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New Lyapunov functions for systems with source terms*

by

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Abstract: Lyapunov functions with exponential weights have been used successfully as a powerful tool for the stability analysis of hyperbolic systems of balance laws. In this paper we extend the class of weight functions to a family of hyperbolic functions and study the advantages in the analysis of 2×2 systems of balance laws. We present cases connected with the study of the limit of stabilizability, where the new weights provide Lyapunov functions that show exponential stability for a larger set of problem parameters than classical exponential weights.

Moreover, we show that sufficiently large time-delays influence the limit of stabilizability in the sense that the parameter set, for which the system can be stabilized becomes substantially smaller.

We also demonstrate that the hyperbolic weights are useful in the analysis of the boundary feedback stability of systems of balance laws that are governed by quasilinear hyperbolic partial differential equations.

Keywords: Lyapunov function, exponential weights, hyperbolic weights, feedback law, stabilization, boundary feedback, Riemann feedback, hyperbolic system

1. Introduction

In Coron (1999), Coron, d'Andrea Novel and Bastin (2007) and related work exponential weights in a quadratic function have been used to obtain a strict Lyapunov function for the stabilization of the Euler equation of incompressible fluids. This valuable tool has been the key to achieve numerous stabilization

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results for various systems, see, for example, the survey paper by Hayat (2012a). In Bastin and Coron (2011), basic quadratic control Lyapunov function for linearized systems was investigated. In Hayat (2021b) also nonlocal source terms are studied for semilinear systems.

In many engineering applications that involve systems, which can be modelled by hyperbolic systems of partial differential equations, the source terms in these equations play an essential role. In order to adapt the Lyapunov function candidates to this situation, in this paper we extend the exponential weights by introducing a family of hyperbolic weight functions. In the analysis, these Lyapunov functions yield additional terms that help to obtain bounds for the size of the admissible source terms, for which the system can be stabilized from the boundary.

To illustrate the results that can be achieved with the hyperbolic weight functions, we look at an example from Bastin and Coron (2016) that illustrates the limits of boundary stabilizability, see also Gugat and Gerster (2019). We show that the hyperbolic weights allow to extend the set, in which the Lyapunov function can be used to prove the exponential stability. The related numerical aspects of the boundary feedback stabilization for semilinear hyperbolic systems have been studied in Gerster et al. (2023). The limits of stabilization of a networked linear hyperbolic system with a circle have been studied in Gugat, Huang and Wang (2023).

This paper has the following structure. In Section 2 we present the example by Bastin and Coron for the limits of stabilizability. Section 3 contains the definition and properties of the hyperbolic weight functions. In Section 4 we show that Lyapunov functions with the hyperbolic weights yield sharper sufficient condition for stabilizability than the exponential weights. In Section 5 we use Lyapunov functions with linear weights to show that for feedback gains that are too large, the system becomes unstable. In Section 6 we study the influence of time delay in the boundary feedback on the stabilizability: We show that for a time-delay that is sufficiently large, the system becomes unstable even if it would be stable without time delay. In Section 7 we show that Lyapunov functions with hyperbolic weights are also useful for the study of the stability of quasilinear systems. In Section 8 we present sufficient conditions for the instability of systems of balance laws for sufiiciently long space intervals.

2. The example by Bastin and Coron

Bastin and Coron (2016) consider the following system in diagonal form:

$$(\delta_+)_t + (\delta_+)_x + \mathcal{M}\,\delta_- = 0, \tag{1}$$
$$(\delta_-)_t - (\delta_-)_x + \mathcal{M}\,\delta_+ = 0. \tag{2}$$

Here, $\mathcal{M} > 0$ is a real parameter, x is in the interval $(0, L)$, and $t \geq 0$. With the boundary conditions

$$\delta_+(t, 0) = k\, \delta_-(t, 0), \tag{3}$$

$$\delta_-(t, L) = \delta_+(t, L) \tag{4}$$

and initial states $\delta_+(0, \cdot), \delta_-(0, \cdot) \in H^1(0, L)$ the system is completed. The following proposition is shown:

Proposition 1 *If*

$$\mathcal{M}\, L \geq \pi,$$

then there is no real value of k such that the closed loop system (1), (2), (3), (4) is exponentially stable.

The example by Bastin and Coron:
A sufficient condition for stabilizability with exponential weights

The following Proposition from Gugat and Gerster (2019) is proven using a Lyapunov function with exponential weights. It states that if $L > 0$ is sufficiently small, the closed loop system is exponentially stable if $|k| < 1$.

Proposition 2 *If for $\lambda > 0$ we have $|k| \leq \mathrm{e}^{-\lambda\, L}$ and $\mathcal{M} < \frac{\lambda}{1+\mathrm{e}^{2\,\lambda\, L}}$, the closed loop system (1), (2), (3), (4) is exponentially stable for all initial states $(\delta_+(0, \cdot), \delta_-(0, \cdot)) \in (H^1(0, L))^2$.*

The proof is presented in Gugat and Gerster (2019) using the Lyapunov function

$$L(t) = \frac{1}{2} \int_0^L \exp(\lambda(L - x))\, \delta_+^2(t, x) + \exp(\lambda(x - L))\, \delta_-^2(t, x)\, dx.$$

Note that the notation in Gugat and Gerster (2019) is different, namely (U, V) instead of (δ_+, δ_-).

Proposition 2 yields stability only if the following inequality holds:

$$\mathcal{M}L < \sup_{z>0} \tfrac{z}{1+\mathrm{e}^{2\,z}} = \frac{W(\exp(-1))}{2} = 0.139... \tag{5}$$

where W is the Lambert W-function.

3. Definition and properties of the hyperbolic weight functions

In this section we define the hyperbolic weight functions that generalize the exponential weights that have been used, for example, in Coron, d'Andrea Novel and Bastin (2007). The exponential weights $\exp(\mp\psi\, x)$ occur naturally, since their derivatives can again be expressed in terms of these weights, that is, they satisfy a linear differential equation. Moreover, they can be used as weight functions since they only attain positive values.

Since $\exp(\pm\psi x) = \cosh(\psi x) \pm \sinh(\psi x)$, a natural perturbation is the weight function

$$h_\pm(x) = \sqrt{\upsilon}\cosh(\psi\, x) \mp \sinh(\psi x)$$

where $\upsilon > 0$ is chosen sufficiently large, so that only positive values are attained.

