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A Jurdjevic-Quinn theorem for stochastic differential systems under weak conditions*

by

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Abstract: The purpose of this paper is to provide sufficient conditions for the stabilizability of weak solutions of stochastic differential systems when both the drift and diffusion are affine in the control. This result extends the well–known theorem of Jurdjevic–Quinn (Jurdjevic and Quinn, 1978) to stochastic differential systems under weaker conditions on the system coefficients than those assumed in Florchinger (2002).

Keywords: stochastic differential system, weak solution of stochastic differential equation, asymptotic stability in probability, Barbashin–Krasovskii theorem, stabilizing state feedback law

1. Introduction

The stabilization of deterministic nonlinear control systems has been the subject of a great stream of research in the past decades by many authors. Among all the results proven in this field of research, it is fair to indicate the work of Jurdjevic and Quinn (1978), which gives sufficient conditions for the existence of stabilizing state feedback laws for deterministic systems affine in the control, provided the control Lie algebra of the system has full rank. This result is at the origin of various publications on this subject, since it appears that many engineering systems are of "Jurdjevic–Quinn type" (see, for instance, Lee and Arapostathis, 1988; Tsinias, 1989; Faubourg and Pomet, 1999; Outbib and Sallet, 1992, or Morin, 1996, and the references therein).

Stochastic versions of Jurdjevic-Quinn theorem have been established by Florchinger (1994) for stochastic differential systems, the drift of which is affine in the control, and later in Florchinger (2002), when both the drift and the

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diffusion are affine in the control, and in Florchinger (2001) for more general nonlinear stochastic differential systems. The technique used in these papers is based on the stochastic Lyapunov analysis, developed by Khasminskii (1980) and the stochastic version of La Salle's invariance principle, proven by Kushner (1972).

However, these results require that the stochastic systems considered in these works admit strong solutions, which is not necessarily the case in the problems of everyday life, in particular in stochastic financial and biological models (see, for instance, Yang, Kloeden and Wu, 2018, and the references therein). Indeed, for such models the system coefficients do not satisfy the local Lipschitz condition, which implies that the existence and uniqueness of the solution cannot be guaranteed by the standard conditions.

To overcome this difficulty, we propose a new approach, based on the results of Li and Liu (2014), which extend the concept of stochastic stability to more general stochastic nonlinear systems that have more than one weak solution. This new approach allows us to consider stochastic differential systems, whose coefficients are only continuous, which is often the case in control theory.

This remainder of the paper is divided in three sections and is organized as follows. In Section 2, we summarize the concepts of stochastic stability in the more general sense, introduced by Li and Liu (2014) to cover the stochastic nonlinear systems having more than one weak solution. In particular, the generalized stochastic Barbashin–Krasovskii theorem, which gives the criteria of stochastic stability for more general stochastic nonlinear systems, is recalled. In Section 3, we introduce the class of stochastic differential systems, affine in the control, that we are dealing with in this paper and introduce some differential operators, associated to this class of stochastic systems that we shall need in the sequel. In Section 4, we prove the main result of the paper, which extends the stochastic version of Jurdjevic-Quinn theorem, stated in Florchinger (2002), to stochastic differential systems that may have more than one weak solution.

2. Asymptotic stability in probability of weak solutions for stochastic differential systems

In this section, in order for the paper to be self-contained, we recall some basic results concerning the existence of weak solutions for stochastic differential systems and the concept of stochastic stability for such systems.

With this aim in mind, we introduce first the following notations, which will be used in the sequel. A function α mapping $\mathbb{R}_+$ into $\mathbb{R}_+$ is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing and $\alpha(0) = 0$. Further, a function α is said to be a class $\mathcal{K}_\infty$ function if α is an unbounded class $\mathcal{K}$ function.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and denote by $(w_t)_{t\geq 0}$ a standard $\mathbb{R}^m$–valued Wiener process, defined on this space.

Consider the stochastic process solution $x_t \in \mathbb{R}^n$ of the stochastic differential equation written in the sense of Itô,

$$x_t = x_0 + \int_0^t f(x_s)ds + \int_0^t g(x_s)dw_s, \tag{1}$$

where x_0 is given in $\mathbb{R}^n$ and f and g are Borel measurable functions, mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively, such that $f(0) = 0$ and $g(0) = 0$.

As it is well–known, in order to guarantee that the stochastic differential equation (1) admits a unique strong solution for any initial condition, the functions f and g need to satisfy a local Lipschitz condition (see. for example, Ikeda and Watanabe, 1989; Karatzas and Shreve, 1991, or Khasminskii, 1980, and the references therein). However, the stochastic differential equation (1) always has weak solutions when the functions f and g are only continuous (see, for example, Cherny, 2002; Cherny and Engelbert, 2018; Hofmanová and Seidler, 2012; Ondreját and Seidler, 2018; Karatzas and Shreve, 1991; or Ikeda and Watanabe, 1989, and the references therein), which is usually the case when modeling stochastic systems in real life, for example in finance and biology (see Yang, Kloeden and Wu, 2018, and the references therein).

