf, \ldots\, G1.r) \tag{11}$$

has its solution on the set D* = {$G_k\,(x)$: $Gk(x) = 0$ *for x::;;* T_k *and* $G_k\,(x) = l$, *for-* $x > T_k, T_k$ E at^1, $k = 1, \ldots,$ N} *of degenerated cdf's.*

Proof: Denote by $G(x)$ = ($G_1(x_1), \ldots, G_N\,(x_N)$) the vector of cdf's used in calculation of (10), and let $A(x) = A(x1, \ldots, x_N)$; $B\,(x) = B(x1, \ldots, x_N)$- For $G(x)$ E W we have

$$\mathbf{J}(\vec{G}) = \frac{A(\tau_1, \ldots, \tau_N)}{B(\tau_1, \ldots, \tau_N)} = \frac{A(\vec{\tau})}{B(\vec{\tau})}.$$

which is a multivariate function with N non-random arguments. Further, let f* and f** be two points where the global minimum and the global maximum of J(G) = A(f)/B(f) are located. They will then correspond to some cdf's G^* and G^{**} in D*, i.e.

$$\boldsymbol{r} = \arg \underset{\mathrm{GErl*}}{\mathrm{Jnf}}\ \mathrm{J(G)} = \arg\ \mathrm{J(G^*)}$$

$$T''^{*} = \arg \underset{\mathrm{GErl*}}{\sup}\ \mathrm{J(G)} = \arg\ \mathrm{J(G^{**})}. \tag{12}$$

It is also possible that some components Tt or Tt* off* and/or f** are equal to oo. However, the following inequalities are satisfied for any vector x

$$A(r)/B(T'')\text{::;;}\ A(x)/B(x)\text{::;;}\ A(r^*)/B(r^*). \tag{13}$$

Multiply the left hand side inequality by the non-negative product $B(T'')B(x)$, and the right hand side one by the product $B(f^{**})B(x)$ to get two equivalent inequalities

$$A(T'')B(x)\text{::;;}\ B(f^*)A(x) \text{ and } B(f^{**})A(.i!)\text{::;;}\ A(T''^*)B(x).$$

Integrate with respect to any cdf $G(x)$, and verify that

$$A(T'') \int_{\mathit{JIJIN}} B(x)dG(x)\text{::;;}\ B(T'') \int_{\mathit{JIJIN}} A(x)dG(x),$$

and

$$B(r^*) \int_{RN} A(x)dG(x) \quad A(r^*) \int_{RN} B(x)dG(x).$$

The above two inequalities are equivalent to the inequality:

$$A(r)/B(r) \quad J(G(x)) \quad A(r^*)/B(r^*). \tag{14}$$

Thus, the statement is true.

Hence, the points of local extremes of the function $A(x^*)/B(x^*)$ are among the solutions of the system

$$\frac{a}{a x_i} \frac{(A(x))}{B(x)}, \; i = 1, 2, \ldots, N, \tag{15}$$

and the absolute extremes can be located at infinity (if some $\tau_i = =$). To find them it is necessary to compare the values of $A(x)/B(x)$ for all those arguments until the extremes are specified. Thus, the stochastic optimization problem (11) turns into (12), a deterministic one. Its solution is given by (15).

We now return to the initial optimization problems for either $R(z)$ from (3), or $L_I(F_y)$ from (7). To have the condition $B(x) > 0$, from (5) and (6) we derive as a sufficient condition, that the inspections starting in working state, during an undiscovered latent failure and after an accident must form an increasing sequence in average. This means that when the natural order

$$\mathbf{E}g_1 \leq \mathbf{E}g_2 \leq \mathbf{E}g_3 \tag{16}$$

holds, and therefore in expressions (3) and (7) it will be true that $B(t) > 0$. Referring to Theorem 1 we conclude that the optimal inspection interval Y^* is non-random, since F_{Y^*} (.T) E D*. Thus for given distributions of the life and the repair times, the optimal inspection has fixed, systematic time interval $Y^* = T^*$.

