sciendo
Control and Cybernetics
vol. **51** (2022) No. 4
pages: 425-444
DOI: 10.2478/candc-2022-0023

Optimality conditions in set-valued optimization using approximations as generalized derivatives*

by

Nazih Abderrazzak Gadhi and Aissam Ichatouhane

LAMA, Sidi Mohamed Ben Abdellah University, Dhar El Mahraz, Department of Mathematics, Fes, Morocco
abderrazzak.gadhinazih@usmba.ac.ma, ichatouhane22@gmail.com

Abstract: The set criterion is an appropriate defining approach regarding the solutions for the set-valued optimization problems. By using approximations as generalized derivatives of set-valued mappings, we establish necessary optimality conditions for a constrained set-valued optimization problem in the sense of set optimization in terms of asymptotical pointwise compact approximations. Sufficient optimality conditions are then obtained through first-order strong approximations of data set-valued mappings.

Keywords: approximations, set-valued optimization, local weak l-minimal solutions, optimality conditions

1. Introduction

During the last decades, the set-valued optimization theory and its applications have been investigated by many authors; see, in particular, Aubin and Cellina (1984), Jahn (2004), Khan, Tammer and Zălinescu (2015), or Klein and Thompson (1984) and the references therein. We may distinguish at least three approaches: vector, lattice, and set approach (Khan, Tammer and Zălinescu, 2015). The most frequently used in the literature is the vector criterion, which is known also as the Pareto efficient solution. Set-valued optimization problems considering this approach have been studied in various frameworks, see, for instance, Alonso and Rodriguez-Marin (2008), Jahn (2004), Khan and Tuan (2011, 2015), Luc (1989), or Mordukhovich (2006), and the references therein. This solution criterion cannot be treated as the appropriate criterion when the decision maker's preference is based on comparing all image sets. It is just what the set criterion does. More realistic order relations for the comparison of

*Submitted: May 2021; Accepted: October 2022.

sets have been introduced to optimization by Kuroiwa (1998), see also Kuroiwa, Tanaka and Ha (1997). Since then, the optimality conditions for set-valued optimisation problems using the set relations have been derived in different ways. Several authors introduced notions of directional derivatives for set-valued maps in order to formulate necessary and sufficient optimality conditions for such problems using the l-less order relation (Kuroiwa, 2009; Rodriguez-Marin and Sama, 2007).

In this paper, we are concerned with the constrained set-valued optimization problem

$$(P):\begin{cases} \min F(x) \\ \text{subject to } G(x)\cap(-Z^{+})\neq\emptyset,\ x\in S, \end{cases}$$

where S is a closed subset of $X,\ F:X\rightrightarrows Y$ and $G:X\rightrightarrows Z$ are set-valued mappings between Banach spaces $X,\ Y,\ Z$, and $Z^{+}\subset Z$ is a closed convex cone with a non empty interior.

Let

$$\Omega=\left\{x\in S:G(x)\cap\left(-Z^{+}\right)\neq\emptyset\right\}$$

be the feasible set of (P) and let $Y^{+}\subset Y$ be a closed convex cone with a non empty interior.

Using the notion of set criterion, introduced by Kuroiwa (2008), together with first order approximations of set-valued mappings, we give necessary optimality conditions for (P) in terms of asymptotical pointwise compact approximations. The concept of asymptotical pointwise compact approximations of a set-valued mapping assumes proper importance in the case of unbounded approximations. Our results are obtained without any convexity assumption. In order to get sufficient optimality conditions, first-order strong approximations of the set-valued mappings F and G are used. To the best of our knowledge, there is no paper studying the optimality conditions for set-valued optimization problems in the sense of set optimization via first-order approximations. Since the set criterion of solution can be viewed as a weaker version of vector criterion, see Rodriguez-Marin and Hernández (2007, Proposition 2.10), our optimality results are sharper than those of Khan and Tuan (2011, 2015), where the notion of an efficient solution was used. We supply appropriate examples to illustrate the advantages of our results over some recent existing ones dealing with necessary optimality conditions using the l-less order relation.

The outline of the paper is as follows: the preliminaries and basic definitions are provided in Section 2; main results are established in Sections 3 and Section 4; a conclusion is given in Section 5.

2. Preliminaries

Our notations are rather standard. $\mathbb{N} = \{1, \ ...., \ n, \ ...\}$ and $\|.\|$ stands for the norm in any normed space (the context makes it clear what space is concerned). We denote by $L(X, Y)$ the set of continuous linear mappings between X and Y, $\mathbb{B}_Y$ denoting the open unit ball of Y centered at the origin, $\overline{\mathbb{B}}_Y$ - the closed unit ball of Y centered at the origin, $\mathbb{S}_Y$ -the unit sphere of Y and X^* - the continuous dual of X. The collection of nonempty subsets of Y will be denoted by $\mathcal{P}(Y)$. We write $\langle ., . \rangle$ for the canonical bilinear form with respect to the duality $\langle X^*, X \rangle$. Let A be a nonempty subset of Y and let D be a nonempty convex subset of Y.

Let C be a nonempty subset of X. The convex hull of C and the closure of C are denoted by *conv* C and *cl* C, respectively. The negative polar cone C° is defined by

$$C^\circ = \{v \in X^* : \quad \langle v, c \rangle \leq 0, \quad \forall c \in C\}.$$

The contingent (or Bouligand) cone of C at $\overline{x}$ is

$$T(C, \overline{x}) = \left\{d \in X : \exists t_n \to 0^+, \ \exists d_n \to d, \ \forall n \in \mathbb{N}, \ \overline{x} + t_n d_n \in C\right\}.$$

For a set-valued mapping $F : X \rightrightarrows Y$, the domain of F and the graph of F are

$$domF = \{x \in X : F(x) \neq \emptyset\} \text{ and } grF = \{(x, y) \in X \times Y : y \in F(x)\}.$$

Let A and B, $A \neq B$, be two nonempty subsets of Y. Let $<^l$ be the following relation defined by

$$B <^l A \Longleftrightarrow A \subseteq B + intY^+$$

where $intY^+$ denotes the topological interior of Y^+. Using the set relation $<^l$, Kuroiwa (2008) introduced the following notion of weakly l-minimal set.

