**Control and Cybernetics**

vol. **47** (2018) No. 3

# Post-optimal analysis for multicriteria integer linear programming problem of finding extreme solutions*

by

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**Abstract:** We consider a multicriteria problem of integer linear programming and study the set of all individual criterion minimizers (extreme solutions) playing an important role in determining the range of Pareto optimal set. In this work, the lower and upper attainable bounds on the stability radius of the set of extreme solutions are obtained in the situation where solution and criterion spaces are endowed with various Hölder's norms. In addition, the case of the Boolean problem is analyzed. Some computational challenges are also discussed.

**Keywords:** sensitivity analysis; multiple criteria; extreme solutions; stability radius; integer linear programming

## 1. Introduction

Multiobjective discrete models have been widely used in decision making, design, management, economics, and many other applied fields. Therefore, the interest of mathematicians regarding multicriteria (vector) discrete optimization problems is far from being lost, which is confirmed by numerous recent publications. One of directions in investigating these problems is the analysis of stability of solutions with respect to perturbations of the initial data (problem parameters). Various notions of stability generate numerous investigation lines.

The terms, such as sensitivity, stability or post-optimal analysis are commonly used for the phase of an algorithm at which a solution (or solutions) of the problem has been already found, and additional calculations are performed in order to investigate how this solution depends on changes in the problem data.

*Submitted: January 2019; Accepted: March 2019

In 1923, Jacques Hadamard recognized the stability problem as one of the central problems in mathematical research. He postulated that in order to be well-posed, a mathematical problem should satisfy three properties: existence of a solution; uniqueness of the solution; and continuous dependence of the solution on the data (Hadamard, 1923). Problems that are not well-posed in the sense of Hadamard are usually termed ill-posed.

Despite the existence of numerous approaches to stability analysis of optimization problems, two major directions can be pointed out: quantitative and qualitative.

Qualitative sensitivity analysis is usually conducted for multicriteria optimization problems with various (linear and nonlinear) criteria. The typical results are necessary and sufficient conditions for different types of stability of one or a set of optimal solutions (see, e.g., Sergienko and Shilo, 2003; Lebedeva and Sergienko, 2008; Lebedeva, Semenova and Sergienko, 2014a,b; Emelichov et al., 2014; Kuzmin, Nikulin and Mäkelä, 2017; Emelichev, Karelkina and Kuzmin, 2012).

Within the scope of the quantitative direction, various measures of stability are investigated. Analytical expressions or (attainable) lower and upper bounds on a quantitative characteristic called stability radius constitute typical results in this area. The results are formulated in the case where parameter space is equipped with various metrics (see, e.g., Leontev, 2007; Gordeev, 2015; Emelichev and Podkopaev, 1998, 2001, 2010; Emelichev et al., 2002; Emelichev and Kuzmin, 2010; Bukhtoyarov and Emelichev, 2015; Emelichev and Nikulin, 2018). In addition to stability radius, some papers are focusing on more general characteristics of stability, for example stability and accuracy functions are analyzed in Libura and Nikulin (2006) and in Nikulin (2009). Sensitivity analysis has been also performed for some problems of scheduling theory, see, e.g., Sotskov et al. (2010) and Nikulin (2014).

This publication follows the ideas of quantitative analysis. It continues a series of publications (Emelichev et a., 2014; Emelichev and Podkopaev, 1998, 2001; Emelichev and Kuzmin, 2007, 2013; Emelichev, Krichko and Nikulin, 2004) seeking the analytical bounds on stability radius for the multicriteria problem of Integer Linear Programming (ILP) with various optimality principles.

In multicriteria optimization and decision making, we deal sometimes with choice functions different from the well-known Pareto optimality principle (Pareto, 1909). Such functions play a crucial role in many real life applications (see, e.g., Podinovskii and Noghin, 1982, and Lotov and Pospelov, 2008). In this paper, we consider the multicriteria problem of ILP with the extreme optimality principle, i.e. with the set of solutions being individual optimizers of all criteria.

