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Necessary optimality conditions for a set valued fractional programming problem in terms of contingent epiderivatives*

by

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Abstract: In this paper, we are concerned with a multi-objective fractional extremal programming problem. Using the concept of subdifferential of cone-convex set valued mappings, introduced by Baier and Jahn (1999), together with the convex separation principle, we give necessary optimality conditions. An example illustrating the usefulness of our results is also provided.

Keywords: fractional optimization, multi-objective optimization, cone-convex mapping, optimality condition, subdifferential

1. Introduction

In very recent years, the analysis and applications of D.C. mappings (difference of convex mappings) have been of considerable interest. Genuinely, nonconvex mappings that arise in nonsmooth optimization are often of this type. Hence, of late, extensive work on the analysis and optimization of D.C. mappings has been carried out. However, much work remains still to be done. For instance, if the data of the objective function of a standard problem are not exactly known, it makes sense to replace the objective by a set-valued objective representing fuzzy outcomes.

Set-valued optimization is a vibrant and expanding branch of mathematics that deals with optimization problems, in which the objective map and/or the constraints maps are set-valued maps, acting between certain spaces. Since set-valued maps generalize the single valued maps, set-valued optimization provides an important extension and unification of the scalar as well as the vector optimization problems.

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Let X be a real normed space, C be a closed subset of X and let $F_i : X \rightrightarrows \mathbb{R}$ and $G_i : X \rightrightarrows \mathbb{R}$, $q \in \mathbb{N}$, $i \in I = \{1, \ ..., \ q\}$ be given $\mathbb{R}^{+}$-convex and continuous set valued mappings such that

$$y_i > 0, \ z_i > 0 \text{ for all } i \text{ and all } y_i \in F_i(x), \ z_i \in G_i(x), \ x \in C.$$

In this paper, we are concerned with the following fractional vector problem involving multiapplications

$$(P) : \begin{cases} Y^{+} - \text{Min } H(x) \\ \text{subject to : } \ x \in C, \end{cases}$$

where $Y^{+} \subset \mathbb{R}^q$ is a pointed ($Y^{+} \cap -Y^{+} = \{0\}$) closed convex cone with nonempty interior Y^{++} introducing a partial order in $Y = \mathbb{R}^q$, and $H : X \rightrightarrows Y$ is the set valued mapping, defined by:

$$h \in H(x) \Leftrightarrow \forall i \in I, \ \exists y_i \in F_i(x), \ z_i \in G_i(x)$$
$$\text{such that } h = \left(\frac{y_1}{z_1}, \ \frac{y_2}{z_2}, \ ..., \ \frac{y_i}{z_i}, \ ..., \ \frac{y_q}{z_q} \right).$$

We pay main attention to deriving the necessary optimality conditions for (P). Let $\overline{x} \in C$ and $\overline{h} \in H(\overline{x})$. The point $(\overline{x}, \overline{h})$ is said to be a weak local Pareto minimal point with respect to Y^{+} of the problem (P) if there exists a neighborhood V of $\overline{x}$ such that $H(V \cap C) \subset \overline{h} + Y \setminus (-Y^{++})$; i.e.

$$h - \overline{h} \in Y \setminus \left(-Y^{++}\right) \qquad \forall h \in V \cap C. \tag{1}$$

For the convenience of the reader, note that when Y^{++} is empty, the use of the relative interior of Y^{+} in place of Y^{++} would be appropriate. Recall that the relative interior of Y^{+}, denoted $ri \ Y^{+}$, is the interior of Y^{+} relative to the closed affine hull of Y^{+}. The non-emptiness of the latter ($ri \ Y^{+} \neq \emptyset$) results from the fact that Y^{+} is convex and the dimension of the space Y is finite. In counterpart, the relative minimality (Bao, Gupta and Mordukhovich, 2010) should be used instead of the above weak Pareto minimality; this can be done simply by replacing Y^{++} by $ri \ Y^{+}$ in (1).

The optimization problem (P) is general and inclusive. It can be seen either as a fractional multiobjective optimization problem or as a set valued optimization problem. Thanks to this structure, it brings together several other problems, previously studied by several authors. In its framework, (P) includes convex and D.C. set valued optimization problems, vector fractional optimization problems, mathematical programming problems, etc. In Gadhi (2008), necessary and sufficient optimality conditions were provided for a special case of (P) using convexificators. Liang, Huang and Pardalos (2001) presented also sufficient optimality conditions and duality results for a special class of (P) (where $p = 1, \ m = 1$ and $k = 1$). In Bao, Gupta and Mordukhovich (2010), using the quotient rule of generalized differentiation from Bao, Gupta and Mordukhovich

(1999), the optimality conditions for a multiobjective fractional program with equilibrium constraints were obtained.