We have the representation

$$h_\pm(x) = \cosh(\psi\, x)\left[\sqrt{\upsilon} \mp \tanh(\psi\, x)\right].$$

Thus, if $\sqrt{\upsilon} > |\tanh(\psi\, L)|$, we have $h_\pm(x) > 0$ for all $x \in [-L,\, L]$ and so the functions $h_\pm(x)$ can be used as weight functions in Lyapunov functions for subintervals of $[-L, L]$. The derivatives of the weight functions can be represented as a linear combination of the weight functions.

In the following Lemma we summarize the properties of $h_\pm(x)$, and state how the derivatives can be expressed in terms of the weight functions. Note that the exponential weights

$$h_+(x) = \exp(-\psi\, x), \quad h_-(x) = \exp(\psi\, x)$$

occur as the special case $\upsilon = 1$. For $\psi \to 0+$ both h_+ and h_- converge to the constant function $\sqrt{\upsilon}$.

Figure 1 shows the graphs of h_+ and h_- for $\psi = L = 1$ and $\upsilon = \frac{3}{2}\,\tanh^2(\psi\, L)$. We have $\sqrt{\upsilon} = 0.9328...$

Figure 2 shows the graphs of h_+ and h_- for $L = 1$, $\psi = \frac{1}{2}$ and $\upsilon \in \{1,\, \tanh^2(\psi\, L)\}$. the case of $\upsilon = 1$ is the case of the exponential weights. In the extremal case of $\upsilon = \tanh^2(\psi\, L)$, the weight functions lose positivity, since $h_+(L) = 0$ and $h_-(-L) = 0$.

Lemma 1 *Let $\psi > 0$ and $\upsilon > \tanh^2(\psi\, L)$ be given. For $x \in [-L,\, L]$, define the functions*

$$h_\pm(x) = \sqrt{\upsilon}\cosh(\psi\, x) \mp \sinh(\psi\, x).$$

a) *The functions h_+ and h_- only attain values in $(0, \infty)$ for $x \in [-L,\, L]$.*

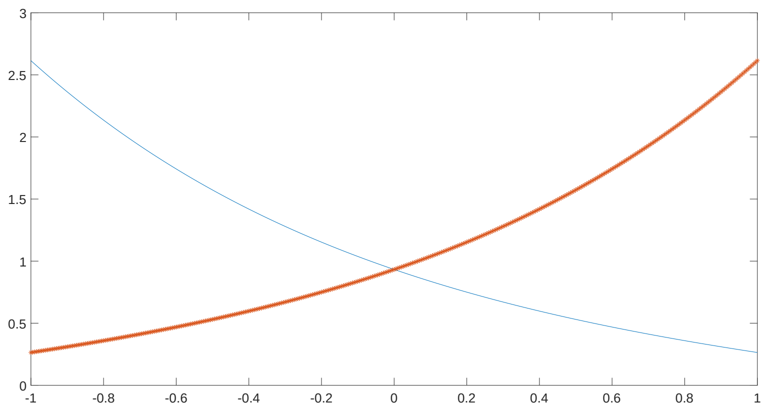

Figure 1. The weight functions h_+ and h_- for $\psi = L = 1$ and $\nu = \frac{3}{2}\tanh^2(\psi\, L)$

Since $h''_{\pm}(x) = \psi^2\, h_{\pm}(x)$, *this implies that the functions* $h_{\pm}(x)$ *are strictly convex on* $[-L,\, L]$.

b) *We have* $h_+(0) = h_-(0) = \sqrt{\upsilon}$, $h'_+(0) = -\psi$ *and* $h'_-(0) = \psi$.

c) *We have*

$$\frac{h_+(x)}{h_-(x)} = \frac{2\sqrt{\upsilon}}{\sqrt{\upsilon} + \tanh(\psi\, x)} - 1.$$

Hence, $\frac{d}{dx}\left(\frac{h_+(x)}{h_-(x)}\right) < 0$. *Thus,* $\frac{h_+(x)}{h_-(x)}$ *is decreasing and* $\frac{h_-(x)}{h_+(x)}$ *is increasing.*

For $x \in (0, L)$ *we have* $0 < \frac{h_+(x)}{h_-(x)} < \frac{h_+(0)}{h_-(0)} = 1$ *and* $\frac{h_-(x)}{h_+(x)} < \frac{h_-(L)}{h_+(L)} = \frac{\upsilon^{\frac{1}{2}} + \tanh(\psi\, L)}{\upsilon^{\frac{1}{2}} - \tanh(\psi\, L)}$. *We then have*

$$\frac{h_+(\frac{L}{2})}{h_-(\frac{L}{2})} = \frac{\sqrt{\upsilon} - \tanh(\psi\,\frac{L}{2})}{\sqrt{\upsilon} + \tanh(\psi\,\frac{L}{2})} = \frac{\sqrt{\upsilon}\left(1 + \frac{1}{\cosh(\psi\, L)}\right) - \tanh(\psi\, L)}{\sqrt{\upsilon}\left(1 + \frac{1}{\cosh(\psi\, L)}\right) + \tanh(\psi\, L)}. \tag{6}$$

d) *For* $x \in [-L, L]$ *there is*

$$h_-(x) = h_+(-x).$$

e) *We have*

$$h'_+(x) = -\psi\left[\frac{\upsilon + 1}{2\sqrt{\upsilon}}\right] h_+(x) + \psi\left[\frac{\upsilon - 1}{2\sqrt{\upsilon}}\right] h_-(x),$$

$$h'_-(x) = -\psi \left[\frac{\upsilon - 1}{2\sqrt{\upsilon}}\right] h_+(x) + \psi \left[\frac{\upsilon + 1}{2\sqrt{\upsilon}}\right] h_-(x).$$

For $\upsilon \in (\tanh^2(\psi L), 1]$, h_+ is decreasing and h_- is increasing on $[-L, L]$.