As an example we consider the exponential case. Let X and Z be independent r.v.'s with $F_x(x) = 1 -$ e-,\x, and $F_z(x) = 1 -$ e-'1"". Consider the availability coefficient $R(z)$ given by (3) - (6) as profit function (it equals to $R(O)$ when $z = 0$). Using Theorem 1 we conclude that when (16) holds, then

$$\sup_{F_Y} R(z) = \sup_{T\,O} A(T,z)/B(T).$$

With the specified forms of $A(t, z)$, $B(t)$, and $1/J(t)$, the inspection interval $T^*(z)$ is the point where the $\sup_{T>O}[A(T, z)/B(T)]$ is located. The ultimate result (after performing the routine-calculus considerations) is as follows:

1. If..\ C 1, then it is always preferable to have inspections in fixed finite time interval $T^*(0)$, scheduled right after any new system has started operation;
2. if..\< 1, and the inequality..\/,> ..\(Eg2 - Eg1) + (1 - ..\)Eg1 holds, then the same type of non-random inspections, as in 1), are optimal, since it gives maximum expected profit per unit time in the long run.

No simple expressions for $7^*(z)$ can be found. Even in the case of explicit failures (i.e. when $P(Z = 0) = 1$ holds), the optimal interval $7^*(0)$ of inspections can not be explicitly given, but as the solution of the equation

$$-\bar{F}x(7) + [Fx(7)/\bar{F}x(7)] \int_0^{7} [1 - \bar{F}x(x)]dx = Egif(Eg3 - Eg1).$$

However, there exist simple numerical algorithms for determination of $7^*(0)$ based on Theorem 1. Several such algorithms in some other particular cases of the reliability maintenance with known probability distributions are given (e.g, Barzilovich and Kashtanov, 1971, 1975; Gnedenko et al., 1983).

2.2. Optimization of maintenance problem under restrictions

We observe in technical maintenance problems that most of the measures of effectiveness have the form

$$I(F) = \int A(t)dF(t) / \int B(t)dF(t) \stackrel{def}{=} U(F)/V(F), \quad (17)$$

where $F(t)$ is a cdf in ^{n}, n 1, and $A(t)$, $B(t)$ are piece-wise continuous and bounded functions. More generally a sufficient supposition is that U(F) in (17) is uniformly bounded on some set 8' of cdf's and V(F) has constant sign on it. Then in related optimization problems the following Lemma (Gnedenko et al., 1983, p.259) can be successfully applied:

Lemma 1 *If there exists* supFES' I(F) = c, *then the subset* S'_c *of all cdf's for which* supFES'c I(F) = c *coincides with the subset of all cdf's in* 8' *for which the maximal value of the expression*

$$J(F) = U(F) - cV(F)$$

is attained.

The proof follows the idea of 'the proof of Theorem 1. When assuming that the Lemma is not true, a contradiction occurs.

Lemma 1 is used in the proof of the next statement, and it is relevant to the optimization of functionals of type (17) under various restrictions. Assume that the cdf $F(x)$ belongs to the class

$$fl(N, Y, A) = \{F: F(y_i) \in A_i, A_i \in {}^1, A_i - \text{closed sets}, \quad i = 1, \dots, N\} \quad (18)$$

of distributions that have values within given numerical subsets $A = \{A_i, i = 1, \dots, N, y_1 < \dots < y_N\}$, whenever the argument y coincides with some of the given y_i. Further, let us denote by $fl^*(N, Y, A)$ the subset of step functions $F(y)$, which are cdf's, and have no more than one jump within each of the intervals $[y_i, y_{i+1})$, i = 1, ... ,N. The following theorem holds:

Theorem 2 For the extremes of any measure of effectiveness I(F) of fraction-linear type (11) it is trv,e that

extremumFED(N,Y,A) I(F) = extremumFED*(N,Y,A) I(F).

Proof. Here we give just the main idea of the proof. Precise details can be easily derived (in analogy with Gnedenko et al., 1983, p.262 - 264). Let Fo(x) be a cdf for which the extremum of I(F) is attained, and let ?ri = F(yi), i = 1, ... , *N*. Since Yi are ordered, and Fo(y) is monotonic, we obtain that 7r1 ?r2 --- 'lrN ..