In order to get the optimality conditions for this complex problem, we had to go through several intermediate problems that are equivalent to it in a certain way. After that, our approach was to use the convex separation principle together with the weak and strong subdifferentials of the set valued mappings, introduced by Sawaragi and Tanino (1980) and Baier and Jahn (1999). Our technique extends the results obtained in the D.C. scalar case by Hiriart-Urruty (1989), for the D. C vector case by Gadhi and Metrane (2004) GM, for the D. C. set valued case by Gadhi (2005), and for the convex set valued case by Baier and Jahn (1999) and Taa (2003).

The rest of the paper is structured as follows. In Section 2, we recall basic definitions and preliminary material. In Section 3, we establish the necessary optimality conditions for a set valued fractional programming problem.

2. Preliminaries

Let C be a nonempty subset of X and let $F : X \rightrightarrows Y$ be a set valued mapping between Banach spaces X and Y. In the sequel, we denote the domain and the graph of F, respectively, by

$$dom\,(F) := \{x \in X : F\,(x) \neq \emptyset\} \text{ and } gr\,(F) := \{(x, y) \in X \times Y : y \in F\,(x)\}\,.$$

If V is a nonempty subset of X, then

$$F\,(V) = \underset{x\in V}{\cup} F\,(x)\,.$$

Let A be a nonempty subset of Y and $\overline{y} \in A$. Then, $\overline{y}$ is said to be a weak Pareto minimal point of A with respect to Y^+ if

$$(A - \overline{y}) \cap \left(-Y^{++}\right) = \emptyset. \tag{2}$$

We shall denote by $W.Min\,(A)$ the set of all weak Pareto minimal points of A. Since cone-convexity plays an important role in the following investigations, we recall the definition of cone-convex mappings.

Definition 1 (Corley, 1988) *Let $C \subset X$ be a convex set. The set valued mapping F from C into Y is said to be Y^+-convex on C, if* $\forall x_1,\ x_2 \in C$, $\forall \lambda \in [0, 1]\,,$

$$\lambda F\,(x_1) + (1 - \lambda)\,F\,(x_2) \subset F\,(\lambda x_1 + (1 - \lambda)\,x_2) + Y^+.$$

The following definition has been introduced by Sawaragi and Tanino (1980).

Definition 2 (Sawaragi and Tanino, 1980) *Let S be a nonempty subset of X and let F be a set valued mapping from S into Y. Considering $\overline{x} \in S$ and*

$\overline{y} \in F(\overline{x})$, a linear mapping $L : X \to Y$ is said to be a weak subgradient for $\overline{y}$ of F at $\overline{x}$ if

$$\overline{y} - L(\overline{x}) \in W.Min \bigcup_{x \in S} (F(x) - L(x)).$$

The set of all weak subgradients for $\overline{y}$ of F at $\overline{x}$ is called the weak subdifferential for $\overline{y}$ of F at $\overline{x}$ and is denoted by $\partial_W F(\overline{x}, \overline{y})$.

Proposition 1 gives a characterization of the above weak subdifferential.

Proposition 1 *Let $F : X \rightrightarrows Y$ be a set valued mapping and let $(\overline{x}, \overline{y}) \in gr(F)$. Suppose that F is Y^+-convex. Then, $L \in \partial_W F(\overline{x}, \overline{y})$ if and only if there exists $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that*

$$\langle y^*, y - \overline{y} - L(x - \overline{x}) \rangle \geq 0 \text{ for all } x \in X \text{ and all } y \in F(x).$$

Proof. The argument is standard. Observing that $\bigcup_{x \in X} (F(x) - L(x))$ is a convex set and that

$$\left(\bigcup_{x \in X} (F(x) - L(x)) - \overline{y} + L(\overline{x}) \right) \cap \left(-int\ Y^+ \right) = \emptyset,$$

the result follows by using the separation theorem. ⊠

Another way of introducing subgradients of set valued mappings is the use of the concept of contingent epiderivative, as given in Jahn and Rauh (1997).