Definition 1 *Let $S \in \mathcal{P}(Y)$. It is said that $A \in S$ is a weakly l-minimal set of S if*

$$\nexists B \in S \text{ such that } B <^l A.$$

We consider the following constrained set-valued optimization problem with a geometric constraint

$$(Q) : \left\{ \begin{array}{c} Y^+ \text{ - Min } F(x) \\ \text{subject to :} \quad x \in S. \end{array} \right.$$

A point $\overline{x} \in S$ is said to be a local weak l-minimal solution of (Q) in set criterion if there exists a neighborhood U of $\overline{x}$ such that for all $x \in S \cap U,\ x \neq \overline{x}$, we have

$$F(x) \not<^l F(\overline{x}).$$

Equivalently, if there exists a neighborhood U of $\overline{x}$ such that for all $x \in S \cap U$, there exists $\overline{y} \in F(\overline{x})$ satisfying

$$\overline{y} \notin F(x) + intY^+.$$

We shall need the following definitions.

Definition 2 (Khanh and Tuan, 2011) *Let $M \in L(X,Y)$ and $M_n \in L(X,Y)$, $n \in \mathbb{N}$. The sequence (M_n) is said to pointwise converge to M and is written $M_n \overset{p}{\to} M$ or $M = p\text{-}\lim\limits_n M_n$ if*

$$\lim_n M_n(x) = M(x), \quad \textit{for all } x \in X.$$

Definition 3 (Kuroiwa, 1998) *A subset $A \subseteq L(X,Y)$ is called (sequentially) asymptotically pointwisely compact, or (sequentially) asymptotically p-compact if*

- *each norm bounded sequence $(M_n) \subseteq A$ has a subsequence (M_{n_k}) and $M \in L(X,Y)$ such that*

$$M = p - \lim_k M_{n_k},$$

- *for each sequence $(M_n) \subseteq A$ with $\lim\limits_{n\to\infty} \|M_n\| = \infty$, the sequence $\left(\dfrac{M_n}{\|M_n\|}\right)$ has a subsequence, which pointwisely converges to some $M \in L(X,Y) \setminus \{0\}$*

For a subset $A \subset L(X,Y)$, let

$$\mathbb{A} = (p\text{-}cl\ A) \cup [(p\text{-}A_\infty) \setminus \{0\}]$$

where

$$p\text{-}cl\ A = \{M \in L(X,Y) : \exists (M_n)_n \subseteq A,\ M = p\text{-}\lim M_n\}$$

and

$$p\text{-}A_\infty = \left\{M \in L(X,Y) : \exists (M_n)_n \subseteq A,\ \exists t_n \to 0^+,\ M = p\text{-}\lim t_n M_n\right\}$$

stand for the p-closure and the p-recession cone of A, respectively. As mentioned in Khanh and Tuan (2011, Remark 3.1), if X and Y are finite dimensional, convergence occurs if and only if the corresponding pointwise convergence does; moreover, there is no difference between the asymptotical p-compactness and the asymptotical compactness.

The following definition of approximation is slightly different from that of Khanh and Tuan (2011, Definition 3.2) since the approximation does not depend on any element $\overline{y} \in F(\overline{x})$.

Definition 4 *Let $\overline{x} \in X$.*

- *A subset $A_F(\overline{x})$ of $L(X,Y)$ is said to be a first-order approximation of F at $\overline{x}$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$(F(x) - F(\overline{x})) \cap (A_F(\overline{x})(x-\overline{x}) + \varepsilon \|x-\overline{x}\| \mathbb{B}_Y) \neq \emptyset, \tag{1}$$

for all $x \in (\overline{x} + \delta \mathbb{B}_X) \cap domF$.

- *A subset $A^s_F(\overline{x})$ of $L(X,Y)$ is said to be a first-order strong approximation of F at $\overline{x}$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$F(x) - F(\overline{x}) \subseteq A^s_F(\overline{x})(x-\overline{x}) + \varepsilon \|x-\overline{x}\| \mathbb{B}_Y, \tag{2}$$

for all $x \in (\overline{x} + \delta \mathbb{B}_X) \cap domF$.

Example 1 *For $\overline{x} = (0,0)$: let $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ be the set-valued mapping defined by*

$$F(x_1,x_2) = \begin{cases} \left\{ y \in \mathbb{R} : 2|x_1|^{\frac{1}{2}} + x_2^4 \leq y \leq \frac{1}{|x_1|+|x_2|} \right\} & \text{if } (x,y) \neq (0,0) \\ \{0\} & \text{if } (x,y) = (0,0). \end{cases}$$

- *The set $A_F(\overline{x}) = \{(0,0)\}$ is a first-order approximation of F at $\overline{x}$.*
- *The set*

$$A^s_F(\overline{x}) = (\mathbb{R}^* \times \{0\}) \cup (\{0\} \times \mathbb{R}^*)$$

is a first-order strong approximation of F at $\overline{x}$.

First-order approximations are not unique, since any set containing an approximation is also an approximation. Evidently, if F is single-valued, the set $A_F(\overline{x})$ collapses to the corresponding notions, defined in Allali and Amahroq (1997) for single-valued maps. In Allali and Amahroq (1997), it is shown that when F is a locally Lipschitz function, it admits as an approximation the Clarke subdifferential of F at $\overline{x}$; i.e.

$$A_F(\overline{x}) = \partial F(\overline{x}) := cl\ co \{\text{Lim } \nabla F(x_n); x_n \in \text{dom}\nabla F \text{ and } x_n \to \overline{x}\}.$$

Notice that, in general, first-order approximations are not closed and may exist even for these set-valued mappings, which are neither upper semicontinuous nor lower semicontinuous at a given point (Khanh and Tuan, 2011); see Example 2.

Example 2 *(Khanh and Tuan, 2011) Consider the real function $F : \mathbb{R} \rightrightarrows \mathbb{R}$, defined by*

$$F(x) = \begin{cases} \{y \in \mathbb{R} \mid y \geq \sqrt{x}\} & \text{if} \quad x > 0, \\ \left\{y \in \mathbb{R} \mid y \leq \dfrac{1}{x}\right\} & \text{if} \quad x < 0, \\ \{0\} & \text{if} \quad x = 0, \end{cases}$$

Even if F is neither upper semicontinous nor lower semicontinuous at $\overline{x} = 0$, it admits the open set $A_F(\overline{x}) =]\alpha, +\infty[$, for any $\alpha > 0$, as a first-order approximation at $\overline{x}$. For more details, see (Khanh and Tuan, 2011, Example 3.1).

Basic calculus rules needed for effective applications of approximations as generalized derivatives were given by Khanh and Tuan (2011, 2015). Below, we recall a rule for compositions of maps; for more details see (Khanh and Tuan, 2015, Proposition 3.2).