This set is used to construct the payoff table, often serving for calculating the ideal point and estimating the nadir point of the Pareto optimal set (see, e.g., Steuer, 1986; Miettinen, 1999; Noghin, 2018; Ehrgott, 2005). We study the type of stability with respect to independent perturbations of linear function coefficients that is a discrete analogue of Hausdorff upper semi-continuity mapping, transforming any set of problem parameters into a set of extreme solutions. In other words, this type of stability guarantees the existence of a neighborhood in problem parameter space such that no new extreme solutions appear, see Emelichev and Podkopaev (1998, 2001, 2010) and Emelichev et al. (2002).

As a result of the parametric analysis performed, the lower and upper bounds on the stability radius are obtained for multicriteria ILP problem with extreme solutions in the case where criterion space is endowed with various Hölder's norms. Attainability of the estimates (both lower and uppers bounds) is demonstrated.

## 2. Problem formulation and basic definitions

We consider an $m$-criteria ILP problem in the following formulation. Let $C = [c_{ij}] \in \mathbf{R}^{m\times n}$ be a real valued $m \times n$ - matrix with rows $C_i \in \mathbf{R}^n,\ i \in N_m = \{1, 2, \dots, m\},\ m \geq 1$. Let also $X \subset \mathbf{Z}^n,\ 1 < |X| < \infty$, be the set of feasible solutions $x = (x_1, x_2, \dots, x_n)^T$, $n \geq 2$. We define a vector criterion

$$Cx = \big(C_1x, C_2x, \dots, C_mx\big)^T \to \min_{x\in X},$$

with linear objective functions.

In this paper, $Z^m(C),\ C \in \mathbf{R}^{m\times n}$, is the problem of finding the set of extreme solutions defined in, e.g., Miettinen (1999) and Branke et al. (2007):

$$E^m(C) = \Big\{x \in X:\ \exists k \in N_m\ \ \forall x' \in X\ \ \big(C_k(x) \leq C_k(x')\big)\Big\}.$$

This set can equivalently be written as follows:

$$E^m(C) = \{x \in X:\ \exists k \in N_m\ \ (E^m_k(x, C_k) = \emptyset)\},$$

where

$$E^m_i(x, C_i) = \Big\{x' \in X:\ C_i(x - x') > 0\Big\},\ i \in N_m,\ x \in X.$$

Thus, the choice of extreme solutions can be interpreted as finding best solutions for each of $m$ criteria, and then combining them into one set. The vector composed of optimal objective values constitutes the ideal vector that is of great importance in theory and methodology of multiobjective optimization (Miettinen, 1999). This also justifies our particular interest in studying some properties of the extreme solutions. Obviously, $E^1(C),\ C \in \mathbf{R}^n$ is the set of optimal solutions for the scalar problem $Z^1(C)$.

We will perturb the elements of matrix $C \in \mathbf{R}^{m\times n}$ by adding elements of the perturbing matrix $C' \in \mathbf{R}^{m\times n}$. Thus, the perturbed problem $Z^m(C + C')$ of finding extreme solutions has the following form:

$$(C + C')x \to \min_{x\in X}.$$

The set of extreme solutions of the perturbed problem is denoted by $E^m(C+C')$. In the solution space $\mathbf{R}^n$, we define an arbitrary Hölder's norm $l_p,\ p \in [1,\infty]$, i.e. the norm of vector $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ is defined as

$$\|a\|_p = \begin{cases} \Big(\sum\limits_{j\in N_n} |a_j|^p\Big)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|a_j| :\ j \in N_n\} & \text{if } p = \infty. \end{cases}$$

In the criteria space $\mathbf{R}^m$, we define another Hölder's norm $l_q,\ q \in [1,\infty]$. The norm of matrix $C \in \mathbf{R}^{m\times n}$ is defined as

$$\|C\|_{pq} = \|(\|C_1\|_p, \|C_2\|_p, \dots, \|C_m\|_p)\|_q.$$

It is easy to see that

$$\|C_i\|_p \le \|C\|_{pq},\ i \in N_m. \tag{1}$$

It is well known that the $l_p$ norm defined in $\mathbf{R}^n$ induces conjugated $l_{p^*}$ norm in $(\mathbf{R}^n)^*$. For $p$ and $p^*$, the following relations hold:

$$\frac{1}{p} + \frac{1}{p^*} = 1, \qquad 1 < p < \infty. \tag{2}$$

In addition, if $p = 1$ then $p^* = \infty$, and, if $p^* = 1$ then $p = \infty$. Notice that $p$ and $p^*$ belong to the same range $[1,\infty]$. We set $\frac{1}{p} = 0$ if $p = \infty$.

