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Efficiency and duality for $E - [0, 1]$ convex multi-objective fractional programming*

by

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Abstract: In this paper, we are interested in the efficiency and duality criteria for multi-objective fractional programming, in which the involved functions are locally Lipschitz. We shall establish the necessary and sufficient conditions for the considered problem under the concept of locally Lipschitz *E-[0,1]* convexity assumptions. Moreover, the dual problem in the sense of Schaible is defined for locally Lipschitz *E-[0,1]* convex multi-objective fractional programming and several duality theorems are established.

Keywords: fractional programming, nondifferentiability, multi-objective, *E-[0,1]* convexity, efficiency, duality

1. Introduction

In recent years, multi-objective fractional programming has attracted the attention of many authors due to the fact that in various programming problems the objective functions are quotients of two functions, see Antczak (2024), Antczak and Verma (2018), Verma (2017a), or Yadav and Gupta (2021). The efficient (Pareto) solution of multi-objective fractional programming is an important concept in mathematical models, economics, decision theory, optimal control, etc., see Ahmad, Kaur and Sharma (2022), Guo et al. (2023), Stancu-Minasian (2017), Treanţă, Singh and Mishra (2025), Treanţă and Dragu (2025), Treanţă, Marghescu and Joshi (2025), Treanţă, Bibic and Bibic (2024), or Verma (2017). Recently, efficiency criteria and duality for multi-objective fractional programs have been studied under several kinds of generalized convexity concepts. In Kim, Kim and Lee (2005), sufficient optimality conditions and duality theorems were derived for the multi-objective fractional programming, involving locally

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Lipschitz invex functions. The concept of generalized invexity for a fractional function was introduced in Kim (2006), and the sufficient optimality conditions and duality results for the considered multi-objective fractional programming involving locally Lipschitz functions were proven. Liu and Feng (2007) derived optimality conditions and duality results for multi-objective fractional programming under the generalized (F, θ, ρ, d)-convexity. In Nobakhtian (2008), multi-objective fractional programming with mixed constraints and the optimality conditions were considered and mixed duality results under generalized invexity hypotheses were established. Mishra and Upadhyay (2012) established necessary and sufficient conditions for a feasible solution to be efficient in a multi-objective fractional programming problem, involving η-pseudolinear functions under η-pseudolinearity assumptions, and duality results for Mond–Weir subgradient dual problem were proved. Lai and Ho (2012) studied a subdifferentiable multi-objective fractional programming and established sufficient optimality conditions and duality results under exponential V-r-invexity assumptions. A significant generalization of convexity is constituted by the concept of *E-[0,1]* convexity. *E-[0,1]* convexity depends on the effect of an operator E on the range of the function and the closed unit interval [0.1]. For more details about *E-[0,1]* convex functions, see Emam (2020), Emam and Ali (2016), Youness, El-Banna and Zorba (2005) and Youness and Emam (2008). Inspired and motivated by the above studies, the purpose of this paper is to study nondifferentiable multi-objective fractional programming under the locally Lipschitz E-$[0, 1]$ convexity concept.

This paper will be organized as follows: In Section 2, we mention some definitions and preliminaries, which will be needed in further reasoning. In Section 3, we establish necessary and sufficient conditions for nondifferentiable multi-objective fractional programming problem, which involves locally Lipschitz E-$[0, 1]$ convex functions. In Section 4, Schaible dual problem for the considered multi-objective fractional problem is formulated. Finally, we establish weakly, strong, and strict-converse duality theorems under locally Lipschitz E-$[0, 1]$ convexity hypotheses.

2. Preliminaries

In this section, we present some definitions and results, which will be needed through the course of this paper.

Definition 1 (Clarke, 1983) *A function $f : R^n \to R$ is said to be Lipschitz near $u \in R^n$, if there exists a positive constant K and a neighborhood N of u such that for any $w, \nu \in N$, we have*

$$|f(w) - f(\nu)| \leq K \parallel w - \nu \parallel .$$

The function f is said to be locally Lipschitz on R^n *if it is Lipschitz near u for each* $u \in R^n$.

Definition 2 (Clarke, 1983) *The Clarke generalized directional derivative of a locally Lipschitz function* f *at* $u \in R^n$ *in the direction* $d \in R^n$*, denoted by* $f^\circ(u,d)$*, is defined as*

$$f^\circ(u,d) = \lim_{t\uparrow 0, u^* \to u} \frac{sup\left(f\left(u^*+td\right)-f\left(u^*\right)\right)}{t} ,$$

where $u^* \in R^n$.

Definition 3 (Clarke, 1983) *The Clarke generalized subdifferential of f at* $u \in R^n$ *is denoted by* $\partial f(u)$*, and is defined as*

$$\partial f(u) = \{\chi \in R^n : f^\circ(u,d) \geq \langle \chi, d \rangle, \forall d \in R^n\}.$$

Definition 4 (Youness, El-Banna and Zorba, 2005) *A real valued function* $f : M \subseteq R^n \to R$ *is said to be E-[0,1] convex function on M, with respect to* $E : R \times [0,1] \to R$*, if M is convex set and*

$$f(\varepsilon_1 x + \varepsilon_2 y) \leq E(f(x), \varepsilon_1) + E(f(y), \varepsilon_2),$$

for each $x, y \in M$ *and* $\varepsilon_1, \varepsilon_2 \in [0,1]$, *s.t.* $\varepsilon_1 + \varepsilon_2 = 1$.

Example 1 (Youness, El-Banna and Zorba, 2005) Let *E-[0,1]* $: R \times [0,1] \to R$ be defined as $E(t,\varepsilon) = \varepsilon + t$, where $t \in R$, $\varepsilon \in [0,1]$. It is clear that

$$f(u) = \begin{cases} u, & 0 \leq u \leq 1 \\ 1, & 1 \leq u \leq 2 \\ u-1, & 2 \leq u \leq 4 \end{cases}$$

is *E-[0,1]* convex function R.

Let us now define the concept of locally Lipschitz E-[0,1] convex functions and its generalizations as follows:

Definition 5 *A locally Lipschitz function* $f : R^n \to R$ *is said to be* E-$[0,1]$ *convex function with respect to* $E : R \times [0,1] \to R$ *such that* $E(t,\varepsilon) = \varepsilon \min\{t, \varepsilon\}, t \in R, \varepsilon \in [0,1]$ *at* $u^* \in R^n$*, if*

$$f(u) - f(u^*) \geq \langle \chi, u - u^* \rangle$$

for each $u \in R^n$ *and every* $\chi \in \partial f(u^*)$.