Proposition 1 *(Khanh and Tuan, 2015) Let $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$, $(x_0, y_0) \in grF$, $(y_0, z_0) \in grG$ and $\operatorname{Im} F \subseteq domG$.*

- *Suppose that F and G admit bounded first-order strong approximation $A_F^S(x_0, y_0)$ and $A_G^S(y_0, z_0)$ at (x_0, y_0) and (y_0, z_0). Then, $A_G^S(y_0, z_0) \circ A_F^S(x_0, y_0)$ is a first-order strong approximation of $G \circ F$ at (x_0, z_0).*
- *Suppose that F and G admit bounded first-order approximation $A_F(x_0, y_0)$ and $A_G(y_0, z_0)$ at (x_0, y_0) and (y_0, z_0). Then, $A_G(y_0, z_0) \circ A_F(x_0, y_0)$ is a first-order approximation of $G \circ F$ at (x_0, z_0).*

3. Necessary optimality conditions

The following theorem gives, in terms of first-order approximations, necessary optimality conditions of the problem (Q).

Theorem 1 *Let $\overline{x}$ be a local weak l-minimal solution of the problem (Q), such that*

$$F(\overline{x}) - F(\overline{x}) \subseteq Y \setminus -intY^+.$$

Suppose that $A_F(\overline{x})$ is an asymptotically p-compact first-order approximation of F at $\overline{x}$. Then, for each $v \in T(S, \overline{x}) \setminus \{0\}$, there exists $A \in \mathbb{A}_F(\overline{x})$ such that

$$A(v) \notin -intY^+.$$

Here,

$$\mathbb{A}_F(\overline{x}) := (p - clA_F(\overline{x})) \cup (p - (A_F(\overline{x}))_\infty \setminus \{0\}).$$

Proof Let $v \in T(S,\overline{x})\backslash\{0\}$ be arbitrary and fixed. By the definition of a contingent cone, there is $(t_n,v_n)\to(0^+,v)$ such that $\overline{x}+t_nv_n\in S$ for all $n\in\mathbb{N}$.

- On the one hand, since $\overline{x}$ is assumed to be a local weak l-minimal solution of the problem (Q), for n large enough, we have

$$F(\overline{x}+t_nv_n)\not\prec^l F(\overline{x}).$$

That is,

$$F(\overline{x})\nsubseteq F(\overline{x}+t_nv_n)+intY^+. \tag{3}$$

Then, there exists $\overline{y}\in F(\overline{x})$ such that

$$\overline{y}-y_n\notin intY^+, \qquad \forall y_n\in F(\overline{x}+t_nv_n). \tag{4}$$

- On the other hand, for n large enough, we have

$$[F(\overline{x}+t_nv_n)-F(\overline{x})]\cap A_F(\overline{x})(t_nv_n)+\varepsilon\|t_nv_n\|\mathbb{B}_Y\neq\emptyset.$$

Then, there exist $\widetilde{y}_n\in F(\overline{x}+t_nv_n)$ and $\widetilde{y}\in F(\overline{x})$ such that

$$\widetilde{y}_n-\widetilde{y}\in A_F(\overline{x})(t_nv_n)+\varepsilon\|t_nv_n\|\mathbb{B}_Y.$$

Consequently, there exist $A_n\in A_F(\overline{x})$ and $b_n\in\mathbb{B}_Y$ such that

$$\widetilde{y}_n=\widetilde{y}+A_n(t_nv_n)+\varepsilon\|t_nv_n\|b_n. \tag{5}$$

From (4), since $\widetilde{y}_n\in F(\overline{x}+t_nv_n)$, we have

$$\widetilde{y}_n\notin\overline{y}-intY^+. \tag{6}$$

Combining (5) and (6), we have

$$A_n(t_nv_n)+\varepsilon\|t_nv_n\|b_n\notin\overline{y}-\widetilde{y}-intY^+. \tag{7}$$

Since

$$F(\overline{x})-F(\overline{x})\subseteq Y\backslash-intY^+$$

we deduce

$$A_n(t_nv_n)+\varepsilon\|t_nv_n\|b_n\notin -intY^+. \tag{8}$$

- If $\{A_n\}$ is norm bounded, one can assume that $A_n\xrightarrow{P}A\in p-clA_F(\overline{x})$. Upon dividing (8) by t_n and passing to the limit, one gets

$$A(v) \notin -intY^{+}.$$

If $\{A_n\}$ is unbounded, one can assume that $\|A_n\| \to \infty$ and $\frac{A_n}{\|A_n\|} \overset{P}{\to} A \in p - (A_F(\overline{x}))_{\infty} \setminus \{0\}$. Dividing (8) by $\|A_n\| t_n$ and passing to the limit leads to

$$A(v) \notin -intY^{+}.$$

■

The following example explains how to employ Theorem 1. Since F is not compact valued, Theorem 1 of Amahroq and Oussarhan (2019) with this property imposed, cannot be employed.

Example 3 *Let* $X = \mathbb{R},\ Y = \mathbb{R},\ S = [0,1],\ Y^{+} = \mathbb{R}_{+},\ \overline{x} = 0$ *and*

$$F(x) = \begin{cases} \left\{y \in \mathbb{R} : y \leq \dfrac{1}{x}\right\} & \text{if} \quad x > 0 \\ \left\{y \in \mathbb{R} : y \geq \sqrt[3]{-x}\right\} & \text{if} \quad x < 0 \\ \{0\} & \text{if} \quad x = 0 \end{cases}.$$

We have

$$F(\overline{x}) - F(\overline{x}) = \{0\} \subseteq \mathbb{R} \setminus -int\mathbb{R}_{+} \text{ and } T(S, \overline{x}) = \mathbb{R}_{+}.$$

In addition, for fixed $a < 0$, we have

$$A_F(\overline{x}) = \left]-\infty,\ a\right[,\ clA_F(\overline{x}) = \left]-\infty,\ a\right],\ A_F(\overline{x})_{\infty} = \left]-\infty,\ 0\right] \text{ and } \mathbb{A}_F(\overline{x}) = \left]-\infty,\ 0\right[.$$

Taking $v = 1 \in T(S, \overline{x})$, we obtain

$$A(v) < 0 \text{ for all } A \in \mathbb{A}_F(\overline{x}).$$

Consequently,

$$A(v) \in -int\mathbb{R}_{+} \text{ for all } A \in \mathbb{A}_F(\overline{x}).$$

Using Theorem 1, we deduce that $\overline{x}$ is not a local weak l-minimal solution of the problem (Q).