It is easy to see that for any vector $\xi = (\xi_1, \xi_2, ..., \xi_n)^T \in \mathbf{R}^n$ with $|\xi_j| = \sigma$, $j \in N_n$, for any $p \in [1,\infty]$ the following equality holds:

$$\|\xi\|_p = n^{\frac{1}{p}}\sigma. \tag{3}$$

For any two real-valued vectors $a$ and $b$ of the same dimension $n$, the following Hölder's inequality is well known:

$$|a^T b| \le \|a\|_p \|b\|_{p^*}, \tag{4}$$

where $p \in [1,\infty]$.

It is also well known (see, e.g., Hardy, Littlewood and Polya, 1988) that Hölder's inequality becomes an equality for $1 < p < \infty$ if and only if

a) one of $a$ or $b$ is the zero vector;
b) the two vectors obtained from non-zero vectors $a$ and $b$ by raising their components' absolute values to the powers of $p$ and $p^*$, respectively, are linearly dependent (proportional), and sign $(a_i b_i)$ is independent of $i$.

When $p = 1$, (4) transforms into the following inequality:

$$|\sum_{i\in N_n} a_i b_i| \le \max_{i\in N_n} |b_i| \sum_{i\in N_n} |a_i|.$$

The last holds as equality if, for example, $b$ is the zero vector or if $a_j \neq 0$ for some $j$ such that $|b_j| = \|b\|_\infty \neq 0$, and $a_i = 0$ for all $i \in N_n \backslash \{j\}$.

When $p = \infty$, (4) transforms into the following inequality:

$$|\sum_{i \in N_n} a_i b_i| \leq \max_{i \in N_n} |a_i| \sum_{i \in N_n} |b_i|.$$

The last holds as equality if, for example, $b$ is the zero vector or if $a_i = \sigma \operatorname{sign}(b_i)$ for all $i \in N_n$ and $\sigma \geq 0$.

From here we deduce that the following formula is valid for $p \in [1, \infty]$:

$$\forall b \in \mathbf{R}^n \quad \forall \sigma > 0 \quad \exists a \in \mathbf{R}^n \quad \Big(|a^T b| = \sigma \|b\|_{p^*} \ \& \ \|a\|_p = \sigma\Big). \tag{5}$$

Given $\varepsilon > 0$, let

$$\Omega_{pq}(\varepsilon) = \Big\{C' \in \mathbf{R}^{m \times n} : \quad \|C'\|_{pq} < \varepsilon\Big\}$$

be the set of perturbing matrices $C' = [c'_{ij}] \in \mathbf{R}^{m \times n}$ with rows $C'_k \in \mathbf{R}^n, \ k \in N_m$.

Denote

$$\Xi_{pq} = \Big\{\varepsilon > 0 : \quad \forall C' \in \Omega_{pq}(\varepsilon) \quad \Big(E^m(C + C') \subseteq E^m(C)\Big)\Big\}.$$

Following Emelichev and Podkopaev (1998, 2001) and Emelichev et al. (2002), the number

$$\rho^m(p,q) = \begin{cases} \sup \ \Xi_{pq} & \text{if } \ \Xi_{pq} \neq \emptyset, \\ 0 & \text{if } \ \Xi_{pq} = \emptyset \end{cases}$$

is called *stability radius* ($T_3$-stability radius in terminology of Sergienko and Shilo, 2003; Lebedeva and Sergienko, 2008; and Emelichev et al., 2014) of problem $Z^m(C)$, $m \in \mathbf{N}$, with Hölder's norms $l_p$ and $l_q$ in the spaces $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Thus, the stability radius of problem $Z^m(C)$ defines the extreme level of perturbations of the elements of matrix $C$ in the metric space $\mathbf{R}^{m \times n}$ such that no new extreme solutions appear in the perturbed problem. The problem $Z^m(C)$ is called stable if and only if the stability radius is positive ($\rho^m(p,q) > 0$).