PROOF:

a) Since $\sqrt{\upsilon} > \tanh(\psi L)$ for all $x \in [-L, L]$, we have

$$h_\pm(x) = \cosh(\psi x) \left[\sqrt{\upsilon} \mp \tanh(\psi x)\right] > 0.$$

e) There is

$$h'_\pm(x) = \psi \sqrt{\upsilon} \sinh(\psi x) \mp \psi \cosh(\psi x).$$

Since

$$\cosh(\psi x) = \frac{1}{2\sqrt{\upsilon}} \left[h_+(x) + h_-(x)\right], \quad \sinh(\psi x) = \frac{1}{2} \left[-h_+(x) + h_-(x)\right]$$

, this yields

$$h'_\pm(x) = \psi \sqrt{\upsilon} \frac{1}{2} \left[-h_+(x) + h_-(x)\right] \mp \psi \frac{1}{2\sqrt{\upsilon}} \left[h_+(x) + h_-(x)\right].$$

Thus, we have

$$h'_\pm(x) = -\psi \left[\frac{\upsilon \pm 1}{2\sqrt{\upsilon}}\right] h_+(x) + \psi \left[\frac{\upsilon \mp 1}{2\sqrt{\upsilon}}\right] h_-(x).$$

Since

$$h'_\pm(x) = \mp\psi \sqrt{\upsilon} \cosh(\psi x) \left[\frac{1}{\sqrt{\upsilon}} \pm \tanh(\psi x)\right]$$

and $\frac{1}{\sqrt{\upsilon}} \geq 1$, we have $h'_+(x) < 0$ for all $x \in [-L, L]$. Moreover, there is $h'_-(x) > 0$ for all $x \in [-L, L]$.

b) We have $h_\pm(0) = \cosh(\psi 0) [\sqrt{\upsilon} \mp \tanh(\psi 0)\ ] = \sqrt{\upsilon}$, and $h'_\pm(0) = \mp\psi$.

c) There is

$$\frac{h_+(x)}{h_-(x)} = \frac{\upsilon^{\frac{1}{2}} - \tanh(\psi x)}{\upsilon^{\frac{1}{2}} + \tanh(\psi x)} = \frac{2\,\upsilon^{\frac{1}{2}}}{\upsilon^{\frac{1}{2}} + \tanh(\psi x)} - 1.$$

d) For $x \in [-L, L]$ we get

$$h_+(-x) = \cosh(-\psi x) \left[\upsilon^{\frac{1}{2}} - \tanh(-\psi x)\right] = \cosh(\psi x) \left[\upsilon^{\frac{1}{2}} + \tanh(\psi x)\right] = h_-(x).$$

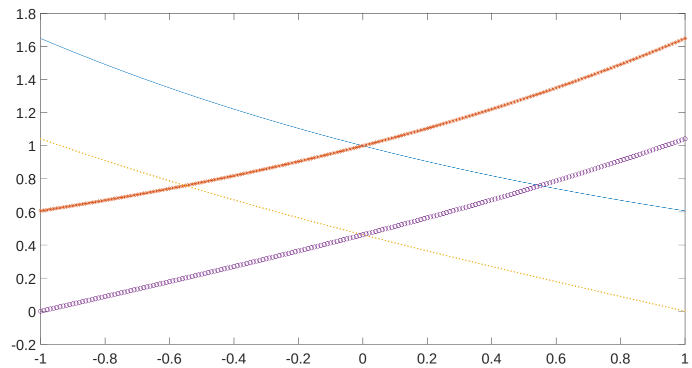

Figure 2. The weight functions h_+ and h_- for $L = 1$, $\psi = \frac{1}{2}$, $\upsilon = 1$ and $\upsilon = \tanh^2(\psi\, L)$. The upper increasing graph corresponds to h_- for the case of $\upsilon = 1$ of the exponential weights, and the upper decreasing line corresponds to h_+ for this case. In the case of $\upsilon = 1$ we have $h_+(0) = h_-(0) = 1$. The lower increasing graph corresponds to h_- for the extremal case of $\upsilon = \tanh^2(\psi L)$, and the lower decreasing line corresponds to h_+ for this extremal case. In this case $h_+(1) = h_-(1) = 0$.

4. The example by Bastin and Coron: a sufficient condition for stabilizability with hyperbolic weights

Let $\psi > 0$ be given. Define the hyperbolic weights $h_+(x) > 0$, $h_-(x) > 0$ ($x \in [-L,\, L]$) such that for $\upsilon > \tanh^2(\psi\, L)$ we have

$$h_\pm(x) = \upsilon^{\frac{1}{2}} \cosh(\psi\, x) \mp \sinh(\psi\, x).$$

For $\upsilon = 1$ we obtain the exponential weights $h_\pm(x) = \exp(\mp\psi x)$.

Define the Lyapunov candidate function

$$\mathcal{E}(t) := \frac{1}{2}\int_0^L h_+(x-L)\,|\delta_+(t,\,x)|^2 + h_-(x-L)\,|\delta_-(t,\,x)|^2\,dx. \tag{7}$$

For the time-derivative we obtain, using (1) and (2),

$$\begin{aligned}\mathcal{E}'(t) &= \int_0^L h_+(x-L)\,\delta_+\,(\delta_+)_t + h_-(x-L)\,\delta_-\,(\delta_-)_t\,dx\\ &= \int_0^L h_+(x-L)\,\delta_+\,(-(\delta_+)_x - \mathcal{M}\,\delta_-) + h_-(x-L)\delta_-((\delta_-)_x - \mathcal{M}\,\delta_+)dx\end{aligned}$$

$$= \int_0^L -h_+(x-L)\left(\tfrac{1}{2}(\delta_+)^2\right)_x + h_-(x-L)\,\left(\tfrac{1}{2}(\delta_-)^2\right)_x$$
$$-\mathcal{M}\left[h_+(x-L) + h_-(x-L)\right]\,\delta_+\,\delta_-\,dx.$$

Integration by parts yields

$$\mathcal{E}'(t) =$$
$$\tfrac{1}{2}\int_0^L h_+'(x-L)\delta_+^2 - h_-'(x-L)(\delta_-)^2 - 2\mathcal{M}\left[h_+(x-L)+h_-(x-L)\right]\delta_+\delta_-dx$$
$$+\left[\tfrac{1}{2}h_-(x-L)\,(\delta_-(t,x))^2 - \tfrac{1}{2}h_+(x-L)\,(\delta_+(t,x))^2\right]\Big|_{x=0}^{x=L}$$
$$= -\tfrac{\psi}{2}\int_0^L \left[\frac{\upsilon+1}{2\sqrt{\upsilon}}\right] h_+(x-L)\,\delta_+^2 + \left[\frac{1-\upsilon}{2\sqrt{\upsilon}}\right] h_-(x-L)\,\delta_+^2$$
$$+\left[\frac{1-\upsilon}{2\sqrt{\upsilon}}\right] h_+(x-L)\,\delta_-^2 + \left[\frac{\upsilon+1}{2\sqrt{\upsilon}}\right] h_-(x-L)\,\delta_-^2\,dx$$
$$-\int_0^L \mathcal{M}\left[h_+(x-L)+h_-(x-L)\right]\,\delta_+\,\delta_-\,dx +$$
$$\left[\tfrac{1}{2}h_-(x-L)\,(\delta_-(t,x))^2 - \tfrac{1}{2}h_+(x-L)\,(\delta_+(t,x))^2\right]\Big|_{x=0}^{x=L}.$$