Definition 6 *A locally Lipschitz function* $f : R^n \to R$ *is said to be strictly* E-$[0,1]$ *convex function with respect to* $E : R \times [0,1] \to R$ *such that* $E(t,\varepsilon) = \varepsilon \min\{t,\varepsilon\}, t \in R, \varepsilon \in [0,1]$ *at* $u^* \in R^n$, *if*

$$f(u) - f(u^*) > \langle \chi, u - u^* \rangle$$

for each $u \in R^n, u \neq u^*$ *and every* $\chi \in \partial f(u^*)$.

Example 2 (Emam and Ali, 2016) Let *E-[0,1]* : $R \times [0,1] \to R$ be defined as $E(t,\varepsilon) = \varepsilon \sqrt[3]{t}$, *where* $t \in R,\ \varepsilon \in [0,1]$ and $M = \{(u_1,u_2) \in R^2 : u_1 + u_2 \leq 3,\ 1 \leq u_2 \leq 3,\ u_1 \geq 0\}$. It is clear that $f(u_1,u_2) = (u_2 - u_1)^3$ is locally Lipschitz *E-[0,1]* convex function on convex set M, because

$$f(u) - f(v) \geq \langle \chi, u - v \rangle$$

for each $u, v \in R^2, u \neq v$ and every $\chi = (-3(u_2 - u_1)^2, 3(u_2 - u_1)^2) \in \partial f(v)$.

Proposition 1 (Youness, El-Banna and Zorba, 2005) *If* $g_i : R^n \to R, i = 1,2,...,m$ *are* E-$[0,1]$ *convex functions with respect to* $E : R \times [0,1] \to R$ *such that* $E(t,\varepsilon) = \varepsilon \min\{t,\varepsilon\}, t \in R, \varepsilon \in [0,1]$, *then the set* $M = \{u \in R^n : g_i(u) \leq 0, i = 1,2,...,m\}$ *is a convex set.*

Example 3 (Emam and Ali, 2016) Let *E-[0,1]* : $R \times [0,1] \to R$ be defined as $E(t,\varepsilon) = \varepsilon \sqrt[3]{t}$, *where* $t \in R,\ \varepsilon \in [0,1]$ and $M = \{(u_1,u_2) \in R^2 : u_1 + u_2 \leq 3,\ 1 \leq u_2 \leq 3,\ u_1 \geq 0\}$. It is clear that $f(u_1,u_2) = (u_2 - u_1)^3$ is *E-[0,1]* convex function on convex set M and

$$E(f(u_1), \varepsilon_1) + E(f(u_2)\ , \varepsilon_2) - f(\varepsilon_1 u_1 + \varepsilon_2 y) \in R^2_{\geqq},$$

for each $u_1, u_2 \in M$ and $\varepsilon_1, \varepsilon_2 \in [0,1]$ such that $\varepsilon_1 + \varepsilon_2 = 1$. Hence, f is $R^2_{\geqq}$ -E-[0,1] convex with respect to E, where $R^2_{\geqq}$ is convex cone.

Definition 7 (Emam, 2017) *A locally Lipschitz function* $f : R^n \to R$ *is said to be pseudo* E-$[0,1]$ *convex function with respect to* $E : R \times [0,1] \to R$ *such that* $E(t,\varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$ *at* $u^* \in R^n$, *if*

$$0 \leq \langle \chi, u - u^* \rangle \Rightarrow E(f(u^*), \varepsilon) \leq E(f(u), 1 - \varepsilon)$$

for each $u \in R^n$ *and some* $\chi \in \partial f(u^*)$.

Definition 8 (Emam, 2017) *A locally Lipschitz function* $f : R^n \to R$ *is said to be strictly pseudo* E-$[0,1]$ *convex function with respect to* $E : R \times [0,1] \to R$ *such that* $E(t,\varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$ *at* $u^* \in R^n$, *if*

$$0 \leq \langle \chi, u - u^* \rangle \Rightarrow E(f(u^*), \varepsilon) < E(f(u), 1 - \varepsilon)$$

for each $u \in R^n, u \neq u^*$ *and some* $\chi \in \partial f(u^*)$.

Definition 9 (Emam, 2017) *A locally Lipschitz function $f : R^n \to R$ is said to be quasi E-$[0,1]$ convex function with respect to $E : R \times [0,1] \to R$ such that $E(t,\varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$ at $u^* \in R^n$, if*

$$E(f(u^*), \varepsilon) \geq E(f(u), 1-\varepsilon) \Rightarrow 0 \geq \langle \chi, u - u^* \rangle$$

for each $u \in R^n$ and every $\chi \in \partial f(u^*)$.

Proposition 2 (Emam, 2017) *If $g_i : R^n \to R, i = 1, 2, ..., m$ are quasi E-$[0,1]$ convex functions with respect to $E : R \times [0,1] \to R$ such that $E(t,\varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$, then the set $M = \{u \in R^n : g_i(u) \leq 0, i = 1, 2, ..., m\}$ is a convex set.*

In this paper, we consider the following multi-objective fractional programming problem formulation

$$(FMP) \qquad min \;\; \varphi(u) = \left(\frac{f_1(u)}{q_1(u)}, \ldots, \frac{f_p(u)}{q_p(u)} \right),$$

subject to

$$g_i(u) \leq 0, \quad i \in I = \{1, \ldots, m\}, \quad u \in R^n,$$

where $f_j(u), q_j(u),\ j = 1, \ldots, p$ and $g_i(u),\ i \in I$ are locally Lipschitz E-$[0,1]$ convex functions from R^n to R, moreover $f_j(u) \geq 0, q_j(u) > 0,\ j = 1, \ldots, p$. The convex feasible set of (FMP) is denoted by $M := \{u \in R^n | g_i(u) \leq 0, \forall i \in I\}$. We also denote by $I(u^*)$ the set of active constraints at $u^* \in M$, that is, $I(u^*) = \{i \in I|\ g_i(u^*) = 0\}$.