Definition 3 (Jahn and Rauh, 1997) *Let S be a nonempty subset of X and let $F : S \rightrightarrows Y$ be a set valued mapping.*
1. The set

$$epi(F) := \left\{ (x, y) \in X \times Y : x \in S,\ y \in F(x) + Y^+ \right\}$$

is called the epigraph of F.
2. A single valued mapping $DF(\overline{x}, \overline{y}) : X \to Y$, whose epigraph equals the contingent cone to the epigraph of F at $(\overline{x}, \overline{y})$, i.e.,

$$epi(DF(\overline{x}, \overline{y})) = T(epi(F), (\overline{x}, \overline{y})),$$

is called the contingent epiderivative of F at $(\overline{x}, \overline{y})$.

Bear in mind that the contingent cone $T(epi(F), (\overline{x}, \overline{y}))$ consists of all tangent vectors

$$h := \lim_{n \to \infty} \lambda_n ((x_n, y_n) - (\overline{x}, \overline{y})),$$

with

$$\lambda_n > 0, \qquad (\overline{x}, \overline{y}) = \lim_{n \to \infty} (x_n, y_n), \qquad (x_n, y_n) \in epi(F), \qquad \text{for all } n \in \mathbb{N}.$$

Properties of the contingent epiderivative can be found in Jahn and Rauh (1997). On the basis of the concept of contingent epiderivatives, Baier and Jahn (1999) introduced a subdifferential of cone-convex set valued mappings.

DEFINITION 4 (BAIER AND JAHN, 1999) *Let $C \subset X$ be a convex set and suppose that F is Y^+-convex on C. If in addition, the contingent epiderivative $DF(\overline{x}, \overline{y})$ of F at $(\overline{x}, \overline{y})$ exists, then*
1. A linear mapping $L : X \to Y$, with

$$DF(\overline{x}, \overline{y}) - L(x) \in Y^+ \text{ for all } x \in X,$$

is called a subgradient of F at $(\overline{x}, \overline{y})$. Moreover, every subgradient L of F at $(\overline{x}, \overline{y})$ fulfills

$$y - \overline{y} - L(x - \overline{x}) \in Y^+ \text{ for all } x \in C \text{ and } y \in F(x). \tag{3}$$

2. The set

$$\partial F(\overline{x}, \overline{y}) := \left\{ L : X \to Y \text{ linear } / \ DF(\overline{x}, \overline{y})(x) - L(x) \in Y^+ \text{ for all } x \in X \right\}$$

of all subgradients L of F at $(\overline{x}, \overline{y})$ is called the subdifferential of F at $(\overline{x}, \overline{y})$.

REMARK 1 *Obviously, the subdifferential is not defined, if the contingent epiderivative does not exist. Conditions ensuring the existence of the contingent epiderivative can be found in Theorem 1 in Jahn and Rauh (1997).*

REMARK 2 *When $F = f$ is only a convex function, $\partial F(\overline{x}, \overline{y})$ reduces to the well known classical subdifferential in the sense of convex analysis*

$$\partial f(\overline{x}) = \{x^* \in X^* : f(x) - f(\bar{x}) \geq \langle x^*, x - \overline{x} \rangle \quad \text{for all} \quad x \in X\}.$$

3. Necessary optimality conditions

In this section, we maintain the notations given in the previous section and we give the necessary optimality conditions for the multi-objective fractional programming problem (P). In the sequel, $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ will be the set-valued mappings, defined by

$$F(x) = (F_1(x), \ ..., \ F_q(x)) = F_1(x) \times F_2(x) \times ... \times F_q(x)$$
$$\text{for all } x \in dom(F) = \bigcap_{i \in I} dom(F_i)$$

and

$$G(x) = (G_1(x), \ ..., \ G_q(x)) = G_1(x) \times G_2(x) \times ... \times G_q(x)$$
$$\text{for all } x \in dom(G) = \bigcap_{i \in I} dom(G_i).$$

Let $\overline{x} \in C$ and let

$$\begin{cases} \overline{y}_i \in F_i(\overline{x}) \qquad i \in I, \\ \overline{z}_i \in G_i(\overline{x}), \qquad i \in I, \\ \overline{h}_i = \dfrac{\overline{y}_i}{\overline{z}_i}, \qquad i \in I, \\ \overline{y} = \left(\overline{y}_1, \ \overline{y}_2, \ ..., \ \overline{y}_q\right), \ \overline{z} = (\overline{z}_1, \ \overline{z}_2, \ ..., \ \overline{z}_q), \ \overline{h} = \left(\overline{h}_1, \ \overline{h}_2, \ ..., \ \overline{h}_q\right), \ \overline{u} = (\overline{x}, \ \overline{y}, \ \overline{z}). \end{cases}$$