If the boundary term in the last line is less than or equal to zero, for $\upsilon \in (\tanh^2(\psi\, L), 1]$, using $|z_1\, z_2| \leq \frac{|z_1|^2}{2} + \frac{|z_2|^2}{2}$, we obtain the inequality

$$\mathcal{E}'(t) \leq -\tfrac{\psi}{2}\int_0^L \left[\frac{\upsilon+1}{2\sqrt{\upsilon}} - \frac{\mathcal{M}}{\psi}\right] h_+(x-L)\delta_+^2 + \left[\frac{1-\upsilon}{2\sqrt{\upsilon}} - \frac{\mathcal{M}}{\psi}\right] h_-(x-L)\delta_+^2$$
$$+\left[\frac{1-\upsilon}{2\sqrt{\upsilon}} - \frac{\mathcal{M}}{\psi}\right] h_+(x-L)\delta_-^2 + \left[\frac{\upsilon+1}{2\sqrt{\upsilon}} - \frac{\mathcal{M}}{\psi}\right] h_-(x-L)\delta_-^2\,dx.$$

If

$$\mathcal{M} \leq \frac{\psi}{2}\,\frac{1-\upsilon}{\sqrt{\upsilon}} = \frac{\psi}{2}\left(\frac{1}{\sqrt{\upsilon}} - \sqrt{\upsilon}\right), \tag{8}$$

due to (7) this yields

$$\mathcal{E}'(t) \leq -\psi\left[\frac{\upsilon+1}{2\sqrt{\upsilon}} - \frac{1-\upsilon}{2\sqrt{\upsilon}}\right]\mathcal{E}(t) = -\psi\,\sqrt{\upsilon}\,\mathcal{E}(t).$$

By Gronwall's inequality this implies $\mathcal{E}(t) \leq \exp(-\psi\,\sqrt{\upsilon}\,t)\,\mathcal{E}(0)$, so it remains to check the negativity assumption on the boundary term. Note that (8) is equivalent to

$$\mathcal{M}L \leq \frac{\psi\, L}{2}\frac{1-\upsilon}{\sqrt{\upsilon}} = \frac{\psi L}{2}\left(\frac{1}{\sqrt{\upsilon}} - \sqrt{\upsilon}\right). \tag{9}$$

For the right–hand side of (9) we have the upper bound

$$\frac{\psi L}{2}\left(\frac{1}{\sqrt{\upsilon}}-\sqrt{\upsilon}\right)\leq \frac{1}{2}\sup_{z>0} z\left(\frac{1}{\tanh(z)}-\tanh(z)\right)=\frac{1}{2}\sup_{z>0}\frac{z}{\cosh(z)\,\sinh(z)}$$

$$=2\sup_{z>0}\frac{z}{\exp(2\,z)-\exp(-2\,z)}=2\sup_{z>0}\frac{z}{4\,z+\frac{16}{6}z^3+.....}=\frac{1}{2}.$$

So, here we obtain an upper bound for $\mathcal{M}L$ that is closer to π than the value from (5), obtained with the exponential weights.

Since $h_+(0)=h_-(0)$, for the boundary term at $x=L$ we get

$$\tfrac{1}{2}(\delta_-)^2(t,\,L)-\tfrac{1}{2}(\delta_+)^2(t,\,L)=0.$$

For the boundary term at $x=0$ we get

$$h_+(-L)\,k^2-h_-(-L)\leq 0$$

if and only if

$$k^2\leq\frac{h_-(-L)}{h_+(-L)}=\frac{h_+(L)}{h_-(L)}.$$

This is the case if $k^2\leq\frac{\sqrt{\upsilon}-\tanh(\psi\,L)}{\sqrt{\upsilon}+\tanh(\psi L)}$. To be precise, in this case it suffices that

$$k^2<B:=\sup_{\psi>0}\;\sup_{\upsilon\in(\tanh^2(\psi\,L),1)}\frac{\sqrt{\upsilon}-\tanh(\psi\,L)}{\sqrt{\upsilon}+\tanh(\psi L)}.$$

With the choice $\sqrt{\upsilon}=\frac{1+\tanh(\psi\,L)}{2}$ we have $\frac{\sqrt{\upsilon}-\tanh(\psi\,L)}{\sqrt{\upsilon}+\tanh(\psi L)}=\frac{1-\tanh(\psi\,L)}{1+3\tanh(\psi L)}$. This yields $B\geq\lim_{\psi\to 0+}\frac{1-\tanh(\psi\,L)}{1+3\tanh(\psi L)}=1$.

Thus, using the Lyapunov function with hyperbolic weights we have shown the following result:

Proposition 3 *If*

$$\mathcal{M}L<\frac{1}{2}$$

and $|k|$ *is sufficiently small (in the sense that* $|k|<1$*) the closed loop system (1), (2), (3), (4) is exponentially stable for all initial states* $(\delta_+(0,\cdot),\delta_-(0,\cdot))\in(H^1(0,\,L))^2$ *with* $\delta_+(0,\,L)=\delta_-(0,L)$ *and* $\delta_+(0,0)=k\,\delta_-(0,0)$.

Note that the stability result in Proposition 3 holds for a larger set of parameters than the bound (5) that is implied by Proposition 2.

5. The example by Bastin and Coron continued: a sufficient condition for instability with affine linear weights

In this section we introduce a Lyapunov function with affine linear weights to show that the system is unstable if $|k|$ is too large. This illustrates further the flexibility of the analysis that is based upon Lyapunov functions. Similarly as for the hyperbolic weights, the derivatives of the affine linear weights can be represented as a linear combination of affine linear weights.