If $E^m(C) = X$, then the inclusion $E^m(C + C') \subseteq E^m(C)$ holds for any perturbing matrix $C'$. Therefore, the stability radius of such a problem is not bounded from above. The problem $Z^m(C)$ with $E^m(C) \neq X$ is referred to as *non-trivial.*

## 3. Bounds on the stability radius

Given the multicriteria ILP problem $Z^m(C)$, $m \in \mathbf{N}$, for any $p \in [1, \infty]$ we set

$$\phi^m(p) = \min_{i \in N_m} \ \min_{x \notin E^m(C)} \ \max_{x' \in X \backslash \{x\}} \frac{C_i(x - x')}{\|x - x'\|_{p^*}},$$

$$\eta^m(p) = \min\{\|C_i\|_p : \ i \in N_m\}.$$

Theorem 1 *Given $p, q \in [1, \infty]$ and $m \in \mathbf{N}$, for the stability radius $\rho^m(p, q)$ of the non-trivial multicriteria ILP problem $Z^m(C)$, the following lower and upper bounds are valid:*

$$0 < \phi^m(p) \le \rho^m(p, q) \le \eta^m(p).$$

*Moreover,*

$$0 < \phi^m(p) \le \rho^m(p, q) \le \min\left\{n^{\frac{1}{p}}\phi^m(\infty), \eta^m(p)\right\}$$

*if the problem is Boolean.*

Proof According to the definition of $E^m(C)$, we have

$$\forall x \notin E^m(C) \ \ \forall i \in N_m \ \ \exists x^0 \in X \ \ \Big(C_i x > C_i x^0\Big),$$

and hence $\phi^m(p) > 0$. Now we prove that

$$\rho^m(p, q) \ge \phi^m(p). \tag{6}$$

Let $C' \in \mathbf{R}^{m \times n}$ be an arbitrary perturbing matrix, and norm

$$\|C'\|_{pq} < \phi^m(p),$$

i.e. $C' \in \Omega_{pq}(\phi^m(p))$. Then, according to the definition of number $\phi^m(p)$ and due to (1), the following statement holds:

$$\forall i \in N_m \ \ \forall x \notin E^m(C) \ \ \exists x^0 \in X \backslash \{x\}$$

$$\Big(\frac{C_i(x - x^0)}{\|x - x^0\|_{p^*}} \ge \phi^m(p) > \|C'\|_{pq} \ge \|C'_i\|_p\Big).$$

Taking into account Hölder's inequalities (4), we deduce that for any index $i \in N_m$ there exists $x^0 \neq x$ such that

$$(C_i + C'_i)(x - x^0) = C_i(x - x^0) + C'_i(x - x^0) \ge$$

$$C_i(x - x^0) - \|C'_i\|_p \|x - x^0\|_{p^*} > 0,$$

i.e. $x \notin E^m(C + C')$ for any $x \notin E^m(C)$.

Hence, the inclusion $E^m(C + C') \subseteq E^m(C)$ holds for any perturbed matrix $C' \in \Omega_{pq}(\phi^m(p))$, so that equation (6) is true.

Further, we prove that $\rho^m(p, q) \le \eta^m(p)$. In order to do that, it suffices to show that $\rho^m(p, q) \le \|C_k\|_p$ for any $k \in N_m$. Let us fix $k \in N_m$ and let matrix $C^0 = [c_{ij}] \in \mathbf{R}^{m \times n}$ with rows $C^0_i \in \mathbf{R}^n$, $i \in N_m$ be constructed as follows:

$$C^0_i = \begin{cases} -C_i & \text{if } i = k, \\ \mathbf{0}^T & \text{if } i \in N_m \backslash \{k\}, \end{cases}$$

where $\mathbf{0}$ is the vector column in $\mathbf{R}^n$, containing all zeroes. Then we get

$$\|C^0\|_{pq} = \|C_k^0\|_p = \|C_k\|_p,$$

$$E^m(C + C^0) = X.$$

Taking into account $X \not\subseteq E^m(C)$, we conclude that $\rho^m(p,q) \leq \|C_k\|_p$. Hence, $\rho^m(p,q) \leq \eta^m(p) = \min\{\|C_i\|_p : \ i \in N_m\}$.