Consider the following problem (P_1) with respect to Y^+ :

$$(P_1): \begin{cases} Y^+ - \text{Minimize } m(x,y,z) = (m_1(x,y,z), \ ..., m_q(x,y,z)) \\ \qquad \text{subject to : } \begin{cases} y_i \in F_i(x), & i \in I, \\ z_i \in G_i(x), & i \in I, \\ \quad x \in C, \end{cases} \end{cases}$$

where

$$m_i(x,y,z) := y_i - \overline{h}_i z_i, \qquad \forall i \in I.$$

The following Lemma compares the set of all weak local Pareto minimal points of the problem (P) and the set of all weak local Pareto minimal points of (P_1).

Lemma 1 *Let $(\overline{x},\overline{h})$ be a weak local Pareto minimal point of the problem (P). Then, there exist $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{x})$ such that $(\overline{x},\overline{y},\overline{z},0)$ is a weak local Pareto minimal point of (P_1) with respect to Y^+.*

Proof. To the contrary, suppose that, for any neighborhood U of $\overline{x}$, there exist $x_0 \in U \cap C,\ y^0 \in F(x_0)$ and $z^0 \in G_i(x_0)$ such that

$$m\left(x^0, y^0, z^0\right) - m(\overline{x},\overline{y},\overline{z}) \in -Y^{++}.$$

Since $\overline{y}_i - \overline{h}_i\overline{z}_i = 0$, one has

$$y_i^0 - \overline{h}_i z_i^0 < 0 \qquad \forall i \in I.$$

Then,

$$\frac{y_i^0}{z_i^0} - \frac{\overline{y}_i}{\overline{z}_i} < 0 \qquad \forall i \in I,$$

a contradiction with the fact that $\overline{x}$ is a weak local Pareto minimal point of the problem (P). ⊠

The following result is a direct consequence of Lemma 1.

Lemma 2 *Let $(\overline{x},\overline{h})$ be a weak local Pareto minimal point of the problem (P). Then $(\overline{x},0)$ is a weak local Pareto minimal point of*

$$(P_2): \begin{cases} Y^+ - \textit{Minimize } F(x) - K(x) \\ \qquad \textit{subject to : } x \in C \end{cases}$$

where

$$K(x) = (K_1(x), \ ..., K_q(x)),$$

and

$$K_i(x) := \overline{h}_i G_i(x) \qquad \forall i \in I.$$

Assuming that the contingent epiderivative $DG(\overline{x},\overline{z})$ (respectively $DG(\overline{x},\overline{k})$) exists, we have the following result. Theorem 1 provides necessary optimality conditions for the multiobjective optimization problem (P).

THEOREM 1 *Suppose that $(\overline{x},\overline{h})$ is a weak local Pareto minimal point of (P). Then, for all $T^* = (T_1^*, \ldots, T_q^*)$, $T_i^* \in \partial G_i(\bar{x},\overline{z}_i)$, there exists $y^* \in (-Y^+)^\circ$ such that $y^* \neq 0_{Y^*}$ and*

$$\sum_{i=1}^{q} \overline{h}_i \; y_i^* T_i^* \in \partial_W \left(\sum_{i=1}^{q} y_i^* F_i \right)(\overline{x}, \langle y^*, \overline{y} \rangle) + N(\overline{x}; C).$$

Moreover, if $\bigcap_{i=1}^{q} int\ epi(F_i) \neq \emptyset$ then

$$\sum_{i=1}^{q} \overline{h}_i \; y_i^* T_i^* \in \sum_{i=1}^{q} y_i^* \partial_W F_i(\overline{x}, \langle y_i^*, \overline{y}_i \rangle) + N(\overline{x}; C)$$

where $N(\overline{x}; C)$ denotes the Clarke's normal cone.