According to Lemma 1, the set for getting the extremes of I(F) coincides with the set where J(F) = 0. By the substitutions ?o = 0, 7rN+l = 1, y_0 = -oo, and YN+l = oo, the functional J(F) can be rewritten in the form

$$J(F) = \sum_{k=0}^{N} \int_{Y_k}^{Y_{k+1}-0} C(x)dF(x), \text{ with } C(x) = A(x) - cB(x).$$

Let xk be the point where C(x*) = $\text{extremum}_{x\,E}$ (Yk,Yk+il C(x). Construct the piece-wise function

Ft(x) = 'Irk, for x E [Yk, xk], and 1rk+1, for x E (xk, Yk+1l- (19)

Then it is easy to see that

J(Fo) = J(Ft),

and from Lemma 1 it follows that I(Fa) = extremumF,{Efl*($_n$,Y,A) I(F). Thus the desired extreme is attained at a step-wise cdf. The omitted details explain the impossibility of attaining the extremum_x E[Yk,Yk+il C(x) outside of the "intervals" [Yk,Yk+l] (note that Yk can be arbitrarily chosen points in R^n).

Theorem 2 determines the form of the extremal cdf Fa *(x),* where the sought extremum is attained. We name this cdf either the most favorable or the most v.nfavorable cdf, pending on the meaning of extremum, whether it is desirable, or not. The practical problem of finding the extremes of any given functional I(*F)'* and Fa *(X)* itself is now transferred from a variational calculus question into a deterministic optimization problem, due to the following equation (we write it for the maximization case):

$$\sup_{F\in D(N,Y,A)} \frac{\int_{-\infty}^{\infty} A(t)dF(t)}{\int_{-\infty}^{\infty} B(t)dF(t)}$$

$$= \sup_{\substack{x_k \in [y_k, Y_{k+1}] \\ k=0,1,\ldots,N}} \frac{\sum_{k=0}^{N} A(x_k)(\pi_{k+1} - \pi_k)}{\sum_{k=0}^{N} B(x_k)(\pi_{k+1} - \pi_k)}. \qquad (20)$$

Thus, by referring to (19), the extremal cdf Fa is found from the solution of (20).

Some peculiarities for the case of optimal inspections of technical maintenance simplify the problems for determination of the extremes in (20). They are based on the contrasts of monotonic behavior of A(x) and B(x) in x. The extreme points for the right hand side of (20) are located at xZ = Yk ± 0.

The results of Theorem 2 are used to determine the optimal inspection interval Y under information that the cdf Fx (x) is in the class D(N, Y, A) (cdf's with known quantiles), where A_i = {1ri}, i = 1, ... ,N. If we assume immediate discovery of the failures, i.e. if $P\{Z = 0\}$ = 1, then the optimization problem is as follows: Find the cdf $G^*(y)$ of the optimal inspection time Y^* which is uniformly good for all the cdf's F^* E D(N, Y, ii') of the life times of the system. Mathematically it is expressed by the equation

$$R(z,F^*,G^*)=\sup_{G\in fl} \inf_{F\in fl(n,Y,ii')} R(z),$$

where the reliability coefficient $R(z)$ is defined by (3) - (6). A detailed solution can be found (Gnedenko et al. 1983, p.293 - 296). More complicated applications with consideration of semi-Markovian models in technical maintenance up to certain extent are also known (Kashtanov, 1987).

2.3. On the optimal control of a non-reliable process

A series of studies (1981-1994), finalized by Kolev (1994), have been devoted to optimal control of unreliable processes by introducing external influence. Let us suppose that one has to perform a job of duration T on an unreliable processor (server). The up and down times of the server form an alternating renewal process. $\{X_n, Y_n\}$, n = 1, 2, ... The interruptions and necessity to restart the job from the origin prolong the actual total processing time of the job. After any server failure the incomplete job must start anew, and it continues until a successful end in some of the up times of duration greater then T occurs.