Definition 10 *A feasible solution $u^* \in M$ is said to be a weakly efficient solution for (FMP) if there is no $u \in M$ such that $\varphi(u) < \varphi(u^*)$.*

Definition 11 *A feasible solution $u^* \in M$ is said to be an efficient solution for (FMP) if there is no $u \in M$ such that, for some $v \in \{1, 2, ..., p\}$,*

$$\varphi_v(u) < \varphi_v(u^*), \quad \varphi_j(u) \leq \varphi_j(u^*), \quad \forall\, j \neq v.$$

Definition 12 *An efficient solution $u^* \in M$ is a properly efficient solution for (FMP) if there exist $\mu > 0$ such that $\forall i,\ i = 1, 2, ..., p$ and $u \in M$ satisfying $\varphi_i(u) < \varphi_i(u^*)$, there exists at least one $j \neq i$ with $\varphi_j(u) > \varphi_j(u^*)$ and*

$$[\varphi_i(u) - \varphi_i(u^*)]/[\varphi_j(u^*) - \varphi_j(u)] \leq \mu.$$

In the objective space of (FMP), the above solutions are said to be nondominated and properly nondominated solutions.

3. Efficiency criteria

In this section, we will derive the necessary and sufficient efficiency conditions for multi-objective fractional programming (FMP) under E-$[0, 1]$ convexity assumptions. We shall use the parametric approach as defined by Crouzeix, Ferland and Schaible (1985), so let us define the auxiliary nonfractional multi-objective programming in the parameter ξ for (FMP) as follows:

$$(MP) \qquad min \quad (f_1(u) - \xi_1 q_1(u), \ldots, f_p(u) - \xi_p q_p(u)),$$

$$subject\ to \qquad u \in M.$$

For the associated multi-objective programming (MP), defined above, the following result is true:

Lemma 1 (Crouzeix, Ferland and Schaible, 1985) *Let u^* be a weakly efficient solution for fractional multi-objective programming (FMP), then u^* is also a weakly efficient solution for the nonfractional multi-objective programming (MP) with $\xi^* = \varphi(u^*)$.*

Theorem 1 *[Necessary conditions theorem] Let u^* be a weakly efficient solution for (FMP) with $\xi^* = \varphi(u^*)$ and the generalized Slater constraint qualification be satisfied at u^*. Then there exist ${\rho_j}^* \geq 0, \sum_{j=1}^{p} {\rho^*}_j = 1$ and ${\beta_i}^* \geq 0,\ i \in I(u^*)$, such that*

$$0 \in \sum_{j=1}^{p} {\rho_j}^* \partial[f_j(u^*) - {\xi_j}^* q_j(u^*)] + \sum_{I(u^*)} {\beta_i}^* \partial g_i(u^*). \tag{1}$$

Proof: Let $u^* \in M$ be a weakly efficient solution of multi-objective fractional programming (FMP) . Hence, by Lemma 1, u^* is also a weakly efficient solution of the associated nonfractional multi-objective programming (MP). Further, by Theorem 2.1 from Antczak (2014), there exist multipliers ${\rho^*}_j \geq 0,$ *and* ${\beta^*}_i \geq 0$ such that condition (1) is fulfilled. This completes the proof. □

Theorem 2 *[Sufficient conditions theorem] Let u^* be feasible for (FMP) and $\xi^* = \varphi(u^*)$. Assume that there exist ${\rho_j}^* \geq 0, \sum_{j=1}^{p} {\rho^*}_j = 1$, and ${\beta^*}_i \geq 0$ for $i \in I(u^*)$, such that condition (1) holds at u^*. If $(f_j - \xi_j q_j), j = 1, 2, ..., p$ and $g_i, i \in I(u^*)$ are locally Lipschitz E-$[0, 1]$ convex at u^* with respect to $E : R \times [0, 1] \to R$ such that $E(t, \varepsilon) = \varepsilon \min\{t, \varepsilon\}, t \in R, \varepsilon \in [0, 1]$, then, u^* is a weakly efficient solution for (FMP).*

Proof: Assume that $u^* \in M$ is not a weakly efficient solution for (FMP). Then, there exist $u \in M$ such that

$$\varphi(u) < \varphi(u^*),$$

and hence, by the definition of φ and $\xi^* = \varphi(u^*)$, the above inequality implies

$$f_j(u) - \xi_j{}^* q_j(u) < f_j(u^*) - \xi_j{}^* q_j(u^*), \quad \forall \quad j = 1, \dots, p. \tag{2}$$

From equation (1), there exist $\chi_j \in \partial(f_j(u^*) - \xi_j{}^* q_j(u^*))$ and $\psi_i \in \partial g_i(u^*)$ such that

$$\sum_{j=1}^{p} \rho^*{}_j \chi_j + \sum_{I(u^*)} \beta^*{}_i \psi_i = 0. \tag{3}$$

Since $(f_j - \xi_j q_j), j = 1, 2, ..., p$ are locally Lipschitz E-$[0,1]$ convex at u^*, then

$$\langle u - u^*, \chi_j \rangle \leq [f_j(u) - \xi_j^* q_j(u)] - [f_j(u^*) - \xi_j^* q_j(u^*)],$$

by using equation (2) and $\rho_j{}^* \geq 0, \sum_{j=1}^{p} \rho^*{}_j = 1$, we get

$$\left\langle u - u^*, \sum_{j=1}^{p} \rho^*{}_j \chi_j \right\rangle < 0. \tag{4}$$

Now, from locally Lipschitz E-$[0,1]$ convexity of $g_i, i \in I(u^*)$ at u^*, we have

$$\langle u - u^*, \psi_i \rangle \leq g_i(u) - g_i(u^*) \leq 0,$$

and, since $\beta^*{}_i \geq 0 \ for \ i \in I(u^*)$, we have

$$\left\langle u - u^*, \sum_{I(u^*)} \beta^*{}_i \psi_i \right\rangle \leq 0. \tag{5}$$

Upon adding equations (4) and (5), we obtain a contradiction with equation (3). Thus, u^* is a weakly efficient solution for (FMP). □

THEOREM 3 *Let u^* be feasible for (FMP) and $\xi^* = \varphi(u^*)$. Assume that there exist $\rho_j{}^* \geq 0, \sum_{j=1}^{p} \rho^*{}_j = 1$, and $\beta^*{}_i \geq 0 \ for \ i \in I(u^*)$, such that condition (1) holds at u^*. If $(f_j - \xi_j q_j), j = 1, 2, ..., p$ are strictly locally Lipschitz E-$[0,1]$ convex at u^* and $g_i, i \in I(u^*)$ are locally Lipschitz E-$[0,1]$ convex at u^* with respect to the same $E : R \times [0,1] \to R$ such that $E(t, \varepsilon) = \varepsilon \min\{t, \varepsilon\}, t \in R, \varepsilon \in [0,1]$, then, u^* is an efficient solution for (FMP) .*