Example 4 *Let* $X = \mathbb{R},\ Y = \mathbb{R}^2,\ S = [0,1],\ Y^{+} = \mathbb{R}^2_{+},\ \overline{x} = 0$ *and*

$$F(x) = \begin{cases} \emptyset & \text{if} \quad x > 0 \\ \left\{(y,z) \in \mathbb{R}^2 : y \geq \sqrt[3]{-x} \text{ and } z = x^2\right\} & \text{if} \quad x < 0 \\ conv\{(0,0),\ (-1,1)\} & \text{if} \quad x = 0 \end{cases}.$$

- *On the one hand, $\overline{x} = 0$ is a local weak l-minimal solution of the problem (P).*

- *On the other hand, we have*

$$F(\overline{x}) - F(\overline{x}) = conv\{(1,-1),\ (-1,1)\} \subseteq \mathbb{R}^2 \backslash -int\mathbb{R}^2_+ \text{ and } T(S,\overline{x}) = \mathbb{R}_+.$$

In addition, for fixed $a < 0$, we have

$$A_F(\overline{x}) =]-\infty,\ a[\times\{0\},\ clA_F(\overline{x}) =]-\infty,\ a]\times\{0\},\ A_F(\overline{x})_\infty =]-\infty,\ 0]\times\{0\}$$

and

$$\mathbb{A}_F(\overline{x}) =]-\infty,\ 0[\times\{0\}.$$

Let $\alpha < 0$. For all $v \in T(S,\overline{x})\backslash\{0\}$, we can find $A = (\alpha, 0) \in \mathbb{A}_F(\overline{x})$ such that

$$A(v) = (\alpha v, 0) \notin -int\mathbb{R}^2_+.$$

The necessary optimality conditions given in Theorem 1 are then satisfied at $\overline{x}$. Notice that, since F is not compact valued, Theorem 1 of Amahroq and Oussarhan (2019) is out of use.

Proposition 2 *Let $\overline{x} \in \Omega$ be a local weak l-minimal solution of the problem (P), such that*

$$G(\overline{x}) - G(\overline{x}) \subseteq Z^+.$$

Then, for any $\overline{z} \in G(\overline{x}) \cap (-Z^+)$, $\overline{x}$ is a local weak l-minimal solution of the problem

$$(P^*) : \left\{ \begin{array}{c} Min\ \ (F,G)(x) \\ subject\ to:\ \ x \in S \end{array} \right.$$

with respect to $Y^+ \times (Z^+ + \overline{z})$.

Proof Reasoning ad absurdum, suppose that there exists $\overline{z} \in G(\overline{x}) \cap (-Z^+)$ such that $\overline{x}$ is not a local weak l-minimal solution of the problem (P^*). Then, there exists $x_n \in S,\ x_n \to \overline{x}$, such that

$$F(\overline{x}) \subseteq F(x_n) + intY^+$$

and

$$G(\overline{x}) \subseteq G(x_n) + int\left(Z^+ + \overline{z}\right). \tag{9}$$

- By (9), we get

  $$G(\overline{x}) \subseteq G(x_n) + Z^+ + \overline{z}.$$

  Then,

  $$0 \in G(x_n) + Z^+ + G(\overline{x}) - G(\overline{x}).$$

  Since

  $$G(\overline{x}) - G(\overline{x}) \subseteq Z^+,$$

  and since Z^+ is a cone, we deduce that

  $$0 \in G(x_n) + Z^+.$$

  Consequently,

  $$G(x_n) \cap \left(-Z^+\right) \neq \emptyset.$$

  Since $x_n \in S$, we deduce that

  $$x_n \in \Omega \text{ and } F(\overline{x}) \subseteq F(x_n) + intY^+.$$

  A contradiction with the fact that $\overline{x}$ is a local weak l-minimal solution of the problem (P) with respect to Y^+.

■

The following theorem gives the necessary optimality conditions of the problem (P).

THEOREM 2 *Let $\overline{x} \in \Omega$ be a local weak l-minimal solution of the problem (P), such that*

$$F(\overline{x}) - F(\overline{x}) \subseteq Y \backslash -intY^+ \text{ and } G(\overline{x}) - G(\overline{x}) \subseteq Z^+.$$

Suppose that $A_F(\overline{x})$ and $A_G(\overline{x})$ are asymptotically p-compact first-order approximations of F and G at $\overline{x}$, respectively. Then, for each $v \in T(S, \overline{x}) \backslash \{0\}$, we can find $A \in \mathbb{A}_F(\overline{x})$ and $B \in \mathbb{A}_G(\overline{x})$ such that $(A(v), B(v)) \notin -int(Y^+ \times Z^+)$.

PROOF Let $v \in T(S, \overline{x}) \backslash \{0\}$ be arbitrary and fixed. By the definition of a contingent cone, there is $(t_n, v_n) \to (0^+, v)$ such that $\overline{x} + t_n v_n \in S$ for all $n \in \mathbb{N}$. Since $\overline{x} \in \Omega$, we have

$$\overline{x} \in S \text{ and } G(\overline{x}) \cap \left(-Z^+\right) \neq \emptyset.$$

We can find $\overline{z} \in G(\overline{x})$ such that $\overline{z} \in -Z^+$.

- By Proposition 2, $\overline{x}$ is a local weak l-minimal solution of the problem (P^*) with respect to $Y^+ \times (Z^+ + \overline{z})$. For n large enough, we have

$$(F(\overline{x}), G(\overline{x})) \nsubseteq (F(\overline{x}+t_n v_n), G(\overline{x}+t_n v_n)) + int\left[Y^+ \times \left(Z^+ + \overline{z}\right)\right].$$

Then, there exist $y_0 \in F(\overline{x})$ and $z_0 \in G(\overline{x})$ such that

$$(y_0 - y_n, z_0 - z_n) \notin int\left[Y^+ \times \left(Z^+ + \overline{z}\right)\right],$$
$$\forall y_n \in F(\overline{x}+t_n v_n),\ \forall z_n \in G(\overline{x}+t_n v_n).$$

Thus, either

$$y_0 - y_n \notin intY^+, \qquad \forall y_n \in F(\overline{x}+t_n v_n), \tag{10}$$

or

$$z_0 - z_n \notin \overline{z} + int\left(Z^+\right), \qquad \forall z_n \in G(\overline{x}+t_n v_n) \tag{11}$$

is satisfied.