Before introducing the extension of the concepts of stochastic stability to weak solutions of stochastic differential equations, we recall first the following definition from Li and Liu (2014).

Definition 1 *If there exists a continuous adapted stochastic process* $(x_t)_{t\geq 0}$ *on a probability space* $(\Omega^x, \mathcal{F}^x, P^x)$*, equipped with a right–continuous filtration* $(\mathcal{F}^x_t)_{t\geq 0}$*, and an* $\mathbb{R}^m$*–valued* $(\mathcal{F}^x_t)_{t\geq 0}$ *adapted standard Wiener process* $(W^x_t)_{t\geq 0}$*, defined on this probability space, such that* $P^x_{x_0} = P_{x_0}$ *and for all* $t \in [0, \tau^x_{+\infty}[$*,*

$$x_t = x_0 + \int_0^t f(x_s)ds + \int_0^t g(x_s)dW^x_s \quad \text{a.s.}$$

then the stochastic process $(x_t)_{t\geq 0}$ *is called a weak solution of the stochastic differential equation (1), where* $\tau^x_{+\infty}$ *is the explosion time of the weak solution* $(x_t)_{t\geq 0}$*; that is* $\tau^x_{+\infty} = \lim_{\epsilon\to+\infty} \inf\{t \geq 0/||x_t|| \geq \epsilon\}$*.*

The above definition shows that given specified probability spaces, filtrations and Wiener processes, weak solutions for the stochastic differential equation (1) may exists on different probability spaces. In fact, for a given Wiener process, defined on a specific probability space, a weak solution for the stochastic differential equation (1) may not exist, but nevertheless, this does not mean that this system has no weak solution for another Wiener process (see Karatzas and Shreve, 1991, for example). Moreover, the following result, stated in Ikeda and Watanabe (1989), gives sufficient conditions, ensuring the existence of weak solutions for the stochastic differential system (1).

Theorem 1 *If the functions f and g are continuous on $\mathbb{R}^n$ then, for any initial distribution μ on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$, the stochastic differential system (1) admits a weak solution $(x_t)_{t\geq 0}$ with initial distribution μ; i.e. $P^x_{x_0} = \mu$.*

In the following, we bring in the extensions, introduced by Li and Liu (2014), of the concepts and criteria for global asymptotic stability in probability of the equilibrium solution of the stochastic differential equation (1), stated in Khasminskii (1980) or Deng, Krstić and Williams (2001), to stochastic differential systems having more than one weak solution rather than a unique strong solution.

Definition 2 *1) The equilibrium solution $x_t \equiv 0$ of the stochastic differential equation (1) is globally stable in probability if for any $\epsilon \in]0,1[$, there exists a class $\mathcal{K}$ function α, such that for any $x_0 \in \mathbb{R}^n$, every weak solution $(x_t)_{t\geq 0}$ of the stochastic differential system (1) satisfies*

$$P^x\left(\sup_{0\leq t} ||x_t|| < \alpha\left(||x_0||\right)\right) \geq 1-\alpha.$$

2) The equilibrium solution $x_t \equiv 0$ of the stochastic differential equation (1) is globally asymptotically stable in probability if it is globally stable in probability and for any $x_0 \in \mathbb{R}^n$, every weak solution $(x_t)_{t\geq 0}$ of the stochastic differential system (1) satisfies

$$P^x\left(\lim_{t\to+\infty} ||x_t|| = 0\right) = 1.$$

Note that in the previous definition the probability measure P^x is associated with the specific weak solution and consequently may differ from the different weak solutions of the stochastic differential system (1). Moreover, it is worth noticing that the class $\mathcal{K}$ function α in Definition 2 does not vary with the solutions of the stochastic differential system (1).

Let us denote by L the infinitesimal generator of the stochastic process solution of the stochastic differential equation (1); that is, the second order differential operator, defined for any function $\varphi \in C^2\left(\mathbb{R}^n;\mathbb{R}\right)$ by

$$L\varphi(x) = \nabla\varphi(x)f(x) + \frac{1}{2}\mathrm{Tr}\left(g(x)g(x)^\tau\nabla^2\varphi(x)\right).$$

Then, the following stochastic version of Barbashin–Krasovskii theorem, which gives sufficient conditions in terms of Lyapunov function for the global asymptotic stability in probability of the equilibrium solution of the stochastic differential equation (1), has been proven by Li and Liu (2014) in Theorem 2.