We then consider the case where $X \subseteq \{0,1\}^n$. All the bounds proven earlier remain valid. All we need to show is that an extra upper bound holds:

$$\rho^m(p,q) \leq n^{\frac{1}{p}} \phi^m(\infty). \tag{7}$$

Indeed, according to the definition of $\phi = \phi^m(\infty)$, there exist a solution $x^0 = (x_1^0, x_2^0, ..., x_n^0)^T \notin E^m(C)$ and an index $k \in N_m$ such that for any solution $x \neq x^0$ the following inequality holds

$$\phi\|x - x^0\|_1 \geq C_k(x^0 - x). \tag{8}$$

Set $\varepsilon > n^{\frac{1}{p}} \phi$, choose $\delta$ such that

$$\phi < \delta < \frac{\varepsilon}{n^{\frac{1}{p}}},$$

and consider the row vector $\xi = (\xi_1, \xi_2, ..., \xi_n)$ with coordinates

$$\xi_j = \begin{cases} -\delta & \text{if } x_j^0 = 1, \\ \delta & \text{if } x_j^0 = 0. \end{cases}$$

Then, according to (3), we get

$$\|\xi\|_p = n^{\frac{1}{p}} \delta.$$

Further, we define a perturbing matrix $C^0 = [c_{ij}] \in \mathbf{R}^{m \times n}$ with rows $C_i^0 \in \mathbf{R}^n$, $i \in N_m$, constructed as follows:

$$C_i^0 = \begin{cases} \xi & \text{if } i = k, \\ \mathbf{0}^T & \text{if } i \in N_m \backslash \{k\}. \end{cases}$$

Then we have

$$\|C^0\|_{pq} = n^{\frac{1}{p}} \phi,$$

$$C^0 \in \Omega_{pq}(\varepsilon).$$

In addition, for any $x \neq x^0$ we have

$$C_k^0(x^0 - x) = -\delta\|x^0 - x\|_1.$$

From the above, using inequality (8), we deduce for any $x \in X \backslash \{x^0\}$:

$$(C_k + C_k^0)(x^0 - x) = C_k(x^0 - x) + C_k^0(x^0 - x) \leq (\phi - \delta)\|x^0 - x\|_1 < 0.$$

This implies that $x^0 \in E^m(C + C^0)$ for $x^0 \notin E^m(C)$. Summing up, we have

$$\forall \varepsilon > n^{\frac{1}{p}} \phi^m(\infty) \;\; \exists C^0 \in \Omega_{pq}(\varepsilon) \;\; \Big( E^m(C + C^0) \not\subseteq E^m(C) \Big),$$

i.e. $\rho^m(p, q) < \varepsilon$ for any number $\varepsilon > n^{\frac{1}{p}} \phi^m(\infty)$. Therefore, inequality (7) is true. □

## 4. Bound attainability

The following corollaries indicate the lower bound attainability $\phi^m(p)$ for the stability radius $\rho^m(p, q)$ of non-trivial ILP problem $Z^m(C)$.

Corollary 1 *Let $m \in \mathbf{N}$. If for a non-trivial multicriteria ILP problem $Z^m(C)$ we have $E^m(C) = \{x^0\}$, then the stability radius $\rho^m(p, q)$ is expressed by the following formula:*

$$\rho^m(p, q) = \min_{i \in N_m} \max_{x \in X \backslash \{x^0\}} \frac{C_i(x - x^0)}{\|x - x^0\|_{p^*}}. \tag{9}$$

Proof Let $\Theta$ denote the right-hand side of (9). According to the definition of $\Theta$, there exist $\hat{x} \in X \backslash \{x^0\}$ and $k \in N_m$ such that the following equality holds:

$$C_k(\hat{x} - x^0) = \Theta \|\hat{x} - x^0\|_{p^*}. \tag{10}$$

Notice that here $\Theta > 0$. Set $\varepsilon > \Theta$ and a number $\gamma$, satisfying

$$\Theta < \gamma < \varepsilon.$$

According to formula (5), there exists a vector $a \in \mathbf{R}^n$ such that

$$a^T(\hat{x} - x^0) = -\gamma \|\hat{x} - x^0\|_{p^*},$$

$$\|a\|_p = \gamma.$$

Further, we define a perturbing matrix $C^0 = [c_{ij}] \in \mathbf{R}^{m \times n}$ with rows $C_i^0 \in \mathbf{R}^n$, $i \in N_m$, constructed as follows:

$$C_i^0 = \begin{cases} a^T & \text{if } i = k, \\ \mathbf{0}^T & \text{if } i \in N_m \backslash \{k\}. \end{cases}$$