- For n large enough, we have

$$[F(\overline{x}+t_n v_n) - F(\overline{x})] \cap A_F(\overline{x})(t_n v_n) + \varepsilon \|t_n v_n\| \mathbb{B}_Y \neq \emptyset$$

and

$$[G(\overline{x}+t_n v_n) - G(\overline{x})] \cap A_G(\overline{x})(t_n v_n) + \varepsilon \|t_n v_n\| \mathbb{B}_Z \neq \emptyset.$$

Then, there exist $\widetilde{y}_n \in F(\overline{x}+t_n v_n)$, $\widetilde{y} \in F(\overline{x})$, $\widetilde{z}_n \in G(\overline{x}+t_n v_n)$, $\widetilde{z} \in G(\overline{x})$ such that

$$\widetilde{y}_n - \widetilde{y} \in A_F(\overline{x})(t_n v_n) + \varepsilon \|t_n v_n\| \mathbb{B}_Y \text{ and } \widetilde{z}_n - \widetilde{z} \in A_G(\overline{x})(t_n v_n) + \varepsilon \|t_n v_n\| \mathbb{B}_Z.$$

Consequently, there exist $A_n \in A_F(\overline{x})$, $a_n \in \mathbb{B}_Y$, $B_n \in A_G(\overline{x})$ and $b_n \in \mathbb{B}_Y$ such that

$$\widetilde{y}_n = \widetilde{y} + A_n(t_n v_n) + \varepsilon \|t_n v_n\| a_n \text{ and } \widetilde{z}_n = \widetilde{z} + B_n(t_n v_n) + \varepsilon \|t_n v_n\| b_n. \tag{12}$$

From (4), since $\widetilde{y}_n \in F(\overline{x}+t_n v_n)$ and $\widetilde{z}_n \in G(\overline{x}+t_n v_n)$, we have

$$\widetilde{y}_n \notin y_0 - intY^+ \text{ or } \widetilde{z}_n \notin z_0 - \overline{z} - int\left(Z^+\right). \tag{13}$$

By combining (5) and (6), we obtain

$$A_n(t_n v_n) + \varepsilon \|t_n v_n\| a_n \notin y_0 - \widetilde{y} - intY^+$$

or

$$B_n(t_n v_n) + \varepsilon \|t_n v_n\| b_n \notin z_0 - \overline{z} - \widetilde{z} - int\left(Z^+\right). \tag{14}$$

Since

$$F(\overline{x}) - F(\overline{x}) \subseteq Y \setminus -intY^{+} \text{ and } G(\overline{x}) - G(\overline{x}) \subseteq Z^{+}$$

we deduce that

$$A_n(t_n v_n) + \varepsilon \|t_n v_n\| a_n \notin -intY^{+} \tag{15}$$

or

$$B_n(t_n v_n) + \varepsilon \|t_n v_n\| b_n \notin -int\left(Z^{+}\right). \tag{16}$$

- If $\{A_n\}$ and $\{B_n\}$ are norm bounded, one can assume that $A_n \overset{P}{\to} A \in p - clA_F(\overline{x})$ and $B_n \overset{P}{\to} B \in p - clA_G(\overline{x})$. Upon dividing (8) by t_n and then passing to the limit one obtains

$$A(v) \notin -intY^{+} \text{ or } B(v) \notin -intZ^{+}.$$

- If $\{A_n\}$ and $\{B_n\}$ are unbounded, one can assume that $\|A_n\| \to \infty$, $\|B_n\| \to \infty$, $\frac{A_n}{\|A_n\|} \overset{P}{\to} A \in p\text{–}(A_F(\overline{x}))_\infty \setminus \{0\}$ and $\frac{B_n}{\|B_n\|} \overset{P}{\to} B \in p\text{–}(A_G(\overline{x}))_\infty \setminus \{0\}$. Dividing (8) by $\|A_n\| t_n$ and (8) by $\|B_n\| t_n$ and passing to the limit leads to

$$A(v) \notin -intY^{+} \text{ or } B(v) \notin -intZ^{+}.$$

- If $\{A_n\}$ is norm bounded and if $\{B_n\}$ is unbounded, one can assume that $A_n \overset{P}{\to} A \in p - clA_F(\overline{x})$ and $\frac{B_n}{\|B_n\|} \overset{P}{\to} B \in p - (A_G(\overline{x}))_\infty \setminus \{0\}$. Upon dividing (8) by t_n and then passing to the limit one obtains

$$A(v) \notin -intY^{+} \text{ or } B(v) \notin -intZ^{+}.$$

- If $\{B_n\}$ is norm bounded and if $\{A_n\}$ is unbounded, one can assume that $B_n \overset{P}{\to} B \in p - clA_G(\overline{x})$ and $\frac{A_n}{\|A_n\|} \overset{P}{\to} A \in p - (A_F(\overline{x}))_\infty \setminus \{0\}$. Dividing (8) by t_n and passing to the limit results in

$$A(v) \notin -intY^{+} \text{ or } B(v) \notin -intZ^{+}.$$

■

The following example explains how to employ Theorem 2.

Example 5 *Let* $X = \mathbb{R},\ Y = \mathbb{R},\ Y^{+} = \mathbb{R}_{+},\ Z^{+} = \mathbb{R}_{+},\ S = \mathbb{R}_{+},\ \overline{x} = 0,$

$$F(x) = \begin{cases} \left\{y \in \mathbb{R} : y < \frac{1}{x}\right\} & if \quad x > 0 \\ \left\{y \in \mathbb{R} : y \geq \sqrt{-x}\right\} & if \quad x < 0 \\ \{0\} & if \quad x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} \{y \in \mathbb{R} : y^3 x \leq 1\} & \text{if} \quad x > 0 \\ \{y \in \mathbb{R} : y^3 + x \leq 0\} & \text{if} \quad x < 0 \\ \{0\} & \text{if} \quad x = 0 \end{cases}$$

We have

$$F(\overline{x}) - F(\overline{x}) = \{0\} \subseteq \mathbb{R} \backslash -int\mathbb{R}_+, \ G(\overline{x}) - G(\overline{x}) = \{0\} \subseteq \mathbb{R} \backslash -int\mathbb{R}_+$$
$$\text{and} \quad T(S, \overline{x}) = \mathbb{R}_+.$$

Moreover, for fixed $a < 0$ *and* $b < 0$*, we have*

$$A_F(\overline{x}) =]-\infty, a[, \ clA_F(\overline{x}) =]-\infty, a], \ A_F(\overline{x})_\infty =]-\infty, 0], \ \mathbb{A}_F(\overline{x}) =]-\infty, 0[$$

and

$$A_G(\overline{x}) =]-\infty, b[, \ clA_G(\overline{x}) =]-\infty, b], \ A_G(\overline{x})_\infty =]-\infty, 0], \ \mathbb{A}_G(\overline{x}) =]-\infty, 0[.$$

By taking $v = 1 \in T(S, \overline{x})$*, we get*

$$(A(v), B(v)) = (A, B) \in -int\left(\mathbb{R}^2_+\right) \text{ for all } A \in \mathbb{A}_F(\overline{x}).$$

Consequently,

$$A(v) \in -int\mathbb{R}_+ \text{ for all } A \in \mathbb{A}_F(\overline{x}) \text{ and } B \in \mathbb{A}_G(\overline{x}).$$

Using Theorem 2, we deduce that $\overline{x}$ *is not a local weak l-minimal solution of the problem* (P)*. Notice that since* F *and* G *are not compact valued, Theorem 3 of Amahroq and Oussarhan (2019) with this property imposed, cannot be employed.*

4. Sufficient optimality conditions

Let us turn to sufficient optimality conditions. Before, let us start by some recalls. The following definition has been introduced by Tanino (1988).