Define the affine linear weights $h_\pm(x) > 0$ by

$$h_\pm(x) = 1 \pm 2\,\mathcal{M}\,x.$$

Note that in the case without the source term, that is, with $\mathcal{M} = 0$, the definition yields $h_\pm(x) = 1$, that is – onstant weights. If $\mathcal{M}\,L < \frac{1}{2}$, we have $h_\pm(x) > 0$ for all $x \in [-L, L]$. We have

$$h_+'(x) = \mathcal{M}(h_+(x) + h_-(x)),\ h_-'(x) = -\mathcal{M}(h_+(x) + h_-(x)).$$

Consider again the Lyapunov candidate function

$$\mathcal{E}(t) := \frac{1}{2}\int_0^L h_+(x-L)\,|\delta_+(t,x)|^2 + h_-(x-L)\,|\delta_-(t,x)|^2\,dx.$$

For the time-derivative we obtain, as above, using integration by parts

$$\begin{aligned}
\mathcal{E}'(t) =& \tfrac{1}{2}\int_0^L h_+'(x-L)\delta_+^2 - h_-'(x-L)\delta_-^2 \\
& - 2\mathcal{M}\left[h_+(x-L) + h_-(x-L)\right]\delta_+\delta_-\,dx \\
& + \left[\tfrac{1}{2}h_-(x-L)\,(\delta_-(t,x))^2 - \tfrac{1}{2}h_+(x-L)\,(\delta_+(t,x))^2\right]\Big|_{x=0}^{x=L} \\
& = \int_0^L \frac{\mathcal{M}}{2}\,h_+(x-L)\,\delta_+^2 + \frac{\mathcal{M}}{2}\,h_-(x-L)\,\delta_+^2 + \frac{\mathcal{M}}{2}h_+(x-L)\,\delta_-^2 + \\
& \frac{\mathcal{M}}{2}h_-(x-L)\,\delta_-^2\,dx \\
& - \int_0^L \mathcal{M}\left[h_+(x-L) + h_-(x-L)\right]\,\delta_+\,\delta_-\,dx \\
& + \left[\tfrac{1}{2}h_-(x-L)\,(\delta_-(t,x))^2 - \tfrac{1}{2}h_+(x-L)\,(\delta_+(t,x))^2\right]\Big|_{x=0}^{x=L}.
\end{aligned}$$

We obtain the equation

$$\mathcal{E}'(t) = \int_0^L \frac{\mathcal{M}}{2}\, h_+(x-L)\,(\delta_+ - \delta_-)^2 + \frac{\mathcal{M}}{2}\, h_-(x-L)(\delta_+ - \delta_-)^2\, dx$$
$$+ \left[\tfrac{1}{2} h_-(x-L)\,(\delta_-(t,x))^2 - \tfrac{1}{2} h_+(x-L)\,(\delta_+(t,x))^2\right]\Big|_{x=0}^{x=L}.$$

This yields

$$\mathcal{E}'(t) \geq \left[\tfrac{1}{2} h_-(x-L)\,(\delta_-(t,x))^2 - \tfrac{1}{2} h_+(x-L)\,(\delta_+(t,x))^2\right]\Big|_{x=0}^{x=L}.$$

Since $h_+(0) = h_-(0)$, for the boundary term at $x = L$ we get

$$\tfrac{1}{2}(\delta_-)^2(t,\, L) - \tfrac{1}{2}(\delta_+)^2(t,\, L) = 0.$$

For the boundary term at $x = 0$ we get

$$h_+(-L)\, k^2 - h_-(-L) \geq 0$$

if and only if

$$k^2 \geq \frac{h_-(-L)}{h_+(-L)} = \frac{h_+(L)}{h_-(L)}.$$

This is the case when $k^2 \geq \frac{1+2\mathcal{M}\, L}{1-2\mathcal{M}\, L}$. Then we have $\mathcal{E}'(t) \geq 0$. Thus, using the Lyapunov function with affine linear weights have shown the following result:

Proposition 4 *If*

$$\mathcal{M} L < \frac{1}{2}$$

and

$$k^2 \geq \frac{1 + 2\,\mathcal{M}\, L}{1 - 2\,\mathcal{M}\, L}$$

the closed loop system (1), (2), (3), (4) is unstable for all initial states $(\delta_+(t,0),\, \delta_-(t,0)) \in (H^1(0,\, L))^2$.

Remark 1 *Note that in the case without source term (that is,* $\mathcal{M} = 0$*) the proof also works and yields instability for all* $|k| \geq 1$ *(see Theorem 2.4 in Bastin and Caron, 2016).*

As pointed out in Bastin and Caron (2016), the results from Lichtner (2008) imply that exponential stabilization can only be achieved if $|k| < 1$*. So, the bound provided in Proposition 4 is no novelty. The novelty is the construction of the Lypunov function with affine linear weights that is used for an easy proof of the statement.*

6. The influence of time delay in the boundary feedback on stabilizability

In this section we discuss the influence of time delay on the stabilizability of the example by Bastin and Coron, that is, the closed loop system (1), (2), (3), (4). In particular, we want to know whether a sufficiently large time delay can lead to non-stabilizability for a system that is stabilizable in the case without time-delay. So we ask the question: Can a sufficiently large time-delay lead to a decrease of the critical length? At this point, it is appropriate to mention Datko's classical contributions to the study of time-delay, see Datko (1988) and Datko, Lagnese and Polis (1986), where it is shown that arbitrarily small time-delay can destabilize a system that is otherwise stable. A recent contribution on the topic for nonlinear systems is Haidar et al. (2017). Our result is of a different type: We show that if the time-delay is sufficiently large, it can make the region where stabilization is possible substantially smaller. So, time-delay influences the limits of stabilizability. This result is related to Gugat and Tucsnak (2011), where it is shown that for certain time delays appropriately chosen sufficiently small feedback gains lead to stability.

Let $\tau > 0$ be a given time delay. For $t \geq 2\,\tau$, we replace the feedback law (3) by

$$\delta_+(t,\, 0) = k\, \delta_-(t - \tau,\, 0). \tag{10}$$

To complete the system, similarly as in Gugat and Dick (2011), a compatible starting phase on the time interval $[0, 2\,\tau)$ has to be added, such that a well-defined regular system state is generated. For $t \in [0,\, 2\tau)$ we define

$$\delta_+(t,\, 0) = k\, \delta_-(t - \zeta(t),\, 0).$$

We choose $\zeta(t)$ as a smooth function with $\zeta(0) = 0$, $\zeta(2\,\tau) = \tau$, $\zeta'(2\tau) = 0$ and $\zeta(t) \leq t$ for all $t \in [0, 2\tau]$. The following proposition provides an affirmative answer to the question posed above:

PROPOSITION 5 *Let $\hat{k} > 0$ be given. If*

$$\mathcal{M}L \in \left(\frac{3}{4}\pi,\, \pi\right) \tag{11}$$

and $\tau > 0$ is sufficiently large, there is no value of $k \in (-\hat{k},\, \hat{k})$ such that the closed loop system (1), (2), (10), (4) is exponentially stable.