PROOF. Since $(\overline{x},\overline{h})$ is a weak local Pareto minimal point of (P), by Lemma 2, $(\overline{x},0)$ is a weak local Pareto minimal point of (P_2). Thus, there exists a neighborhood V of $\overline{x}$ such that for all $x \in V \cap \Omega$

$$F(x) - K(x) \subset Y \setminus (-Y^{++}). \tag{4}$$

Let $T^* = (T_1^*, \ldots, T_q^*)$, $T_i^* \in \partial G_i(\bar{x},\overline{z}_i)$, and let $L^* = (\overline{h}_1 T_1^*, \ldots, \overline{h}_q T_q^*)$. Consider the following set

$$\Delta := \{p \in Y : \exists x \in C \cap V \text{ such that } (F(x) - L^*(x-\overline{x}) - p) \cap (\overline{y} - Y^{++}) \neq \emptyset\}.$$

The proof of this theorem consists of several steps. First, we prove two important properties of this set Δ and then we apply a separation theorem in order to obtain the multiplier rule.

- $\Delta \neq \emptyset$. Indeed, $r \in \Delta$ for all $r \in Y^{++}$.
- $0 \notin \Delta$. By contrary, suppose that there exists $x \in V \cap C$ such that $(F(x) - L^*(x-\overline{x})) \cap (\overline{y} - Y^{++}) \neq \emptyset$.

  Consequently, for all $i \in I$, there exist $y_i \in F_i(x)$ such that

  $$y_i - \overline{y}_i - \overline{h}_i T_i^*(x-\overline{x}) < 0. \tag{5}$$

  Since $T_i^* \in \partial G_i(\bar{x},\overline{z}_i)$, one has

  $$z_i - \overline{z}_i - T_i^*(x-\overline{x}) \geq 0, \qquad \forall x \in X,\ \forall z_i \in G_i(x). \tag{6}$$

  Combining (5) and (6), one gets

  $$y_i - \overline{h}_i z_i < \overline{y}_i - \overline{h}_i \overline{z}, \qquad \forall i \in I, \tag{7}$$

  a contradiction with (4).

- Let us prove that Δ is a convex subset of Y. Let $\lambda \in [0,1]$, $p_1 \in \Delta$ and $p_2 \in \Delta$. From the definition, there exist $x_1 \in V \cap C$ and $x_2 \in V \cap C$ such that

$$\left(F(x_1) - L^*(x_1 - \overline{x}) - p_1\right) \cap \left(\overline{y} - Y^{++}\right) \neq \emptyset$$
and
$$\left(F(x_2) - L^*(x_2 - \overline{x}) - p_2\right) \cap \left(\overline{y} - Y^{++}\right) \neq \emptyset.$$

Consequently, there exist $z_1,\ z_2 \in \overline{y} - Y^{++}$ such that

$$z_1 \in F(x_1) - L^*(x_1 - \overline{x}) - p_1,\ z_2 \in F(x_2) - L^*(x_2 - \overline{x}) - p_2.$$

Fix $x := \lambda x_1 + (1-\lambda) x_2,\ z := \lambda z_1 + (1-\lambda) z_2$ and $p := \lambda p_1 + (1-\lambda) p_2$. From the convexity assumption of F, one gets

$$z \in F(x) - L^*(x - \overline{x}) - p + Y^+.$$

On the one hand, there exist $u_0 \in Y^+$ and $v_0 \in Y^+$ such that $z - u_0 \in F(x) - L^*(x - \overline{x}) - p$. On the other hand,

$$mz - u_0 \in \overline{y} - Y^+ - Y^{++} \subset \overline{y} - Y^{++}.$$

Consequently, $(F(x) - L^*(x - \overline{x}) - p) \cap (\overline{y} - Y^{++}) \neq \emptyset$; which means that $p \in \Delta$.

- In the next step of the proof we show that Δ is open.
Indeed, consider $p \in \Delta$. From the definition, there exists $x \in V \cap C$ such that $(F(x) - L^*(x - \overline{x}) - p) \cap (\overline{y} - Y^{++}) \neq \emptyset$. Consequently, there exist $z \in \overline{y} - Y^{++}$ such that $z \in F(x) - L^*(x - \overline{x}) - p$. Since Y^{++} is an open set, there exists $\delta > 0$ such that

$$z - a \in \overline{y} - Y^{++} \text{ and } z - a \in F(x) - L^*(x - \overline{x}) - (p + a),\ \forall a \in \mathbb{B}_Y(0, \delta).$$

Consequently,

$$\left(F(x) - L^*(x - \overline{x}) - (p + a)\right) \cap \left(\overline{y} - Y^{++}\right) \neq \emptyset,\ \forall a \in \mathbb{B}_Y(0, \delta).$$

Then, $p + \mathbb{B}_Y(0, \delta) \subset \Delta$. Thus, Δ is open.