Then we have

$$\|C^0\|_{pq} = \gamma,$$

$$C^0 \in \Omega_{pq}(\varepsilon),$$

$$C_k^0(\hat{x} - x^0) = -\gamma\|\hat{x} - x^0\|_{p^*}.$$

From the above, using inequality (10), we deduce

$$(C_k + C_k^0)(\hat{x} - x^0) = C_k(\hat{x} - x^0) - \gamma\|\hat{x} - x^0)\|_{p^*} = (\Theta - \gamma)\|\hat{x} - x^0\|_{p^*} < 0.$$

This implies that $x^0 \notin E_k^m(\hat{x}, C_k + C_k^0)$. If $E_k^m(\hat{x}, C_k + C_k^0) = \emptyset$, then $\hat{x} \in E^m(C + C^0)$. If $E_k^m(\hat{x}, C_k + C_k^0) \neq \emptyset$, then there exists $\tilde{x} \in E_k^m(\hat{x}, C_k + C_k^0)$ such that $\tilde{x} \in E^m(C + C^0)$ and $\tilde{x} \neq x^0$.

Summing up, we have that for any $\varepsilon > \Theta$ there exists a perturbing matrix $C^0 \in \Omega_{pq}(\varepsilon)$ such that one can specify $x' \in X\backslash\{x^0\}$ satisfying the condition $x' \in E^m(C+C^0)$. This implies that $E^m(C+C^0) \not\subseteq E^m(C)$. Hence $\rho^m(p,q) < \varepsilon$ for any number $\varepsilon > \Theta$, i.e. $\rho^m(p,q) \leq \Theta$.

Taking into account the lower bound $\rho^m(p,q) \geq \Theta$, proven earlier in Theorem 1, we get formula (9). □

In the case of a Boolean non-trivial problem, the following corollary results from Theorem 1 and indicates the lower bound attainability for the stability radius $\rho^m(\infty, q)$.

Corollary 2 *Given $m \in \mathbf{N}$ and $q \in [1,\infty)$, the stability radius $\rho^m(\infty,q)$ of a non-trivial multicriteria Boolean problem $Z^m(C)$ is expressed by the following formula:*

$$\rho^m(\infty,q) = \phi^m(\infty) = \min_{i \in N_m} \min_{x \notin E^m(C)} \max_{x' \in X\backslash\{x\}} \frac{C_i(x - x')}{\|x - x'\|_1}. \tag{11}$$

Further, we show that for any number $p \in [1,\infty]$, the upper bound $n^{\frac{1}{p}}\phi^m(\infty)$ for the stability radius of the Boolean problem is attainable when $m = 1$.

Theorem 2 *Given $p, q \in [1,\infty]$, there exists a class of scalar Boolean problems $Z^1(C)$, $C \in \mathbf{R}^n$, such that the stability radius $\rho^1(p,q)$ of any problem belonging to the class is expressed by the following formula:*

$$\rho^1(p,q) = n^{\frac{1}{p}}\phi^1(\infty). \tag{12}$$

Proof Due to Theorem 1, in order to prove (12) it suffices to find a class of problems satisfying $\rho^1(p,q) \geq n^{\frac{1}{p}}\phi^1(\infty)$. Let $X = \{x^0, x^1, ..., x^n\} \in \mathbf{E}^n$, where $x^0 = (0,0,...,0)^T \in \mathbf{R}^n$, $x^i = e^j$, $j \in N_n$. Here $e^j$ is the $j$-th column of the $n \times n$ basis matrix (basic column vector). We set $C = (-a,-a,...,-a) \in \mathbf{R}^n$, $a > 0$. Then

$$E^1(C) = X\backslash\{x^0\},$$

$$\phi^1(\infty) = a.$$

Let $C' = (c'_1, c'_2, ..., c'_n)$ be an arbitrary perturbing row vector belonging to $\Omega_{pq}(n^{\frac{1}{p}}a)$. Reasoning by contradiction, it is easy to see that there exists at least one index $k \in N_m$ such that $|c'_k| < a$. Therefore, we get

$$(C + C')(x^0 - x^k) = a - c'_k > 0,$$

i.e. $x^0 \notin E^1(C + C')$ for any perturbing row $C' \in \Omega_{pq}(n^{\frac{1}{p}}\phi^1(\infty))$. Hence, due to $x^* \notin E^1(C)$, we get $\rho^1(p, q) \geq n^{\frac{1}{p}}\phi^1(\infty)$. □

The numerical example, given below, shows that all three bounds for the stability radius of a non-trivial Boolean problem can also be attainable in the single criterion case.