The external control is introduced by a sequence of checkpoints. Each checkpoint is an action of saving parameters (job status, copies, forms, models etc.) that make possible repetitive job start from the attained level of job performance, and thus save the job already done. A checkpoint needs some time 0 to be completed. It is usually scheduled for a moment η_k when the accumulated job volume attains a certain level, measured since the preceding (k-1)-st checkpoint, if no interruption occurs. The job restarts after a failure begining from the job level collected up to the nearest previous checkpointed state. The total execution time of a job that needs time T to be completed when served on a non-reliable server without checkpoint is (set $Y_0 = 0$ below)

$$T(T) = \sum_{n=1}^{N} (X_n + Y_{n+1}), \tag{21}$$

where

$$N = \min_{k\geq 1} \{k;\ X_k > T\}.$$

When checkpoints are introduced, the total execution time is given by the equation

$$T(T; \{\eta_k\}) = \sum_{n=1}^{M_T} [\tau(\eta_k) + \theta_k] + \tau(T - \sum_{k=1}^{M_T} \eta_k), \tag{22}$$

where $\{\theta_k\}$ are the checkpoint durations, the random variable

$$M_T = \max_{n \geq 1}\{n;\ \eta_1 + \ldots + \eta_n \leq T\}.$$

represents the number of checkpoints to be scheduled in time of duration T, and $\tau(.)$ is defined by (21). An intensive, explicit study of the total execution time $T(T; \{\eta_k\})$ under various assumptions about the nature of failures, and several reliability structures, is made by Dimitrov, Khalil, Kolev, and Petrov in a series of papers (1986 - 1994). The profit function due to external influence can be defined by the expected duration to complete the job with checkpoints. This is

$$\mathbf{J}(G_1, G_2, \ldots) = \mathbf{E}\left(\tau(T; \{\eta_k\})\right),$$

where $G_1, G_2, \ldots$ are the cdf's of the intervals between two consecutive checkpoints. Using Theorem 1 one can see that the optimal checkpoint schedule is deterministic, not a random one, since any $G_k(x)$ must be a degenerate cdf. This fact simplifies the study of optimal checkpointing. This is usually assumed in this kind of studies without proof (as in Dimitrov and Petrov, 1987). Kolev (1994) proves this fact clearly. Moreover, the checkpoint schedule is uniform (equidistant) when the failure rate of the server is constant and job duration T is either deterministic or exponential. The optimal checkpoint schedule is not uniform when the failure rate vary in time. Nothing is known about the influence of the distribution of T on the form of the checkpoint schedule.

3. Optimization problems under moment restrictions

3.1. General theory

Here we give a brief review of the Markov-Tchebysheff moments problem using notations and terminology from Carlin and Studden (1968) and Krein and Nudelman (1973). Let $\{u_k(t)\}_{k=0}^{n}$ be a system of continuous linearly independent functions on the interval $[a, b]$. We say that this sequence forms a Tchebysheff system on $[a, b]$, briefly T-system iff the determinant of order $n+1$

$$U\begin{pmatrix} 0, & 1, & \ldots, & n \\ t_0, & t_1, & \ldots, & t_n \end{pmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \ldots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \ldots & u_1(t_n) \\ \ldots & \ldots & \ldots & \ldots \\ u_n(t_0) & u_n(t_1) & \ldots & u_n(t_n) \end{vmatrix}$$

is strictly positive for any a $to < ti < \ldots < t_n$ b The moment space M_{n+l} is the set of vectors in the form $C = (co,, c1, \ldots, e_n)$ E R^{n+l}, where

$$ci = \int_a^b V_i(t) dq(t), \quad i = 0, 1, \ldots, n, \tag{23}$$

and $CJ(t)$ belongs to the set of all non-decreasing right-continuous functions with bounded variation. We say the measure $CJ(t)$ belongs to the class $V(c)$ iff for the chosen and fixed C E M_{n+l} it satisfies the conditions (23).

The Markov-Tchebysheff optimization problem (MTOP) is stated as follows: For a given function $D.(t)$ and c E M_{n+i}, find two measures Cj^* and Cj^{**} in $V(c)$ for which there is either

$$\sup_{a\text{-}EV(c)} \int_a^b D.(t)dCJ(t) = \int_a^b D.(t)dCJ^{**}(t), \tag{24}$$

or

$$\inf_{a\text{-}EV(c)} \int_a^b D.(t)dCJ(t) = \int_a^b D.(t)dCJ^*(t) \tag{25}$$

correspondingly.