PROOF: To represent the state for $t \geq 2\tau$ we consider the separation ansatz

$$\delta_+(t,\, x) = \exp(\sigma\, t)\, f(x), \quad \delta_-(t,\, x) = \exp(\sigma\, t)\, g(x)$$

for the solution of (1), (2). For $\sigma \in (0,\, \mathcal{M})$ define $\omega = \sqrt{\mathcal{M}^2 - \sigma^2} > 0$. The pdes (1) and (2) imply $f''(x) + (\mathcal{M}^2 - \sigma^2) f(x) = 0$, and $g''(x) + (\mathcal{M}^2 - \sigma^2) g(x) = 0$.

Hence, the solutions have the form $f(x) = A\, \sin(\omega\, x) + B\, \cos(\omega\, x)$, $g(x) = C\, \sin(\omega\, x) + D\, \cos(\omega\, x)$ with real numbers A, B, C, D. The pdes (1) and (2) imply

$$\begin{pmatrix} \sigma & -\omega & \mathcal{M} & 0 \\ \omega & \sigma & 0 & \mathcal{M} \\ \mathcal{M} & 0 & \sigma & \omega \\ 0 & \mathcal{M} & -\omega & \sigma \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{12}$$

The feedback law (10) implies $B = f(0) = k\, \exp(-\sigma\, \tau)\, g(0) = k\, \exp(-\sigma\, \tau)\, D$. We set $D = -\omega$. Then, $B = -k \exp(-\sigma\, \tau)\, \omega$ and (12) yields

$$\begin{pmatrix} \sigma & \mathcal{M} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} \omega\, B \\ -\,\sigma B + \mathcal{M} D \end{pmatrix} = \begin{pmatrix} -k \exp(-\sigma\, \tau)\, \omega^2 \\ k \exp(-\sigma\, \tau)\, \sigma\, \omega + \mathcal{M}\omega \end{pmatrix}. \tag{13}$$

This yields $A = k \exp(-\sigma\, \tau)\, \sigma + \mathcal{M}$ and $C = -k \exp(-\sigma\, \tau)\, \mathcal{M} - \sigma$. Note that

$$\begin{pmatrix} \mathcal{M} & \sigma \\ 0 & -\omega \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -\omega\, D \\ -\mathcal{M} B - \sigma D \end{pmatrix}, \tag{14}$$

hence, (12) is satisfied. Thus, we have

$$\begin{aligned} f(x) &= (\mathcal{M} + k\, e^{-\sigma \tau} \sigma)\, \sin(\omega\, x) - k\, e^{-\sigma \tau} \omega\, \cos(\omega\, x), \\ g(x) &= -(\sigma + k\, e^{-\sigma \tau} \mathcal{M})\, \sin(\omega\, x) - \omega\, \cos(\omega\, x). \end{aligned}$$

The boundary condition (4) is equivalent to $f(L) = g(L)$.

With the choice $k = -e^{\sigma \tau}$ we have $f(L) = g(L)$ if $\sigma \in (0,\, \mathcal{M})$ is such that

$$0 = \cos(\sqrt{\mathcal{M}^2 - \sigma^2}\, L).$$

This is possible if $\mathcal{M}\, L$ is sufficiently large in the sense that

$$\mathcal{M}\, L > \pi/2$$

which follows from assumption (11). Note that in this case we have $|k| = e^{\sigma \tau} > 1$.

For $k \neq -e^{\sigma \tau}$ there is $f(L) = g(L)$ if $\sigma \in (0,\, \mathcal{M})$ is such that

$$H(\sigma, k, \tau) := (\sigma + \mathcal{M})\, \frac{\tan(\sqrt{\mathcal{M}^2 - \sigma^2}\, L)}{\sqrt{\mathcal{M}^2 - \sigma^2}} - \frac{k\, e^{-\sigma \tau} - 1}{k\, e^{-\sigma \tau} + 1} = 0. \tag{15}$$

We have

$$\lim_{\sigma \to \sqrt{\mathcal{M}^2 - \left(\frac{\pi}{2L}\right)^2}-} H(\sigma, k, \tau) = -\infty.$$

Since $\frac{\sigma + \mathcal{M}}{\sqrt{\mathcal{M}^2 - \sigma^2}} = \sqrt{\frac{\mathcal{M}+\sigma}{\mathcal{M}-\sigma}}$ for $\sigma_0 \in (0,\, \mathcal{M})$ and $k \neq -e^{\sigma_0 \tau}$, we get

$$\lim_{\sigma \to \sigma_0 +} H(\sigma, k, \tau) = \sqrt{\frac{\mathcal{M} + \sigma_0}{\mathcal{M} - \sigma_0}} \tan\left(\sqrt{\mathcal{M}^2 - \sigma_0^2}\, L \right) - \frac{k\, e^{-\sigma_0 \tau} - 1}{k\, e^{-\sigma_0 \tau} + 1}.$$

Consider the auxiliary function

$$G_{\sigma_0}(k) = \frac{k\, e^{-\sigma_0 \tau} - 1}{k\, e^{-\sigma_0 \tau} + 1}.$$

Then, G_{σ_0} is continuously differentiable on $(-e^{\sigma_0 \tau},\, \infty)$ with the derivative

$$G'_{\sigma_0}(k) = \frac{2e^{-\sigma_0 \tau}}{(k\, e^{-\sigma_0 \tau} + 1)^2} > 0.$$