Theorem 2 *Assume that the functions f and g are continuous and that there exist a function V in $C^2(\mathbb{R}^n;\mathbb{R})$ and class $\mathcal{K}_\infty$ functions α and β such that*

1. $\alpha(||x||) \leq V(x) \leq \beta(||x||)$,
2. $LV(x) \leq 0$,
3. *for any initial distribution, no nonzero weak solution of the stochastic differential system (1) completely belongs to the set $\{x \in \mathbb{R}^n / LV(x) = 0\}$ almost surely.*

Then the equilibrium solution $x_t \equiv 0$ of the stochastic differential equation (1) is globally asymptotically stable in probability.

To conclude this section, we recall a result from Khalil (1996), connecting proper Lyapunov functions and class $\mathcal{K}_\infty$ functions.

Proposition 1 *Let V be a proper Lyapunov function defined on $\mathbb{R}^n$ (i.e. a radially unbounded function V in $C^2(\mathbb{R}^n;\mathbb{R})$ such that $V(0) = 0$ and $V(x) > 0$ for any $x \in \mathbb{R}^n$, $x \neq 0$), then there exist class $\mathcal{K}_\infty$ functions α and β such that for any $x \in \mathbb{R}^n$,*

$$\alpha(||x||) \leq V(x) \leq \beta(||x||).$$

3. Problem setting and notations

In this section, we introduce the class of stochastic differential control systems we are dealing with in this paper.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and denote by $(w_t)_{t\geq 0}$ a standard $\mathbb{R}^m$–valued Wiener process, defined on this space.

Consider the stochastic process solution $x_t \in \mathbb{R}^n$ of the multi–input stochastic differential system written in the sense of Itô,

$$x_t = x_0 + \int_0^t \left(f_0(x_s) + \sum_{k=1}^{p} u^k f_k(x_s) \right) ds + \sum_{i=1}^{m} \int_0^t \left(g_{i,0}(x_s) + \sum_{k=1}^{p} u^k g_{i,k}(x_s) \right) dw_s^i \tag{2}$$

where

1. x_0 is given in $\mathbb{R}^n$,
2. u is a measurable $\mathbb{R}^p$-valued control law,
3. f_k, $0 \leq k \leq p$, and $g_{i,k}$, $1 \leq i \leq m$, $0 \leq k \leq p$, are smooth continuous functions mapping $\mathbb{R}^n$ into $\mathbb{R}^n$, vanishing in the origin.

The aim of this paper is to design a state feedback law u such that the equilibrium solution of the closed–loop system deduced from the stochastic differential system (3) is globally asymptotically stable in probability in the sense of Definition 2.

With this purpose in mind, we introduce the following differential operators that we shall need in the sequel.

Denote by L the infinitesimal generator of the stochastic process solution of the unforced stochastic differential system deduced from (3); that is, L is the second order differential operator, defined for any function φ in $C^2(\mathbb{R}^n;\mathbb{R})$ by

$$L\varphi(x) = \sum_{i=1}^{n} f_0^i(x)\frac{\partial\varphi(x)}{\partial x_i} + \frac{1}{2}\sum_{k,r=1}^{n}\sum_{j=1}^{m} g_{j,0}^k(x)g_{j,0}^r(x)\frac{\partial^2\varphi(x)}{\partial x_k\partial x_r}.$$

Furthermore, for any $i \in \{1,...,p\}$, denote by L_i the second order differential operator, defined for any function φ in $C^2(\mathbb{R}^n;\mathbb{R})$ by

$$L_i\varphi(x) = \sum_{k=1}^{n} f_i^k(x)\frac{\partial\varphi(x)}{\partial x_k} + \sum_{k,r=1}^{n}\sum_{j=1}^{m} g_{j,0}^k(x)g_{j,i}^r(x)\frac{\partial^2\varphi(x)}{\partial x_k\partial x_r}$$

and, for any $i, j \in \{1,...,p\}$, denote by L_{ij} the second order differential operator, defined for any function φ in $C^2(\mathbb{R}^n;\mathbb{R})$ by

$$L_{ij}\varphi(x) = \frac{1}{2}\sum_{k,r=1}^{n}\sum_{\nu=1}^{m} g_{\nu,i}^k(x)g_{\nu,j}^r(x)\frac{\partial^2\varphi(x)}{\partial x_k\partial x_r}.$$

To conclude, for any $i \in \{1,...,m\}$, denote by G_i the first order differential operator, defined for any function φ in $C^1(\mathbb{R}^n;\mathbb{R})$ by

$$G_i\varphi(x) = \sum_{k=1}^{n} g_{i,0}^k(x)\frac{\partial\varphi(x)}{\partial x_k}.$$

4. The main result

In this section, we state sufficient conditions ensuring the existence of a stabilizing state feedback law for the stochastic differential system (3). This result extends the well–known theorem of Jurdjevic–Quinn (Jurdjevic and Quinn, 1978) to the framework considered in this paper.