- In this step, we prove the theorem. Since Δ is an open convex subset of Y with $0 \notin \Delta$ (due to (4)), using a separation theorem, one can find $0_{Y^*} \neq y^* \in Y^*$ such that

$$\langle y^*, p \rangle \geq 0 \text{ for all } p \in \Delta.$$

Let $x \in V \cap C,\ y \in F(x),\ r \in Y^{++}$ and $\varepsilon > 0$.
Taking $p = y - \overline{y} - L^*(x - \overline{x}) + \varepsilon r$, one obtains that $p \in \Delta$; then

$$\langle y^*, y - \overline{y} - L^*(x - \overline{x}) + \varepsilon r \rangle \geq 0. \tag{8}$$

For $x = \overline{x},\ y = \overline{y}$ and $h = \overline{h}$, we have

$$\varepsilon \langle y^*, r \rangle \geq 0, \tag{9}$$

as $\varepsilon > 0$, (9) yields $\langle y^*, r \rangle \geq 0$. Since r is arbitrary in Y^+, one deduces that $y^* \in (-Y^+)^\circ$.
Now, reconsidering (8), we have (ε is arbitrary)

$$\langle y^*, y - \overline{y} - L^* (x - \overline{x}) \rangle \geq 0. \tag{10}$$

Consequently, by Proposition 1, $y^* \circ L^* \in \partial_W (y^* \circ F)(\overline{x}, \langle y^*, \overline{y} \rangle) + N(\overline{x}; C)$. Then,

$$\sum_{i=1}^{q} \overline{h}_i\ y_i^* T_i^* \in \partial_W \left(\sum_{i=1}^{q} y_i^* F_i \right) (\overline{x}, \langle y^*, \overline{y} \rangle) + N(\overline{x}; C).$$

- Now, suppose that $\bigcap_{i=1}^{q} int\ epi\,(F_i) \neq \emptyset$. Using the generalized Moreau-Rockafellar theorem (see Taa, 2003),

$$\sum_{i=1}^{q} \overline{h}_i\ y_i^* T_i^* \in \sum_{i=1}^{q} y_i^* \partial_W (F_i)(\overline{x}, \langle y_i^*, \overline{y}_i \rangle) + N(\overline{x}; C).$$

The proof is thus finished. ⊠

With the following example, we illustrate the usefulness of our result.

Example 1 *Let Ω be a closed subset of $\mathbb{R}^n$, and let $f_1 : \mathbb{R}^n \to \mathbb{R}$, ..., $f_q : \mathbb{R}^n \to \mathbb{R}$ and $g_1 : \mathbb{R}^n \to \mathbb{R}$, ..., $g_q : \mathbb{R}^n \to \mathbb{R}$ be given convex continuous functions such that $f_i(x) > 0$ and $g_i(x) > 0$ for all $x \in \Omega$. We consider the set valued mappings F_i and $G_i : \mathbb{R}^n \rightrightarrows \mathbb{R}$ with*

$$F_i(x) := \{ y \in \mathbb{R} : f(x) - y \leq 0 \} \text{ and } G_i(x) := \{ z \in \mathbb{R} : g(x) - z \leq 0 \},\ \ \forall i \in I.$$

Under these assumptions, we investigate the optimization problem

$$(P^{\rhd}) : \begin{cases} \left(\mathbb{R}_+^q\right) - Min\ H(x, y, z) = \left(\dfrac{y_1}{z_1}, \ ..., \ \dfrac{y_q}{z_q} \right) \\ Subject\ to: \quad y_i \in F_i(x),\ z_i \in G_i(x), \quad i \in I, \\ \qquad x \in C. \end{cases}$$

This is a special case of the general type (P). In this example, the values of the objective may vary depending of the values of several known functions.
Next, assume that $\overline{u} = (\overline{x}, \overline{y}, \overline{z})$ is a weak local Pareto minimal point of the problem $(P^{\rhd})$. Then, there exists $y^* \in (-Y^+)^\circ$ such that $y^* \neq 0_{Y^*}$ and $\sum_{i=1}^{q} \dfrac{f_i(\overline{x})}{g_i(\overline{x})} y_i^* \partial g_i(\bar{x}, \overline{z}) \subseteq \partial \left(\sum_{i \in I} y_i^* f_i \right)(\overline{x}) + N(\overline{x}; C)$.

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