PROOF: Assume that $u^* \in M$ is not an efficient solution for (FMP). Then, there exists $u \in M$ such that

$$\varphi(u) \leq \varphi(u^*),$$

and hence, by the definition of φ and $\xi^* = \varphi(u^*)$, there exists an index v such that

$$\begin{aligned} &f_v(u) - {\xi_v}^* q_v(u) < f_v(u^*) - {\xi_v}^* q_v(u^*), \\ &f_j(u) - {\xi_j}^* q_j(u) \le f_j(u^*) - {\xi_j}^* q_j(u^*), \forall j \ne v. \end{aligned} \tag{6}$$

Since $(f_j - \xi_j q_j), j = 1, 2, ..., p$ are strictly locally Lipschitz E-$[0,1]$ convex at u^*, then there exist $\chi_j \in \partial(f_j(u^*) - {\xi_j}^* q_j(u^*))$, such that

$$\langle u - u^*, \chi_j\rangle < [f_j(u) - \xi_j^* q_j(u)] - [f_j(u^*) - \xi_j^* q_j(u^*)],$$

and by using equation (6) and ${\rho_j}^* \ge 0, \sum_{j=1}^{p} {\rho^*}_j = 1$, we get

$$\left\langle u - u^*, \sum_{j=1}^{p} {\rho^*}_j \chi_j \right\rangle < 0. \tag{7}$$

Now, we can proceed as in Theorem 2, by adding equations (5) and (7), which leads to a contradiction with equation (3) to prove that u^* is an efficient solution for (FMP). $\square$

Theorem 4 *Let u^* be feasible for (FMP) and $\xi^* = \varphi(u^*)$. Assume that there exist ${\rho_j}^* \ge 0, \sum_{j=1}^{p} {\rho^*}_j = 1$, and ${\beta^*}_i \ge 0$ for $i \in I(u^*)$, such that condition (1) holds at u^*. If $\sum_{j=1}^{p} {\rho^*}_j [f_j - \xi_j q_j]$ is locally Lipschitz pseudo E-$[0,1]$ convex at u^* and $g_i, i \in I(u^*)$ are locally Lipschitz quasi E-$[0,1]$ convex at u^* with respect to the same $E : R \times [0,1] \to R$ such that $E(t,\varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$, then, $E(\sum_{j=1}^{p} {\rho^*}_j [f_j(u^*) - {\xi_j}^* q_j(u^*)], \varepsilon)$ is a properly nondominated solution for (FMP).*

Proof: Since $g_i, i \in I(u^*)$ are locally Lipschitz quasi E-$[0,1]$ convex at u^* and $E(t,\varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$, then there exist $\psi_i \in \partial g_i(u^*)$ such that

$$E(g_i(u), 1 - \varepsilon) \le E(g_i(u^*), \varepsilon) = 0 \Rightarrow \langle u - u^*, \psi_i\rangle \le 0,$$

and since ${\beta^*}_i \ge 0 \ for \ i \in I(u^*)$, we get

$$\left\langle u - u^*, \sum_{I(u^*)} {\beta^*}_i \psi_i \right\rangle \le 0. \tag{8}$$

By using (3), we have

$$\left\langle u - u^*, \sum_{j=1}^{p} {\rho^*}_j \chi_j \right\rangle \ge 0, \tag{9}$$

where $\chi_j \in \partial(f_j(u^*) - \xi_j{}^* q_j(u^*))$ and from locally Lipschitz pseudo E-$[0,1]$ convexity of $\sum_{j=1}^{p} \rho^*{}_j[f_j - \xi q_j]$, we get

$$E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u) - \xi_j{}^* q_j(u)], 1-\varepsilon) \geq E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon),$$

and from the definition of E-$[0,1]$, we get

$$\sum_{j=1}^{p} \rho^*{}_j[f_j(u) - \xi_j{}^* q_j(u)] \geq E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon),$$

which implies that $E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon)$ is the minimizer in the objective space of $\sum_{j=1}^{p} \rho^*{}_j[f_j(u) - \xi_j{}^* q_j(u)]$ under the constraint $g(x) \leq 0$. Hence, from Theorem (4.11) of Chankong and Haimes (2008), $E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u^*)$ $-\xi_j{}^* q_j(u^*)], \varepsilon)$ is a properly nondominated solution for (FMP). $\square$

Theorem 5 *Let u^* be feasible for (FMP) and $\xi^* = \varphi(u^*)$. Assume that there exist $\rho_j{}^* \geq 0, \sum_{j=1}^{p} \rho^*{}_j = 1$, and $\beta^*{}_i \geq 0$ for $i \in I(u^*)$, such that condition* (1) *holds at u^*. If $\sum_{j=1}^{p} \rho^*{}_j[f_j - \xi_j q_j]$ is strictly locally Lipschitz pseudo E-$[0,1]$ convex at u^* and $g_i, i \in I(u^*)$ are locally Lipschitz quasi E-$[0,1]$ convex at u^* with respect to the same $E : R \times [0,1] \to R$ such that $E(t,\varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$, then, $E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon)$ is a nondominated solution for (FMP).*

Proof: Assume that $E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon)$ is a dominated solution for (FMP), then it is not a minimizer of $\sum_{j=1}^{p} \rho^*{}_j[f_j(u) - \xi_j{}^* q_j(u)]$, i.e.

$$\sum_{j=1}^{p} \rho^*{}_j[f_j(u) - \xi_j{}^* q_j(u)] \leq E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon),$$

where $u \in M$ and, since $E(t,\varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$, we have

$$E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u) - \xi_j{}^* q_j(u)], 1-\varepsilon) \leq \sum_{j=1}^{p} \rho^*{}_j[f_j(u) - \xi_j{}^* q_j(u)].$$

But $\sum_{j=1}^{p} \rho_j^*(f_j - \xi_j{}^* q_j)$ is strictly locally Lipschitz pseudo E-$[0,1]$ convex at u^*, so that there exist $\chi_j \in \partial(f_j(u^*) - \xi_j{}^* q_j(u^*))$ such that

$$E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u) - \xi_j{}^* q_j(u)], 1-\varepsilon) \leq E(\sum_{j=1}^{p} \rho^*{}_j[f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon)$$

$$\Rightarrow \left\langle u - u^*, \sum_{j=1}^{p} \rho_j \chi_j \right\rangle < 0. \tag{10}$$