Thus, G_{σ_0} is strictly increasing and for $k > -e^{\sigma_0 \tau}$ with $|k| < \hat{k}$ we have $G_{\sigma_0}(k) < G_{\sigma_0}(\hat{k})$. This implies

$$\begin{aligned} \lim_{\sigma \to \sigma_0 +} H(\sigma, k, \tau) \quad &= \sqrt{\tfrac{\mathcal{M}+\sigma_0}{\mathcal{M}-\sigma_0}} \tan(\sqrt{\mathcal{M}^2 - \sigma_0^2}\, L) - G_{\sigma_0}(k) \\ &> \sqrt{\tfrac{\mathcal{M}+\sigma_0}{\mathcal{M}-\sigma_0}} \tan(\sqrt{\mathcal{M}^2 - \sigma_0^2}\, L) - G_{\sigma_0}(\hat{k}) \\ &= \sqrt{\tfrac{\mathcal{M}+\sigma_0}{\mathcal{M}-\sigma_0}} \tan(\sqrt{\mathcal{M}^2 - \sigma_0^2}\, L) - \tfrac{\hat{k} e^{-\sigma_0 \tau} - 1}{\hat{k} e^{-\sigma_0 \tau} + 1}. \end{aligned}$$

Hence, if

$$\sqrt{\frac{\mathcal{M} + \sigma_0}{\mathcal{M} - \sigma_0}} \tan\left(\sqrt{\mathcal{M}^2 - \sigma_0^2}\, L \right) \geq \frac{\hat{k} e^{-\sigma_0 \tau} - 1}{\hat{k} e^{-\sigma_0 \tau} + 1}, \tag{16}$$

the above argument implies that for $k > -e^{\sigma_0 \tau}$ with $|k| < \hat{k}$ we have $\lim_{\sigma \to \sigma_0 +} H(\sigma, k, \tau) > 0$.

Due to (11) there is

$$\tan(\mathcal{M} L) \in (-1,\, 0).$$

For $s \in \left[0,\, \sqrt{\mathcal{M}^2 - \left(\frac{\pi}{2L}\right)^2}\right)$ define the function

$$F(s) = \sqrt{\frac{\mathcal{M} + s}{\mathcal{M} - s}} \tan\left(\sqrt{\mathcal{M}^2 - s^2}\, L \right).$$

Then, F is strictly decreasing and for $\sigma_0 \in \left(0,\ \sqrt{\mathcal{M}^2 - \left(\frac{\pi}{2\,L}\right)^2}\right)$ we have $F(\sigma_0) < F(0) = \tan(\mathcal{M}\,L) < 0$. Thus, the range of F for $\sigma_0 \in \left(0,\ \sqrt{\mathcal{M}^2 - \left(\frac{\pi}{2\,L}\right)^2}\right)$ is the interval $(-\infty,\ \tan(\mathcal{M}\,L))$. Since $\tan(\mathcal{M}\,L) + 1 > 0$, we can choose $\sigma_0 > 0$ sufficiently small such that we have $F(\sigma_0) + 1 > 0$.

Then we have

$$\lim_{\tau\to\infty} F(\sigma_0) - \frac{\hat{k}e^{-\sigma_0\tau} - 1}{\hat{k}e^{-\sigma_0\tau} + 1} = F(\sigma_0) + 1 > 0.$$

Hence, if $\tau > 0$ is sufficiently large, we get

$$F(\sigma_0) - \frac{\hat{k}e^{-\sigma_0\tau} - 1}{\hat{k}e^{-\sigma_0\tau} + 1} > 0.$$

Thus, (16) holds. Moreover, we can choose $\tau > 0$ sufficiently large, such that $e^{\sigma_0\tau} < \hat{k}$.

Therefore, for all $\sigma > \sigma_0$ and $|k| < \hat{k}$ we have $\lim_{\sigma\to\sigma_0+} H(\sigma, k, \tau) > 0$ and

$$\lim_{\sigma\to\sqrt{\mathcal{M}^2 - \left(\frac{\pi}{2\,L}\right)^2}-} H(\sigma, k, \tau) = -\infty.$$

Hence, due to the continuity of $H(\cdot, k, \tau)$, Bolzano's intermediate value theorem implies that for all $k \in (-\hat{k},\ \hat{k})$ we can find a number $\sigma \geq \sigma_0$ such that $H(\sigma, k, \tau) = 0$. This finishes the proof of Proposition 5. $\square$

7. Hyperbolic weights for quasilinear systems: stabilization

In this section we discuss how Lyapunov functions with hyperbolic weights can be used to obtain estimates for the stability domains of systems that are governed by quasilinear PDEs.

We consider a system that is governed by the isothermal Euler equations with the Riemann invariants (R_+, R_-). For the stabilization of a stationary state $(\bar{R}_+,\ \bar{R}_-)$ we consider $\delta_\pm = R_\pm - \bar{R}_\pm$. We obtain the following system in diagonal form (see Gugat and Herty, 2011):

$$\begin{pmatrix} \delta_+ \\ \delta_- \end{pmatrix}_t + \begin{pmatrix} \lambda_+(\delta_+, \delta_-) & 0 \\ 0 & \lambda_-(\delta_+, \delta_-) \end{pmatrix} \begin{pmatrix} \delta_+ \\ \delta_- \end{pmatrix}_x = \begin{pmatrix} G_+(\delta_+, \delta_-) \\ G_-(\delta_+, \delta_-) \end{pmatrix}. \tag{17}$$

The system is completed with the boundary conditions

$$\delta_+(t,\ 0) = k_0\,\delta_-(t,\ 0) \tag{18}$$

and

$$\delta_-(t,\,L) = k_L\,\delta_+(t,\,L). \tag{19}$$

The theory of semi-global solutions (see, for example, Li, Wang and Gu, 2016; Wang, 2006) states that for a given time horizon $T > 0$ and $\varepsilon_0 > 0$ there exists $\varepsilon_T > 0$ such that for all initial states that have a C^1-norm, which is less than or equal to $\varepsilon_T > 0$ and are C^1-compatible with the feedback laws (18), (19), the system has a classical solution on $[0,\,T]$ satisfying the inequalities

$$|\partial_x\left(\lambda_\pm(\cdot)\right)| \le \varepsilon_0 \tag{20}$$

and for some $d \ge c > 0$ we have

$$-d \le \lambda_-(\cdot) \le -c, \;\; c \le \lambda_+(\cdot) \le d. \tag{21}$$

Moreover, for the source term we assume that

$$|G_\pm(\delta_+,\delta_-)| \le \mathcal{M}\left(|\delta_+| + |\delta_-|\right). \tag{22}$$