Example 1 *Let* $X = \{x^0, x^1\} \subset \mathbf{E}^n$ *where* $x^0 = (0, 0, ..., 0)^T$, $x^1 = (1, 1, ..., 1)^T$, *and* $C = (1, 1, ..., 1)$. *Then, we have*

$$Cx^0 = 0,\ Cx^1 = n,$$

$$E^1(C) = \{x^0\},\ X \backslash E^1(C) = \{x^1\},$$

$$\rho^1(p, q) \leq \|C\|_p.$$

*Moreover, by taking into account (2) and (3), we obtain the equalities*

$$\phi^1(p) = n^{\frac{1}{p}} = \|C\|_p.$$

*Then, according to Theorem 1,*

$$\rho^1(p, q) = \|C\|_p,\ p, q \in [1, \infty].$$

*In addition, we notice that*

$$\phi^1(p) = \|C\|_p = n^{\frac{1}{p}}\phi^1(\infty),$$

*i.e. all the three bounds are attainable in the scalar case of* $m = 1$.

## 5. Conclusion

In this paper, the lower and upper attainable bounds on the stability radius of the set of extreme solutions were obtained in the situation where solution and criterion spaces are endowed with various Hölder's norms. As corollaries, analytical formulae for the stability radius are specified in the case of the Boolean set of feasible solutions.

One of the biggest challenges in this field is to construct efficient algorithms to calculate the analytical expressions. To the best of our knowledge, there are not so many results known in that area, and, moreover, some of those results, which have been already known, put more questions than answers. As it was pointed out in Nikulin, Karelkina and Mäkelä (2013), calculating exact values of stability radii is an extremely difficult task in general, so one could concentrate either on finding easily computable classes of problems or on developing general metaheuristic approaches.

Estimations of stability radius obtained in this paper, are based on the enumeration of the set of feasible solutions, whose cardinality may grow exponentially with $n$. In the case of a single objective function, an approach to calculating the stability radius of an $\varepsilon$-optimal solution to the linear problem of 0-1 programming in polynomial time has been given in Chakravarti and Wagelmans (1999). These authors assumed that the objective function is minimized, the feasible solution set is fixed and a given subset of the objective function coefficients is perturbed. The approach requires that the original single objective optimization problem be polynomially solvable, for example it can be one of the well-known graph theoretic problems, such as minimum spanning tree or shortest path problems. Another approach, based on $k$-best solutions, was proposed in Libura et al. (1998) for NP-hard problems, such as traveling salesman problem. In Emelichov and Podkopaev (2010), it has been shown how analytical formulae similar to (9) can be transformed into polynomial type calculation procedure in the case of Boolean variables, Chebyshev norm and polynomial solvability of the problem. However, for multicriteria case the question of existence of the polynomial time procedures remains open. As it is well known that the presence of multiple criteria increases the level of complexity, for example, polynomially solvable single objective problems become intractable even in bicriteria case, see, e.g., Ehrgott (2005), finding polynomial methods seems to be unlikely in general. For some particular challenging combinatorial problems, it has been proven that the problem of finding the radii of every type of stability is intractable unless $P = NP$ (Kuzmin, 2015). An application of inverse optimization allows for reducing the calculation of stability radius to a logarithmic number of mixed integer programs for multi-objective combinatorial problems, where each objective function is a maximum sum and the coefficients are restricted to natural numbers (Roland, Smet and Figueira, 2012).

## 6. Acknowledgement

The authors thank two referees for their helpful comments and suggestions, which substantially improved the presentation and the content of this paper. The authors are also grateful to Kirill Kuzmin (Georgia State University, USA) for the fruitful discussion on some parts of the article.

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