DEFINITION 5 *(Tanino, 1988) A base* Q *of* Y^+ *is a nonempty subset of* Y^+ *with* $0 \notin cl(Q)$ *and such that every* $c \in Y^+ \backslash \{0\}$ *has a unique presentation as follows*

$$c = rq \text{ with } r > 0 \text{ and } q \in cl(Q).$$

If Q *is compact, we say that* Y^+ *has a compact base* Q*.*

REMARK 1 *(Tanino, 1988) If* Y *is finite dimentional, then* Y^+ *has a compact base.*

Remark 2 *(Shi, 1991) The cone Y^+ has a compact base Q if and only if $Y^+ \cap \mathbb{S}_Y$ is compact.*

Theorem 3 *Let $\overline{x} \in S$ and $\overline{z} \in G(\overline{x}) \cap (-Z^+)$. Suppose that X is finite dimensional, that $Y^+ \times Z^+$ has a compact base and that $A_F^S(\overline{x})$ and $A_G^S(\overline{x})$ are compact first-order strong approximations of F and G at $\overline{x}$ respectively. Impose further that*

$$\left(A_F^S(\overline{x}),\ A_G^S(\overline{x})\right)(0) \cap -\left(Y^+ \times T\left(Z^+, -\overline{z}\right)\right) = \{(0,0)\} \tag{17}$$

and

$$\begin{aligned}&\left(A_F^S(\overline{x}),\ A_G^S(\overline{x})\right)(v) \cap -\left(Y^+ \times T\left(Z^+, -\overline{z}\right)\right) = \emptyset,\\ &\forall v \in \left(domA_F(\overline{x}) \cap domA_G(\overline{x})\right) \diagdown \{0\}.\end{aligned} \tag{18}$$

Then, $\overline{x}$ is a local weak l-minimal solution of the problem (P) with respect to Y^+.

Proof To the contrary, suppose that $\overline{x}$ is not a local weak l-minimal solution of the problem (P) with respect to Y^+. Then, there exists $x_n \in \Omega,\ x_n \to \overline{x}$, such that

$$F(\overline{x}) \subseteq F(x_n) + intY^+.$$

Consequently, for all $\overline{y} \in F(\overline{x})$ there exists $y_n \in F(x_n)$ such that $\overline{y} - y_n \in intY^+$; and thus $\overline{y} - y_n \neq 0$. Since $x_n \in \Omega$, we have

$$G(x_n) \cap \left(-Z^+\right) \neq \emptyset \text{ and } x_n \in S.$$

Then, we can find $z_n \in G(x_n)$ such that $-z_n \in Z^+$; which implies

$$(\overline{y} - y_n, \overline{z} - z_n) \neq (0,0),\ (y_n, z_n) \in (F,G)(x_n)$$

and

$$(\overline{y} - y_n, \overline{z} - z_n) \in Y^+ \times \left(Z^+ + \overline{z}\right).$$

By setting

$$t_n := \|x_n - \overline{x}\| \text{ and } v_n := \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|}$$

we get

$$x_n = \overline{x} + t_n v_n,\ x_n \to \overline{x},\ t_n \to 0^+ \text{ and } v_n \to v \in \mathbb{S}_X.$$

Since

$$F(\overline{x}+t_nv_n)-\overline{y}\subset A_F^S(\overline{x})(t_nv_n)+\varepsilon\|t_nv_n\|\mathbb{B}_Y$$

and

$$G(\overline{x}+t_nv_n)-\overline{z}\subset A_G^S(\overline{x})(t_nv_n)+\varepsilon\|t_nv_n\|\mathbb{B}_Z$$

we have

$$y_n-\overline{y}\in A_F^S(\overline{x})(t_nv_n)+\varepsilon\|t_nv_n\|\mathbb{B}_Y \text{ and } z_n-\overline{z}\in A_G^S(\overline{x})(t_nv_n)+\varepsilon\|t_nv_n\|\mathbb{B}_Z. \quad (19)$$

- Let

  $$\alpha_n=\|(\overline{y}-y_n,-z_n)\|,\ b_n=\frac{1}{\alpha_n}(\overline{y}-y_n) \text{ and } q_n=\frac{-z_n}{\alpha_n}.$$

  Then,

  $$\|(b_n,q_n)\|=1,\ \overline{y}-y_n=\alpha_nb_n \text{ and } -z_n=\alpha_nq_n. \quad (20)$$

  From (19), we obtain

  $$-\alpha_nb_n\in A_F^S(\overline{x})(t_nv_n)+\varepsilon\|t_nv_n\|\mathbb{B}_Y \text{ and } -\overline{z}-\alpha_nq_n\in A_G^S(\overline{x})(t_nv_n)+\varepsilon\|t_nv_n\|\mathbb{B}_Z.$$

  Thus, there exist $M_n\in A_F^S(\overline{x})$, $N_n\in A_G^S(\overline{x})$, $\beta_n\in\mathbb{B}_Y$ and $\gamma_n\in\mathbb{B}_Z$ such that

  $$M_n(t_nv_n)+\varepsilon\|t_nv_n\|\beta_n=-\alpha_nb_n \text{ and } N_n(t_nv_n)+\varepsilon\|t_nv_n\|\gamma_n=-\overline{z}-\alpha_nq_n \quad (21)$$

  and

  $$-\alpha_nb_n\in -Y^+ \text{ and } -z_n\in Z^+. \quad (22)$$

  Notice that by compactness of the approximations $A_F^S(\overline{x})$ and $A_G^S(\overline{x})$, there exist $M\in A_F^S(\overline{x})$ and $N\in A_G^S(\overline{x})$ such that $M_n\to M$ and $N_n\to N$. Taking a subsequence, if necessary, we can assume that $\beta_n\to\beta\in\overline{\mathbb{B}}_Y$ and $\gamma_n\to\gamma\in\overline{\mathbb{B}}_Z$ when n tends to $+\infty$.