Remark 1 *The fv,nctions $v,i(t) = t^i$, $i = 0, 1, \ldots, n$ form a T-system. With them and with $CJ(t)$ a probability measure, the MTOP transforms into an* im*portant problem in probability theory and mathematical statistics with a lot of interesting applications. $V(c)$ is a class of cdf's with first n moments given. Functionals of type (24) and (25) appear as criterions of effectiveness under conditions of incomplete information abov,t probability distributions.*

A general solution of the problems (24), (25) is given by the following two theorems:

Theorem 3 *(Carlin and Studden, 1916, p.88) If $\{uk(t)\}k=0$ and the enlarged system $\{v,o,v,1,\cdots,u_n; D(t)\}$ are T-systems then (24) is attained only for the measures Cj^{**} corresponding to the upper principal representation of the point c, and (25) is attained only for the measures Cj^* corresponding to the lower principal representation of c.*

The concept of principal representation of points in R^n is given in the cited source.

Theorem 4 *(Krein and Nudelman, 1913, p.115) Under the conditions of Theorem 3:*

i) The following equations hold

$$\inf_{\sigma \in V(c)} \int_a^b \phi(t)d\sigma(t) = \sup_{\{a_k\}} \left(\sum_{k=0}^{n} a_k v_k(t);\ \sum_{k=0}^{n} a_k u_k(t) \geq \phi(t) \right);$$

$$\sup_{\sigma \in V(c)} \int_a^b \phi(t)d\sigma(t) = \inf_{\{a_k\}} \left(\sum_{k=0}^{n} a_k v_k(t);\ \sum_{k=0}^{n} a_k u_k(t) \leq \phi(t) \right)'$$

ii) The necessary and sufficient condition for a measure $\sigma^* \in V(c)$ *to be extremal in (24) is the existence of a generalized polynomial* $P_0(t) = a_0 u_0(t) + \ldots + a_n u_n(t)$, *which satisfies the inequality* $P_0(t) \leq \phi(t)$, *and coincides with* $\phi(t)$ *at all the points of growth of the measure* $\sigma_0(t)$ *(for the extremal measure of (25) the inequality* $P_0(t) \geq \phi(t)$ *is requested) for all* $t \in [a, b]$.

The last two theorems give the form of all optimal measures σ^{**} and σ^*, and also propose methods for its determination. However, in applications even the smallest deviations from the above theoretical conditions generate specific problems which need specific techniques to get efficient explicit final results. Several such applications are presented in Carlin & Studden (1968), and in Krein & Nudelman (1973), (e.g. applications to approximation theory, to sums of random variables, to inequality problems, to experimental design and others). Our experience shows that reliability related problems offer other specifics where general theoretical results barely work.

3.2. Applications

Here we give some results related to the warranty analysis and inventory theory under incomplete information, analogous to MTOP.

A Warranty Analysis Problem:

The newly formed branch of Operations Research known recently as Warranty Analysis, is an intensively developing area of optimization methods (see Blischke and Murthy, 1996). However, there are very limited studies in this area which discuss optimization problems under the restrictions existing in practice, and which are of the above described area. In 1986 Chukova considered the following model in warranties: A product has random life time with cdf $F(t)$, net cost c, market price d and a warranty period (WP) of duration T. The probability for this product to be sold is p_T, and depends on the assigned WP. If the product fails during the warranty existence, then the user claims the item, and this costs the producer an amount e. The expected producers profit is then given by the expression

$$E[g(T)] = dp_T - p_T F(T) - c. \tag{26}$$

The incompleteness of information means that the cdf $F(t)$ is known only through its first n initial moments, i.e. $F(t)$ satisfies

$$1 = \int_0 dF(x) \text{ and } m_k = \int_0 x^k dF(x), k = 1, \ldots, n \qquad (27)$$

The optimal warranty problem is in most cases stated as follows: To assign that duration T_0 to the WP, which gives maximal producer's profit under the 'worst' feasible cdf $F(t)$ that fits the available information. Mathematically T_0 is defined as the maximal solution to the equation

$$T_0 = arg \max_T \min_F E[g(T)],$$

where E[g(T)] and the class of feasible cdf's are defined by (26) and (27) correspondingly.

Here the general theory does not help. Chukova applied a direct approach and got the following results: The most unfavorable distribution function $F(t)$ has the form

$$F^*(t) = \begin{cases} , & for \quad t < \frac{m_1 T - m_2}{T - m_1} = a(T); \\ a, & for \quad a(T) \leq t < T; \\ 1, & for \quad t \geq T, \end{cases}$$

where $a = (T-m_1)^2/T-2m_1T+m_2)$. The optimal value of T is either $T_0 = a(t)$, when $[p_7 - P_a(_7 J]/(a.p_7) < e/d$, or $T_0 = T$ otherwise.