THEOREM 3 *Assume that there exists a proper smooth Lyapunov function V defined on $\mathbb{R}^n$ such that*

1) $LV(x) \leq 0$ for every $x \in \mathbb{R}^n$, *2) the set*

$$\mathcal{H} = \{x \in \mathbb{R}^n \ / \ G_{i_0}^{q_0} L^{r_0} ... G_{i_k}^{q_k} L^{r_k} L_j V(x) = 0 \text{ and } G_{i_0}^{q_0} L^{r_0} ... G_{i_k}^{q_k} L^{r_k+1} V(x) = 0,$$

$$\forall j \in \{1,..,p\}, \forall k \in \mathbb{N}, \forall i_0,..,i_k \in \{1,...,m\}, \forall q_0, r_0,.., q_k, r_k \in \{0,..,k\}$$

$$\text{s.t. } \sum_{i=0}^{k} (q_i + r_i) = k\}$$

is reduced to $\{0\}$.

Then, the control law, u, defined on $\mathbb{R}^n$ *by*

$$u^j(x) = -\frac{L_j V(x)}{\gamma(x)}, \ 1 \leq j \leq p, \tag{3}$$

where $\gamma(x) = 1 + \left(\sup_{1 \leq i,j \leq p} L_{ij} V(x) \right)^2$, *renders the stochastic differential system (2) globally asymptotically stable in probability.*

Remark 1 *In the definition of the stabilizing control law u, given in (3), one can actually use any positive function* γ *mapping* $\mathbb{R}^n$ *into* $\mathbb{R}$ *such that*

$$L_{ij} V(x) < \gamma(x)$$

for any $x \in \mathbb{R}^n$ *and* $i, j \in \{1,...,p\}$.

Proof of Theorem 3. First, note that since V is a proper Lyapunov function, Proposition 1 implies that there exist class $\mathcal{K}_\infty$ functions α and β such that for any $x \in \mathbb{R}^n$,

$$\alpha(||x||) \leq V(x) \leq \beta(||x||).$$

Further, denoting by $\mathcal{L}$ the infinitesimal generator of the stochastic process solution of the closed–loop system deduced from (2) with the state feedback law u given by (3), one gets for every $x \in \mathbb{R}^n$,

$$\mathcal{L}V(x) = LV(x) - \frac{1}{\gamma(x)} \sum_{i=1}^{p} (L_i V(x))^2 + \frac{1}{\gamma(x)^2} \sum_{i,j=1}^{p} L_i V(x) L_j V(x) L_{ij} V(x). \tag{4}$$

Then, since $LV(x) \leq 0$ for every $x \in \mathbb{R}^n$, by taking into account the definition of the function γ, one has

$$\mathcal{L}V(x) \leq 0$$

for every $x \in \mathbb{R}^n$ and therefore the two first assertions in Theorem 2 are fulfilled.

Furthermore, it is obvious from (4) that $\mathcal{L}V(x_t) \equiv 0$ for every $t \geq 0$ if, and only if, $LV(x_t) \equiv 0$ for every $t \geq 0$ and $L_iV(x_t) \equiv 0$ for every $t \geq 0$ and $i \in \{1, ..., p\}$.

Then, by applying Itô's formula to the stochastic processes $LV(x_t)$ and $L_iV(x_t)$, $i \in \{1, ..., p\}$, it appears easily that if $LV(x_t) \equiv 0$ for every $t \geq 0$ and $L_iV(x_t) \equiv 0$ for every $t \geq 0$ and $i \in \{1, ..., p\}$, one has $L^2V(x_t) \equiv 0$, $G_iLV(x_t) \equiv 0$, $1 \leq i \leq m$, $LL_iV(x_t) \equiv 0$, $1 \leq i \leq p$, and $G_iL_jV(x_t) \equiv 0$, $1 \leq i \leq m$, $1 \leq j \leq p$, for every $t \geq 0$.

Therefore, by inductive applications of Itô's formula, one can prove that if $\mathcal{L}V(x_t) \equiv 0$ for every $t \geq 0$, one has $x_t \in \mathcal{H}$ for every $t \geq 0$ and, consequently, according with the second hypothesis, $x_t \equiv 0$ for every $t \geq 0$.

Consequently, no nonzero weak solution of the stochastic differential system (2) in conditions of applying the state feedback law u, given by (2), completely belongs to the set $\{x \in \mathbb{R}^n / \mathcal{L}V(x) = 0\}$ almost surely and hence, according to Theorem 2, the equilibrium solution $x_t \equiv 0$ of the closed–loop stochastic differential system deduced from (2) when applying the state feedback law u given by (3), is globally asymptotically stable in probability. □

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