  - From (20) and (22), we have

    $$\|(b_n,q_n)\|=1,\ b_n\in Y^+ \text{ and } q_n\in Z^+.$$

    Consequently,

    $$M_n(v_n)+\varepsilon\|v_n\|\beta_n\in -Y^+.$$

    By Remark 2, since $Y^+\times Z^+$ has a compact base, there exist $b\in Y^+$ and $q\in Z^+$ such that

    $$\|(b,q)\|=1,\ b_n\to b \text{ and } q_n\to q. \quad (23)$$

– Since Z^+ is convex and since $-\overline{z} \in Z^+$, we have $Z^+ \subset -\overline{z} + T(Z^+, -\overline{z})$. Since $-z_n \in Z^+$, we get

$$\overline{z} - z_n \in T\left(Z^+, -\overline{z}\right).$$

By (20), we have

$$\overline{z} + \alpha_n q_n \in T\left(Z^+, -\overline{z}\right). \tag{24}$$

Consequently,

$$N_n(v_n) + \varepsilon \|v_n\| \gamma_n \in -T\left(Z^+, -\overline{z}\right).$$

The rest of the proof consists of several steps.

- Suppose that $(\alpha_n)_{n\in\mathbb{N}}$ has no convergent subsequence. Then,

$$\alpha_n \to +\infty.$$

Dividing (21) and (22) by α_n yields

$$-b_n = M_n\left(\frac{t_n v_n}{\alpha_n}\right) + \varepsilon \left\|\frac{t_n v_n}{\alpha_n}\right\| \beta_n \text{ and } -b_n \in Y^+ \tag{25}$$

and

$$N_n\left(\frac{t_n v_n}{\alpha_n}\right) + \varepsilon \left\|\frac{t_n v_n}{\alpha_n}\right\| \gamma_n = \frac{-\overline{z}}{\alpha_n} - q_n \text{ and } \frac{-\overline{z}}{\alpha_n} - q_n \in -T\left(Z^+, -\overline{z}\right). \tag{26}$$

From (26), letting $n \to \infty$, we have

$$M(0) = -b \in -Y^+ \text{ and } N(0) = -q \in -T\left(Z^+, -\overline{z}\right). \tag{27}$$

By combining (23) and (27), we obtain

$$\left(A_F^S(\overline{x}),\ A_G^S(\overline{x})\right)(0) \cap -\left(Y^+ \times T\left(Z^+, -\overline{z}\right)\right) \setminus \{(0,0)\} \neq \emptyset$$

which contradicts (17).

- Suppose that $(\alpha_n)_{n\in\mathbb{N}}$ has a convergent subsequence, that we note also $(\alpha_n)_{n\in\mathbb{N}}$, converging to some $\alpha \in [0,\ +\infty[$.

  – Suppose that $\alpha \neq 0$. Dividing (21) and (22) by $t_n \alpha_n$ we get

$$-\frac{b_n}{t_n} = M_n\left(\frac{v_n}{\alpha_n}\right) + \varepsilon \left\|\frac{v_n}{\alpha_n}\right\| \beta_n \text{ and } -\frac{b_n}{t_n} \in Y^+$$

and

$$\frac{-\overline{z}}{t_n\alpha_n}-\frac{q_n}{t_n}=N_n\left(\frac{v_n}{\alpha_n}\right)+\varepsilon\left\|\frac{v_n}{\alpha_n}\right\|\gamma_n \text{ and } \frac{-\overline{z}}{t_n\alpha_n}-\frac{q_n}{t_n}\in -T\left(Z^+,-\overline{z}\right).$$

Consequently,

$$M_n\left(\frac{v_n}{\alpha_n}\right)+\varepsilon\left\|\frac{v_n}{\alpha_n}\right\|\beta_n\in Y^+ \text{ and } N_n\left(\frac{v_n}{\alpha_n}\right)+\varepsilon\left\|\frac{v_n}{\alpha_n}\right\|\gamma_n\in -T\left(Z^+,-\overline{z}\right).$$

Letting $n\to\infty$, leads to

$$M\left(\frac{v}{\alpha}\right)+\varepsilon\left\|\frac{v}{\alpha}\right\|\beta\in -Y^+ \text{ and } N\left(\frac{v}{\alpha}\right)+\varepsilon\left\|\frac{v}{\alpha}\right\|\gamma\in -T\left(Z^+,-\overline{z}\right).$$

Letting $\varepsilon\to 0$, we get

$$M\left(\frac{v}{\alpha}\right)\in -Y^+ \text{ and } N\left(\frac{v}{\alpha}\right)\in -T\left(Z^+,-\overline{z}\right).$$

Consequently,

$$\left(A_F^S\left(\overline{x}\right),\ A_G^S\left(\overline{x}\right)\right)\left(\frac{v}{\alpha}\right)\cap -\left(Y^+\times T\left(Z^+,-\overline{z}\right)\right)\neq\emptyset \text{ with } \frac{v}{\alpha}\neq 0.$$

A contradiction with (18).

- Suppose that $\alpha=0$. We have two cases
  * Case $1:\ \overline{z}\neq 0$
    Letting $n\to\infty$ in (21) and (22), we obtain

$$0=M\left(0\right)\in -Y^+,\ -\overline{z}=N\left(0\right)\in -T\left(Z^+,-\overline{z}\right) \text{ and } \left(0,-\overline{z}\right)\neq\left(0,0\right).$$

    Then,

$$\left(0,-\overline{z}\right)\in\left(A_F^S\left(\overline{x}\right),\ A_G^S\left(\overline{x}\right)\right)\left(0\right)\cap -\left(Y^+\times T\left(Z^+,-\overline{z}\right)\right)\setminus\left\{\left(0,0\right)\right\}$$

    which contradicts (17).
  * Case $2:\ \overline{z}=0$
    From (21), (22) and (24), we have

$$M_n\left(t_nv_n\right)+\varepsilon\left\|t_nv_n\right\|\beta_n=-\alpha_nb_n \text{ and } N_n\left(t_nv_n\right)+\varepsilon\left\|t_nv_n\right\|\gamma_n=-\alpha_nq_n$$

    and

$$-\alpha_nb_n\in -Y^+ \text{ and } -\alpha_nq_n\in -T\left(Z^+,-\overline{z}\right).$$