Inventory Control Problems:

Every inventory problem is an optimization problem. Usually all participating probability distributions are assumed completely known. Models with limited information are rare. In the classical inventory theory (non-periodic models, models homogeneous in time) one can frequently observe expressions of the form

$$R(Q, F) = C \int^Q (Q - x)dF(x) + D[E(X) - Q] \qquad (28)$$

which must be optimized in Q (the quantity to be supplied). Here $F(x)$ is the cdf of the demand X during the considered time interval, C and D are known cost coefficients. Usually $F(x)$ is assumed known. However, under incomplete information the cdf $F(x)$ is determined by the knowledge that it belongs to a class V(m.) of cdf's where the only known is a set of initial moments $m = (m_1, \ldots, m_n)$, mi being the i-th moment of $F(x)$. The inventory optimization problem is how to find the optimal supply Q^* under the "worst" demand (according to the best of our knowledge about it). Therefore, it then turns info the following mathematical optimization problem: To find

$$Q^* = arg \min_Q \max_{F \in V(m)} R(Q,F).$$

Here we also have a deviation from the theoretical conditions given in Theorems 3 and 4, since the functions $1, t, t^2, \ldots, t^n, (Q - t)$ do not form a T-system. Moreover, the integral and the integrand in (28) are mutually dependent (case not discussed in the theory).

In Fu (1987) some theoretical elements for that case are developed. One of his results is the following:

Let $P_n(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial with real coefficients, and let $K[P_n, Q]$ be the maximal number of points of intersection for $P_n(x)$ and the function $(Q - x)_+ = \max(0, Q - x)$, for $x > 0$, when the condition $P_n(x) \geq (Q - x)$ is fulfilled. Then the following is true.

Theorem 5 *For any $Q \geq 0$, $n \geq 2$ and $a_n > 0$ there is*

$K[P_n, Q] \leq (n+1)/2$, *when* $P_n(0) \neq Q$;

$K[P_n, Q] \leq (n+1)/2 + 1$, *when* $P_n(0) = Q$.

For $a_i = m_i$, $i = 1, \ldots, n$ this theorem brings additional precision to Theorem 4, and formally solves the problem of how to obtain the extremal cdf $F^*(x)$ for the functionals of type (28), when $F \in V(m_1, \ldots, m_n)$. Explicit results are obtained for the case n=2 as stated in the next theorem.

Theorem 6 *The most unfavorable demand distribution function in the class $V(m_1, m_2)$ which maximizes (28), has one of the following forms:*

(i) *In the case of $Q < m_2/(2m_1)$:*

$$F^*(x) = \begin{cases} 0, & \text{for } x < 0; \\ (m_2 - m_1)/m_1, & \text{for } 0 \leq x < m_2/(2m_1); \\ 1, & \text{for } x \geq m_2/(2m_1). \end{cases}$$

(ii) *In the case of $Q \geq m_2/(2m_1)$:*

$$F^*(t) = \begin{cases} 0, & \text{for } x < a; \\ p, & \text{for } a \leq x < b; \\ 1, & \text{for } x \geq b; \end{cases}$$

where $a = Q - (Q^2 - 2Qm_1 + m_2)^{1/2}$, $b = Q + (Q^2 - 2Qm_1 + m_2)^{1/2}$, $p = a/(b - Q)$;

(iii) *If $F \in V(m_1, \ldots, m_n)$, and $n \geq 3$, then $F^*(x)$ is a step-wise cdf with $[(n+1)/2] + 1$ points of growth. In addition, when n is an odd number, then $F^*(x)$ has a jump (point of growth) at x=0.*

The solution of the inventory optimization problem stated by (28), and under demand obeying the cdf $F^*(x)$ is $Q^* = 0$, when $(C - D)/(C + D) > m_1/m_2$, and $Q^* = m_1 + \varepsilon\sigma[k/(1-k)]^{1/2}$, otherwise, where $\sigma^2 = m_2 - m_1^2$, $k = [(C - 2D)/C]^2$, and $\varepsilon = sgn(2D - C)$.