Now, from locally Lipschitz quasi E-$[0,1]$ convexity of $g_i, i \in I(u^*)$ at u^*, we conclude that there exist $\psi_i \in \partial g_i(u^*)$ such that

$$E(g_i(u), 1-\varepsilon) \leq E(g_i(u^*), \varepsilon) = 0 \Rightarrow \langle u - u^*, \psi_i \rangle \leq 0,$$

and, since $\beta^*{}_i \geq 0 \ for \ i \in I(u^*)$, we get

$$\left\langle u - u^*, \sum_{I(u^*)} \beta^*{}_i \psi_i \right\rangle \leq 0. \tag{11}$$

By adding equations (10) and (11), we obtain a contradiction with equation (3). Thus, $E(\sum_{j=1}^{p} \rho^*{}_j [f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon)$ is a nondominated solution for (FMP). □

Theorem 6 *Let u^* be feasible for (FMP) and $\xi^* = \varphi(u^*)$. Assume that there exist $\rho_j{}^* \geq 0, \sum_{j=1}^{p} \rho^*{}_j = 1$, and $\beta^*{}_i \geq 0 \ for \ i \in I(u^*)$, such that condition (1) holds at u^*. If $\sum_{j=1}^{p} \rho^*{}_j [f_j - \xi_j q_j]$ is locally Lipschitz quasi E-$[0,1]$ convex at u^* and $g_i, i \in I(u^*)$ are strictly locally Lipschitz pseudo E-$[0,1]$ convex at u^* with respect to the same $E : R \times [0,1] \to R$ such that $E(t, \varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$, then, $E(\sum_{j=1}^{p} \rho^*{}_j [f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon)$ is a nondominated solution for (FMP).*

Proof: Assume that $E(\sum_{j=1}^{p} \rho^*{}_j [f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon)$ is a dominated solution for (FMP), then it is not a minimizer of $\sum_{j=1}^{p} \rho^*{}_j [f_j(u) - \xi_j{}^* q_j(u)]$, i.e.

$$\sum_{j=1}^{p} \rho^*{}_j [f_j(u) - \xi_j{}^* q_j(u)] \leq E(\sum_{j=1}^{p} \rho^*{}_j [f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon),$$

$\forall u \in M$, and, since $E(t, \varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$, then we have

$$E(\sum_{j=1}^{p} \rho^*{}_j [f_j(u) - \xi_j{}^* q_j(u)], 1-\varepsilon) \leq \sum_{j=1}^{p} \rho^*{}_j [f_j(u) - \xi_j{}^* q_j(u)].$$

But $\sum_{j=1}^{p} \rho_j^* (f_j - \xi_j{}^* q_j)$ is locally Lipschitz quasi E-$[0,1]$ convex at u^*, so that there exist $\chi_j \in \partial(f_j(u^*) - \xi_j{}^* q_j(u^*))$ such that

$$E(\sum_{j=1}^{p} \rho^*{}_j [f_j(u) - \xi_j{}^* q_j(u)], 1-\varepsilon) \leq E(\sum_{j=1}^{p} \rho^*{}_j [f_j(u^*) - \xi_j{}^* q_j(u^*)], \varepsilon)$$

$$\Rightarrow \left\langle u - u^*, \sum_{j=1}^{p} \rho_j \chi_j \right\rangle \leq 0. \tag{12}$$

Now, from strictly locally Lipschitz pseudo E-$[0,1]$ convexity of $g_i, i \in I(u^*)$ at u^*, we conclude that there exist $\psi_i \in \partial g_i(u^*)$ such that

$$E(g_i(u), 1-\varepsilon) \leq E(g_i(u^*), \varepsilon) = 0 \Rightarrow \langle u - u^*, \psi_i \rangle < 0,$$

and, since ${\beta^*}_i \geq 0$ for $i \in I(u^*)$, we get

$$\left\langle u - u^*, \sum_{I(u^*)} {\beta^*}_i \psi_i \right\rangle < 0. \tag{13}$$

By adding equations (12) and (13), we obtain a contradiction with equation (3). Thus, $E(\sum_{j=1}^{p} {\rho^*}_j [f_j(u^*) - {\xi_j}^* q_j(u^*)], \varepsilon)$ is a nondominated solution for (FMP). $\square$

Example 4 Let *E-[0,1]* $: R \times [0,1] \to R$ be defined as $E(t,\varepsilon) = \varepsilon \sqrt[3]{t}$, $where\, t \in R$, $and\, \varepsilon \in [0,1]$. Consider the following bi-objective fractional programming:

$min\ \varphi_1(u,w) = \frac{u^3}{w}$

$min\ \varphi_2(u,w) = \frac{(w-u)^3}{w}$

$s.t.\ \ (u,w) \in M = \{(u,w) \in R^2 : u + w \leq 3,\ 1 \leq w \leq 3,\ u \geq 0\}$,

where $\varphi_j = f_j - \xi_j q_j$, $j = 1,2$ are locally Lipschitz E-$[0,1]$ convex functions on convex set M. By applying the conditions (1), we deduce that there exist $\chi_j \in \partial(f_j - \xi_j q_j)$ and $\psi_i \in \partial g_i$ such that

$$\sum_{j=1}^{2} \rho_j \chi_j + \sum_{i=1}^{4} \beta_i \psi_i = 0,$$

where $\chi_1 = (3u^2, -\xi_1), \chi_2 = (-3(w-u)^2, 3(w-u)^2 - \xi_2)$ and $\psi_1 = (1,1), \psi_2 = (0,1), \psi_3 = (0,-1), \psi_4 = (-1,0)$, i.e.

$$3\rho_1(u^*)^2 - 3\rho_2(w^* - u^*)^2 + \beta_1 - \beta_4 = 0,$$

$$-\rho_1\xi_1 + \rho_2[3(w^* - u^*)^2 - \xi_2] + \beta_1 + \beta_2 - \beta_3 = 0,$$

which gives the efficient solution $(u^*, w^*) = (2,1)$ for $\rho_1 = \rho_2 = \frac{1}{2}, \beta_1 = 0, \beta_2 = 1, \beta_3 = 2, \beta_4 = \frac{9}{2}, \xi_1 = 1, \xi_2 = 2$ and for other $\rho_1, \rho_2 \geq 0$, $\rho_1 + \rho_2 = 1$ and $\beta_i \geq 0$, $i = 1,2,3,4$, we conclude that the set of all efficient solutions is:

$$(u^*, 1) \in M : 0 \leq u^* \leq 2 \ and\ (0, w^*) \in M : 1 \leq w^* \leq 3.$$

4. Duality criteria

In this section, we consider the dual problem in the sense of Schaible (Crouzeix, Ferland and Schaible, 1985) for the here considered fractional multi-objective programming (FMP) problem, as follows:

$$\max \quad \xi = (\xi_1, \dots, \xi_p), \qquad (DP)$$

subject to

$$0 \in \sum_{j=1}^{p} \rho_j \partial(f_j(y) - \xi_j q_j(y)) + \sum_{i=1}^{m} \beta_i \partial g_i(y), \tag{14}$$

$$f_j(y) - \xi_j q_j(y) \geq 0, \quad j = 1, \dots, p, \tag{15}$$

$$\sum_{i=1}^{m} \beta_i g_i(y) \geq 0, \tag{16}$$

$$\rho_j \geq 0, \sum_{j=1}^{p} \rho_j = 1, \text{ and } \beta_i \geq 0, i = 1, \dots, m. \tag{17}$$

Firstly, let us establish the weakly duality theorems for the pair (FMP) and (DP).