    By setting

$$w_n=\frac{t_nv_n}{\alpha_n} \text{ and } \varpi_n=\frac{1}{\left\|w_n\right\|}=\frac{\alpha_n}{t_n\left\|v_n\right\|}=\frac{\alpha_n}{t_n},$$

we obtain

$$M_n(w_n)+\varepsilon\|w_n\|\beta_n=-b_n \text{ and } N_n(w_n)+\varepsilon\|w_n\|\gamma_n=-q_n \quad (28)$$

and

$$-b_n\in -Y^+ \text{ and } -q_n\in -T\left(Z^+,-\overline{z}\right). \quad (29)$$

▷ Suppose that the sequence (w_n) has a convergent subsequence, that we note also $(w_n)_{n\in\mathbb{N}}$, to some $w\in X$. Letting $n\to\infty$ and $\varepsilon\to 0$, we get

$$M(w)=-b,\ N(w)=-q,\ -b\in -Y^+ \text{ and } -q\in -T\left(Z^+,-\overline{z}\right).$$

Then,

$$-b\in M(w)\cap -Y^+ \text{ and } -q\in N(w)\cap -T\left(Z^+,-\overline{z}\right).$$

Thus,

$$\left(A_F^S(\overline{x}),\ A_G^S(\overline{x})\right)(w)\cap -\left(Y^+\times T\left(Z^+,-\overline{z}\right)\right)\setminus\{(0,0)\}\neq\emptyset.$$

We have a contradiction with (17) if $w=0$ or (18) if $w\neq 0$.
▷ Suppose that the sequence (w_n) has no convergent subsequence; that is, $\varpi_n\to 0$. We have

$$\frac{\alpha_n}{t_n}=\varpi_n\to 0.$$

Dividing (28) and (29) by t_n, yields

$$\begin{cases} M_n(v_n)+\varepsilon\beta_n=-\alpha_n\dfrac{b_n}{t_n}=-\varpi_n b_n, \\ N_n(v_n)+\varepsilon\gamma_n=-\alpha_n\dfrac{q_n}{t_n}=-\varpi_n q_n \\ -\varpi_n b_n=-\alpha_n\dfrac{b_n}{t_n}\in -Y^+ \\ -\varpi_n q_n=-\alpha_n\dfrac{q_n}{t_n}\in -T(Z^+,-\overline{z}). \end{cases}$$

By letting $n\to\infty$ and $\varepsilon\to 0$, we get

$$M(v)=0,\ N(v)=0,\ 0\in -Y^+ \text{ and } 0\in -T\left(Z^+,-\overline{z}\right).$$

We deduce that

$$\left(A_F^S(\overline{x}),\ A_G^S(\overline{x})\right)(v)\cap -\left(Y^+\times T\left(Z^+,-\overline{z}\right)\right)\neq\emptyset$$

which contradicts (18).

■

5. Conclusion

Using the notion of set criterion, introduced by Kuroiwa (2008) together with first order approximations of set-valued mappings, we gave necessary optimality conditions for a set-valued optimization problem (P) in terms of asymptotical pointwise compact approximations. With the help of first order strong approximations, we also proposed sufficient optimality conditions. For future research, it would be interesting to investigate second order optimality conditions.

Acknowledgments

Our sincere acknowledgments to the anonymous referees for their insightful remarks and valuable suggestions, which have helped to improve the paper.

References

Allali, K. and Amahroq, T. (1997) Second order approximations and primal and dual necessary optimality conditions. *Optimization* 3, 229-246.

Alonso, M. and Rodríguez-Marín, L. (2008) Optimality conditions for a non convex set-valued optimization problem. *Comput. Math. Appl.* 56, 82-89.

Amahroq, T. and Oussarhan, A. (2019) Lagrange multiplier rules for weakly minimal solutions of compact-valued set optimization problems. *Asia-Pacific Journal of Operational Research* 36, 1-22.

Aubin, J.-P. and Cellina, A. (1984) Differential Inclusions. Set-Valued Maps and Viability Theory. *Grundlehren der Mathematischen Wissenschaften* **264**, Springer-Verlag, Berlin.

Jahn, J. (2004) *Vector Optimization. Theory, Applications, and Extensions.* Springer, Berlin.

Khanh, P.Q. and Tuan, N.D. (2011) Optimality conditions without continuity in multivalued optimization using approximations as generalized derivatives. In: S.K. Mishra, ed., *Recent Contributions in Nonconvex Optimization*, 47-61. Springer, New York.

Khanh, P.Q., Tuan, N.D. (2015) First and second-order optimality conditions without differentiability in multivalued vector optimization. *Positivity* 19, 817-841.

Khan, A.A., Tammer, C., Zălinescu, C. (2015) *Set-valued Optimization. An Introduction with Applications.* Springer, Berlin.

Klein, E., Thompson, A.C. (1984) *Theory of Correspondences. Including Applications to Mathematical Economics. Canadian Mathematical Society Series of Monographs and Advanced Texts*, New York.

Kuroiwa, D. (2008) On Derivatives of Set-Valued Maps in Set Optimization. Manuscript, Shimane University, Japan.

Kuroiwa, D. (1988) Natural criteria of set-valued optimization. Manuscript, Shimane University, Japan.

Kuroiwa, D. (2009) On derivatives of set-valued maps and optimality conditions for set optimization. *Journal of Nonlinear and Convex Analysis* 10, 41-50.

Kuroiwa, D., Tanaka, T. and Ha, T. X. D. (1997) On cone convexity of set-valued maps. *Nonlinear Analysis* 30, 1487-1496.

Luc, D.T. (1989) *Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems, Theory of Vector Optimization* **319**, Springer-Verlag, Berlin.

Mordukhovich, B.S. (2006) *Variational Analysis and Generalized Differentiation, II: Applications*. Springer, Berlin.

Rodríguez-Marín, L. and Sama, M. (2007) (Λ, C)-contingent derivatives of set-valued maps. *Journal of Mathematical Analysis and Applications* 335, 974-989.

Rodríguez-Marín, L. and Hernández, E. (2007) Nonconvex scalarization in set optimization with set-valued maps. *Journal of Mathematical Analysis and Applications* 325, 1-18.

Shi, D.S. (1991) Contingent derivative of perturbed map in multiobjective optimization. *Journal of Optimization Theory and Applications* 70, 385-395.

Tanino, T. (1998) Sensitivity analysis in multiobjective optimization. *Journal of Optimization Theory and Applications* 56, 479-499.