Other results concerning the noted inventory models under incomplete information (reduced to knowledge of the class $V(m_1, m_2)$ of cdf's), are reported in Dimitrov and Fu (1987), and were explicitly studied by Fu (1987). In addition, stability problems related to the IFR class of distributions are considered in continuation of the original optimization inventory problems. However, the problem still exists for a number of inventory models, waiting for an explicit analysis.

4. Further developments

The methods of solution related to MTOP essentially use the convex properties of the considered class of distributions. However, the applications offer a number of optimization problems (in particular with restrictions on the moments), where these properties do not hold. Thus the classical methods of analysis cannot be applied to these cases. This is probably the reason for seeing considerations in the literature concerning mainly the particular cases where low number of moments is involved. Important classes of distributions like the IFR and the DFR classes do not form any convex class.

An extension exists of Markov-Tchebysheff problem of the classical theory to a class of cdf's called by Danielian and Tatalian (1987a, 1987b, 1988, 1994) a majorizing class. They assume, a cdf F to be k-majorizing ($k > 0$ is an integer) to another cdf G and write $G \prec_k F$ if there exist k subsets $A_1 < \dots < A_k$ (notation $A < B$ means that any $x \in A$ is less than any $y \in B$) such that
(i) $(-1)^{i-1}[G(t) - F(t)] > 0$ for $t \in A_i \cup \dots \cup A_k$, $i = 1, \dots, k$, and
(ii) $G(t) - F(t) = 0$, for $t \in R^1 \setminus \{A_1 \cup \dots \cup A_k\}$.

For $F = G$ it is true that $G \prec_0 F$. A superscript of a cdf $F(x)$ is introduced as the minimal integer k for which $F \in \{G : G \prec_k F, k \geq 1\}$ is fulfilled. The inverse relation $F \prec_k G$ defines the lower index of $F(x)$.

This definition is extended to classes of cdf's, say $D_1 \prec_k D_2$, iff for any $G \in D_1$, and for any $F \in D_2$, there is $G \prec_m F$ for all $m = 0, 1, \dots, k$.

The optimization problem is: To find the extremes (maximum or minimum) of the integral

$$J(F) = \int_a^b D(t)\,dF(t)$$

with respect to the cdf $F(x)$ from the class $V(c,d)$ defined by the conditions

$$c_i \leq \int_a^b u_i(t)\,dF(t) \leq d_i, \quad i = 0, 1, \dots, n.$$

Here the continuous functions $u_0(t) = 1, u_1(t), \dots, u_n(t), D(t)$, are given and fixed.

Danielian and Tatalian (1988) give the solution of this problem in the case when $V(c, d)$ is a majorizing class, i.e. when it satisfies the following:

1. $V(c, d)$ consists of uniformly bounded cdf's;
2. There exist distributions in $V(c, d)$ with arbitrarily large indices;

3. For any given cdf in $V(c, d)$ the corresponding index cannot be simultaneously an upper and lower one;
4. For any cdf F of index $k > 1$ there exists a continuous $(k-1)$-parametric family of cdf's $\{F_a\}$, $a \in V \subset \Re^{k-1}$, and all of them are of index k, so that $F \in \{F_a\}$, and $V(c,d)$ is an open set in $\Re^{k-1}$.

The same authors show in (1988) that:

a) If the functions $u_0(t), \ldots, u_n(t)$, $D(t)$ form a special type of Tchebysheff system, then the extreme of $J(F)$ is attained for a unique cdf $F^*(x)$ in $V(c, d)$;

b) There exists a majorizing structure in the class of IFR cdf's, and the extreme cdf $F^*(x)$ has as index $n+1$ with specified construction;

c) The same qualitative results as in b) are valid also in the case of IFRA class of cdf's.

5. Concluding remarks

Stochastic optimization problems with restrictions on distribution functions are important and frequently occur in practice. Their solution can be found in a narrow class of cdf's of simple construction, depending on the type of restrictions and on target functions. Where known general methods do not work specific approaches are applied. However, the forms of those cdf's which give the optimal solution are the same as that of the narrowed class, have some piece-wise structure, and change the original variational calculus problem into a conventional optimization problem. This leads us to believe that there must be some common general approach to these problems which contains all particular results.

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