Theorem 7 *Let u be feasible for (FMP) and (y, ρ, β, ξ) be feasible for (DP). If $(f_j - \xi_j q_j), j = 1, 2, ..., p$ and $\beta_i g_i$ are locally Lipschitz E-$[0,1]$ convex at y with respect to $E : R \times [0,1] \to R$ such that $E(t, \varepsilon) = \varepsilon \min\{t, \varepsilon\}, t \in R, \varepsilon \in [0,1]$, then the following cannot hold:*

$$\varphi(u) < \xi.$$

Proof: Let u be feasible for (FMP) and (y, ρ, β, ξ) be feasible for (DP), then from (14), there exist $\chi_j \in \partial(f_j(y) - \xi_j q_j(y))$ and $\psi_i \in \partial g_i(y)$ such that

$$\sum_{j=1}^{p} \rho_j \chi_j + \sum_{i=1}^{m} \beta_i \psi_i = 0. \tag{18}$$

Suppose, contrary to the result, that

$$\varphi(u) < \xi,$$

which gives, by the definition of φ and the constraint (15), that

$$f_j(u) - \xi_j q_j(u) < f_j(y) - \xi_j q_j(y), \ \forall \ j = 1, \dots, p. \tag{19}$$

Since $(f_j - \xi_j q_j), j = 1, 2, ..., p$ are locally Lipschitz E-$[0,1]$ convex at y, then

$$\langle u - y, \chi_j \rangle \leq [f_j(u) - \xi_j q_j(u)] - [f_j(y) - \xi_j q_j(y)],$$

and by using (19) and $\rho_j \geq 0, \sum_{j=1}^{p} \rho_j = 1$, we get

$$\left\langle u - y, \sum_{j=1}^{p} \rho_j \chi_j \right\rangle < 0. \tag{20}$$

Now, since u is feasible for (FMP) and (y, ρ, β, ξ) is feasible for (DP) , then $\beta_i g_i(u) \leq \beta_i g_i(y)$ and, from locally Lipschitz E-$[0,1]$ convexity of $\beta_i g_i$ at y, we obtain

$$\langle u - y, \beta_i \psi_i \rangle \leq \beta_i g_i(u) - \beta_i g_i(y) \leq 0,$$

i.e.

$$\left\langle u - y, \sum_{i=1}^{m} \beta_i \psi_i \right\rangle \leq 0. \tag{21}$$

By adding equations (20) and (21), we obtain a contradiction with equation (18). Then, $\varphi(u) < \xi$ cannot hold. □

Theorem 8 *Let u be feasible for (FMP) and (y, ρ, β, ξ) be feasible for (DP). If $(f_j - \xi_j q_j), j = 1, 2, ..., p$ are strictly locally Lipschitz E-$[0,1]$ convex at y and $\beta_i g_i$ are locally Lipschitz E-$[0,1]$ convex at y with respect to the same $E : R \times [0,1] \to R$ such that $E(t, \varepsilon) = \varepsilon \min\{t, \varepsilon\}, t \in R, \varepsilon \in [0,1]$, then the following cannot hold:*

$$\varphi_v(u) < \xi_v, \quad \varphi_j(u) \leq \xi_j, \ \forall j \neq v,$$

for some $v \in \{1, 2, ..., k\}$.

Proof: Suppose, contrary to the result, that

$$\varphi_v(u) < \xi_v, \quad \varphi_j(u) \leq \xi_j, \ \forall j \neq v,$$

for some $v \in \{1, 2, ..., k\}$, which gives, by the definition of φ and the constraint (15), that

$$\begin{aligned} &f_v(u) - \xi_v q_v(u) < f_v(y) - \xi_v q_v(y), \\ &f_j(u) - \xi_j q_j(u) \leq f_j(y) - \xi_j q_j(y), \ \forall j \neq v. \end{aligned} \tag{22}$$

Since $(f_j - \xi_j q_j), j = 1, 2, ..., p$ are strictly locally Lipschitz E-$[0,1]$ convex at y, then

$$\langle u - y, \chi_j \rangle < [f_j(u) - \xi_j q_j(u)] - [f_j(y) - \xi_j q_j(y)],$$

and by using (22) and $\rho_j \geq 0, \sum_{j=1}^{p} \rho_j = 1$, we get

$$\left\langle u - y, \sum_{j=1}^{p} \rho_j \chi_j \right\rangle < 0. \tag{23}$$

Now we can proceed as in Theorem (7) by adding equations (21) and (23) and thus obtaining a contradiction with equation (18). Hence,

$$\varphi_v(u) < \xi_v, \quad \varphi_j(u) \leq \xi_j, \ \forall j \neq v,$$

cannot hold, for some $v \in \{1, 2, ..., k\}$. □

Theorem 9 *Let u be feasible for (FMP) and (y, ρ, β, ξ) be feasible for (DP). If $\sum_{j=1}^{p} \rho_j(f_j - \xi_j q_j)$ is strictly locally Lipschitz pseudo E-$[0,1]$ convex at y and $\sum_{i=1}^{m} \beta_i g_i$ is locally Lipschitz quasi E-$[0,1]$ convex at y with respect to the same $E : R \times [0,1] \to R$ such that $E(t, \varepsilon) = \min\{t, \varepsilon\}, t \in R, \varepsilon \in [0,1]$, then the following cannot hold:*

$$\sum_{j=1}^{p} \rho_j \varphi_j(u) \leq E(\sum_{j=1}^{p} \rho_j \varphi_j(y), \varepsilon).$$

Proof: Suppose, contrary to the result, that

$$\sum_{j=1}^{p} \rho_j \varphi_j(u) \leq E(\sum_{j=1}^{p} \rho_j \varphi_j(y), \varepsilon),$$

and since $E(t, \varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$, we get

$$E(\sum_{j=1}^{p} \rho_j \varphi_j(u), 1 - \varepsilon) \leq \sum_{j=1}^{p} \rho_j \varphi_j(u) \leq E(\sum_{j=1}^{p} \rho_j \varphi_j(y), \varepsilon).$$

Since $\sum_{j=1}^{p} \rho_j(f_j - \xi_j q_j)$ is strictly locally Lipschitz pseudo E-$[0,1]$ convex at y and by the definition of φ, we conclude that there exist $\chi_j \in \partial(f_j(y) - \xi_j q_j(y))$ such that

$$E(\sum_{j=1}^{p} \rho_j [f_j(u) - \xi_j q_j(u)], 1 - \varepsilon) \leq E(\sum_{j=1}^{p} \rho_j [f_j(y) - \xi_j q_j(y)], \varepsilon)$$

$$\Rightarrow \left\langle u - y, \sum_{j=1}^{p} \rho_j \chi_j \right\rangle < 0. \tag{24}$$

Now, since u is feasible for (FMP) and (y, ρ, β, ξ) is feasible for (DP), then $\sum_{i=1}^{m} \beta_i g_i(u) \leq 0$ and $E(\sum_{i=1}^{m} \beta_i g_i(y), \varepsilon) \geq 0$, i.e.

$$E(\sum_{i=1}^{m} \beta_i g_i(u), 1-\varepsilon) = \sum_{i=1}^{m} \beta_i g_i(u) \leq E(\sum_{i=1}^{m} \beta_i g_i(y), \varepsilon),$$

and from the locally Lipschitz quasi E-$[0,1]$ convexity of $\sum_{i=1}^{m} \beta_i g_i$ at y, we obtain

$$\left\langle u - y, \sum_{i=1}^{m} \beta_i \psi_i \right\rangle \leq 0. \tag{25}$$

Upon adding equations (24) and (25), we arrive at a contradiction with equation (18). Then,

$$\sum_{j=1}^{p} \rho_j \varphi_j(u) \leq E(\sum_{j=1}^{p} \rho_j \varphi_j(y), \varepsilon).$$

cannot hold. □

Theorem 10 *Let u be feasible for (FMP) and (y, ρ, β, ξ) be feasible for (DP). If $\sum_{j=1}^{p} \rho_j (f_j - \xi_j q_j)$ is locally Lipschitz quasi E-$[0,1]$ convex at y and $\sum_{i=1}^{m} \beta_i g_i$ is strictly locally Lipschitz pseudo E-$[0,1]$ convex at y with respect to the same $E : R \times [0,1] \to R$ such that $E(t, \varepsilon) = \min\{t, \varepsilon\}, t \in R, \varepsilon \in [0,1]$, then the following cannot hold:*

$$\sum_{j=1}^{p} \rho_j \varphi_j(u) \leq E(\sum_{j=1}^{p} \rho_j \varphi_j(y), \varepsilon).$$

Proof: Suppose, contrary to the result, that

$$\sum_{j=1}^{p} \rho_j \varphi_j(u) \leq E(\sum_{j=1}^{p} \rho_j \varphi_j(y), \varepsilon),$$

and, since $E(t, \varepsilon) = \min\{\varepsilon, t\}, t \in R, \varepsilon \in [0,1]$, we get

$$E(\sum_{j=1}^{p} \rho_j \varphi_j(u), 1-\varepsilon) \leq \sum_{j=1}^{p} \rho_j \varphi_j(u) \leq E(\sum_{j=1}^{p} \rho_j \varphi_j(y), \varepsilon).$$

Since $\sum_{j=1}^{p} \rho_j (f_j - \xi_j q_j)$ is locally Lipschitz quasi E-$[0,1]$ convex at y and by the definition of φ, we conclude that there exist $\chi_j \in \partial(f_j(y) - \xi_j q_j(y))$ such that

$$E(\sum_{j=1}^{p} \rho_j [f_j(u) - \xi_j q_j(u)], 1-\varepsilon) \leq E(\sum_{j=1}^{p} \rho_j [f_j(y) - \xi_j q_j(y)], \varepsilon)$$

$$\Rightarrow \left\langle u - y, \sum_{j=1}^{p} \rho_j \chi_j \right\rangle \leq 0. \tag{26}$$

Now, since u is feasible for (FMP) and (y, ρ, β, ξ) is feasible for (DP), then $\sum_{i=1}^{m} \beta_i g_i(u) \leq 0$ and $E(\sum_{i=1}^{m} \beta_i g_i(y), \varepsilon) \geq 0$, i.e.

$$E(\sum_{i=1}^{m} \beta_i g_i(u), 1 - \varepsilon) = \sum_{i=1}^{m} \beta_i g_i(u) \leq E(\sum_{i=1}^{m} \beta_i g_i(y), \varepsilon),$$

and from strictly locally Lipschitz pseudo E-$[0,1]$ convexity of $\sum_{i=1}^{m} \beta_i g_i$ at y, we get

$$\left\langle u - y, \sum_{i=1}^{m} \beta_i \psi_i \right\rangle < 0. \tag{27}$$

Upon adding equations (26) and (27), we arrive at a contradiction with equation (18). Then,

$$\sum_{j=1}^{p} \rho_j \varphi_j(u) \leq E(\sum_{j=1}^{p} \rho_j \varphi_j(y), \varepsilon).$$

cannot hold. □

Secondly, let us establish strong duality theorems for (FMP) and (DP).

Theorem 11 *Let u^* be a weakly efficient solution for (FMP),at which the generalized Slater constraint is satisfied at u^*. If the locally Lipschitz E-$[0,1]$ convexity assumptions of Theorem (7) are satisfied, then there exists (ρ^*, β^*) such that $(u^*, \rho^*, \beta^*, \xi^*)$ is a weakly efficient solution for (DP) and the objective values are equal.*

Proof: Since u^* is a weakly efficient solution for (FMP) with $\varphi(u^*) = \xi^*$ and the generalized Slater constraint is satisfied at u^*, then there exists (ρ^*, β^*) such that $(u^*, \rho^*, \beta^*, \xi^*)$ is feasible for (DP).

From Theorem 7, $\varphi(u^*) < \xi$ cannot hold and since $\varphi(u^*) = \xi^*$, then, $(u^*, \rho^*, \beta^*, \xi^*)$ is a weakly efficient solution for (DP) and the objective values of (FMP) and (DP) are equal at u^*. □

Theorem 12 *Let u^* be an efficient solution for (FMP), at which the generalized Slater constraint is satisfied at u^*. If the strictly locally Lipschitz E-$[0,1]$ convexity assumptions of Theorem 8 are satisfied, then there exists (ρ^*, β^*) such that $(u^*, \rho^*, \beta^*, \xi^*)$ is an efficient solution for (DP) and the objective values are equal.*

Proof: The proof is similar to the proof of Theorem 11 except for the fact that we use the result of Theorem 8 instead of Theorem 7.

Theorem 13 *Let $E(\sum_{j=1}^{p} \rho^{*}{}_{j}\varphi_j(u^*), \varepsilon^*)$ be a nondominated solution for (FMP), at which the generalized Slater constraint is satisfied at u^*. If the assumptions of Theorem 9 are satisfied, then there exists (ρ^*, β^*, ξ^*) such that $E(\sum_{j=1}^{p} \rho^{*}{}_{j}\xi_j^*, \varepsilon^*)$ is a nondominated solution for (DP).*

Proof: Since $E(\sum_{j=1}^{p} \rho^{*}{}_{j}\varphi_j(u^*), \varepsilon^*)$ is a nondominated solution for (FMP) with a $\varphi(u^*) = \xi^*$ and the generalized Slater constraint is satisfied at u^*, then there exists (ρ^*, β^*) such that $(u^*, \rho^*, \beta^*, \xi^*)$ is feasible for (DP). From Theorem 9, we conclude that the following cannot hold:

$$\sum_{j=1}^{p} \rho_j\varphi_j(u) \leq E(\sum_{j=1}^{p} \rho_j\varphi_j(u^*), \varepsilon^*).$$

Since $\varphi(u^*) = \xi^*$, then, $E(\sum_{j=1}^{p} \rho^{*}{}_{j}\xi_j^*, \varepsilon^*)$ is a nondominated solution for (DP). □

Theorem 14 *Let $E(\sum_{j=1}^{p} \rho^{*}{}_{j}\varphi_j(u^*), \varepsilon^*)$ be a nondominated solution for (FMP), at which the generalized Slater constraint is satisfied at u^*. If the assumptions of Theorem 10 are satisfied, then there exists (ρ^*, β^*, ξ^*) such that $E(\sum_{j=1}^{p} \rho^{*}{}_{j}\xi_j^*, \varepsilon^*)$ is nondominated solution for (DP).*

Proof: The proof is similar to the proof of Theorem 13 except for the fact that we use the result of Theorem 10 instead of Theorem 9.
Finally, let us establish strict converse duality theorem for (FMP) and (DP).

Theorem 15 *Let u^* be a weakly efficient solution for (FMP) at which the generalized Slater constraint is satisfied at u^*. Let $(f_j - \xi_j q_j), j = 1, 2, ..., p$ and $\beta_i g_i$ be locally Lipschitz E-$[0,1]$ convex at u^* with respect to $E : R \times [0,1] \to R$ such that $E(t, \varepsilon) = \varepsilon \min\{t, \varepsilon\}, t \in R, \varepsilon \in [0,1]$. If $(\tilde{u}, \rho, \beta, \xi)$ is a weakly efficient solution for (DP) and $(f_j - \xi_j q_j), j = 1, 2, ..., p$ are strictly locally Lipschitz E-$[0,1]$ convex at $\tilde{u}$, then $\tilde{u} = u^*$.*

Proof: Assume that $\tilde{u} \neq u^*$, and $(\tilde{u}, \rho, \beta, \xi)$ are feasible for (DP), then from (14), there exist $\chi_j \in \partial(f_j(\tilde{u}) - \xi_j q_j(\tilde{u}))$ and $\psi_i \in \partial g_i(\tilde{u})$ such that

$$\sum_{j=1}^{p} \rho_j\chi_j + \sum_{i=1}^{m} \beta_i\psi_i = 0. \tag{28}$$

The constraint (15) gives

$$f_j(u^*) - \xi_j q_j(u^*) \leq f_j(\tilde{u}) - \xi_j q_j(\tilde{u}), \ \forall \ \ j = 1, \ldots, p, \tag{29}$$

Since $(f_j - \xi_j q_j), j = 1, 2, ..., p$ are strictly locally Lipschitz E-$[0, 1]$ convex at $\tilde{u}$, then

$$\langle u^* - \tilde{u}, \chi_j \rangle \leq [f_j(u^*) - \xi_j q_j(u^*)] - [f_j(\tilde{u}) - \xi_j q_j(\tilde{u})],$$

and by using (29) and $\rho_j \geq 0, \sum_{j=1}^{p} \rho_j = 1$, we get

$$\left\langle u^* - \tilde{u}, \sum_{j=1}^{p} \rho_j \chi_j \right\rangle < 0. \tag{30}$$

Now, since u^* is a weakly efficient solution for (FMP) , $(\tilde{u}, \rho, \beta, \xi)$ is a weakly efficient solution for (DP) and then $\beta_i g_i(u^*) \leq \beta_i g_i(\tilde{u})$ and, from locally Lipschitz E-$[0, 1]$ convexity of $\beta_i g_i$ at $\tilde{u}$, we deduce that

$$\langle u^* - \tilde{u}, \beta_i \psi_i \rangle \leq \beta_i g_i(u^*) - \beta_i g_i(\tilde{u}) \leq 0,$$

i.e.

$$\left\langle u^* - \tilde{u}, \sum_{i=1}^{m} \beta_i \psi_i \right\rangle \leq 0. \tag{31}$$

Adding equations (30) and (31) leads to a contradiction with equation (28). Thus, $\tilde{u} = u^*$. $\square$

5. Conclusion and future work

In this paper, we establish necessary and sufficient conditions for nondifferentiable multi-objective fractional programming problem, which involves locally Lipschitz E-$[0, 1]$ convex functions. Schaible dual problem for the considered multi-objective fractional problem is formulated. Finally, we establish weakly, strong, and strict-converse duality theorems under locally Lipschitz E-$[0, 1]$ convexity hypotheses. There are many open points in this area, which should be studied in the future, for example

1. To discuss the potential applications in various fields, such as economics, engineering, and data science, which impliy E-$[0, 1]$ convex multi-objective programming.

2. To construct applications that appropriately capture the generality and abstract nature of the theoretical results of E-$[0, 1]$ convex multi-objective programming.

3. To perform a parametric study for the new kind of generalized E-$[0,1]$ convex functions in multi-objective programming.

4. To find the numerical methods for solving multi-objective programming involving E-$[0,1]$ convex functions.

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