For $\psi > 0$ and $\upsilon \in (\tanh^2(\psi\,L), 1]$ with the hyperbolic weights that were introduced in Section 3 define the Lyapunov candidate function

$$\mathcal{E}(t) := \frac{1}{2}\int_0^L h_+(x-\tfrac{L}{2})\,|\delta_+(t,\,x)|^2 + h_-(x-\tfrac{L}{2})\,|\delta_-(t,\,x)|^2\,dx. \tag{23}$$

For the time-derivative of $\mathcal{E}(t)$, due to (17) for $t \in [0,\,T]$, we obtain

$$\begin{aligned}
\mathcal{E}'(t) &= \int_0^L h_+(x-\tfrac{L}{2})\,\delta_+\,(\delta_+)_t + h_-(x-\tfrac{L}{2})\,\delta_-\,(\delta_-)_t\,dx\\
&= \int_0^L h_+(x-\tfrac{L}{2})\,\delta_+\left[-\lambda_+(\delta)\,(\delta_+)_x + G_+(\delta)\right]\\
&+ h_-(x-\tfrac{L}{2})\,\delta_-\left[-\lambda_-(\delta)\,(\delta_-)_x + G_-(\delta)\right]dx\\
&= \int_0^L -h_+(x-\tfrac{L}{2})\,\lambda_+(\delta)\,\left(\tfrac{\delta_+^2}{2}\right)_x - h_-(x-\tfrac{L}{2})\,\lambda_-(\delta)\,\left(\tfrac{\delta_-^2}{2}\right)_x\\
&+ \left[\delta_+\,G_+(\delta)\,h_+(x-\tfrac{L}{2}) + \delta_-\,G_-(\delta)\,h_-(x-\tfrac{L}{2})\right]\,dx.
\end{aligned}$$

Now, (22) and $|\delta_+\,\delta_-| \le \frac{|\delta_+|^2}{2} + \frac{|\delta_-|^2}{2}$ yield the inequality

$$\begin{aligned}
&\int_0^L \left[\delta_+\,G_+(\delta)\,h_+(x-\tfrac{L}{2}) + \delta_-\,G_-(\delta)\,h_-(x-\tfrac{L}{2})\right]\,dx\\
&\le \int_0^L \mathcal{M}\,h_+(x-\tfrac{L}{2})\left(\tfrac{3}{2}\delta_+^2 + \tfrac{1}{2}\delta_-^2\right) + \mathcal{M}\,h_-(x-\tfrac{L}{2})\left(\tfrac{1}{2}\delta_+^2 + \tfrac{3}{2}\delta_-^2\right)\,dx.
\end{aligned}$$

With integration by parts and Lemma 1 e) the above inequalities yield

$$
\begin{aligned}
\mathcal{E}'(t) = & \int_0^L \left[h_+'(x-\tfrac{L}{2})\,\lambda_+(\delta) + h_+(x-\tfrac{L}{2})\,\partial_x\lambda_+(\delta)\right]\tfrac{\delta_+^2}{2} \\
& + \left[h_-'(x-\tfrac{L}{2})\,\lambda_-(\delta) + h_-(x-\tfrac{L}{2})\,\partial_x\lambda_-(\delta)\right]\tfrac{\delta_-^2}{2}dx \\
& + \left[-h_+(x-\tfrac{L}{2})\,\lambda_+(\delta)\,\tfrac{\delta_+^2}{2} - h_-(x-\tfrac{L}{2})\,\lambda_-(\delta)\,\tfrac{\delta_-^2}{2}\Big|_{x=0}^{x=L}\right] \\
& + \int_0^L \left[\delta_+\,G_+(\delta)\,h_+(x-\tfrac{L}{2}) + \delta_-\,G_-(\delta)\,h_-(x-\tfrac{L}{2})\right]\,dx \\
& \leq -\tfrac{\psi}{2}\int_0^L \left[\frac{1+\upsilon}{2\sqrt{\upsilon}}\right] h_+(x-\tfrac{L}{2})\,\lambda_+(\delta)\,\delta_+^2 + \left[\frac{1-\upsilon}{2\sqrt{\upsilon}}\right] h_-(x-\tfrac{L}{2})\lambda_+(\delta)\,\delta_+^2 \\
& + \left[\frac{1-\upsilon}{2\sqrt{\upsilon}}\right] h_+(x-\tfrac{L}{2})\,|\lambda_-(\delta)|\,\delta_-^2 + \left[\frac{1+\upsilon}{2\sqrt{\upsilon}}\right] h_-(x-\tfrac{L}{2})\,|\lambda_-(\delta)|\,\delta_-^2\,dx \\
& + \int_0^L h_+(x-\tfrac{L}{2})\,\partial_x\lambda_+(\delta)\,\tfrac{\delta_+^2}{2} + h_-(x-\tfrac{L}{2})\,\partial_x\lambda_-(\delta)\,\tfrac{\delta_-^2}{2}\,dx \\
& + \int_0^L \mathcal{M}\,h_+(x-\tfrac{L}{2})\left(\tfrac{3}{2}\delta_+^2 + \tfrac{1}{2}\delta_-^2\right) + \mathcal{M}\,h_-(x-\tfrac{L}{2})\left(\tfrac{1}{2}\delta_+^2 + \tfrac{3}{2}\delta_-^2\right)\,dx \\
& + \left[-h_+(x-\tfrac{L}{2})\,\lambda_+(\delta)\,\tfrac{\delta_+^2}{2} - h_-(x-\tfrac{L}{2})\,\lambda_-(\delta)\,\tfrac{\delta_-^2}{2}\Big|_{x=0}^{x=L}\right].
\end{aligned}
$$

If the boundary term in the last line is less than or equal to zero, since $\upsilon \in (\tanh^2(\psi\,L), 1]$, due to Lemma 1 and (20), we obtain the inequality

$$
\begin{aligned}
& \mathcal{E}'(t) \leq \\
& -\tfrac{\psi}{2}\int_0^L \left[c\frac{1+\upsilon}{2\sqrt{\upsilon}} - 3\frac{\mathcal{M}}{\psi} - \frac{\varepsilon_0}{\psi}\right] h_+(x-\tfrac{L}{2})\,\delta_+^2 + \left[c\,\frac{1-\upsilon}{2\sqrt{\upsilon}} - \frac{\mathcal{M}}{\psi}\right] h_-(x-\tfrac{L}{2})\,\delta_+^2 \\
& + \left[c\frac{1-\upsilon}{2\sqrt{\upsilon}} - \frac{\mathcal{M}}{\psi}\right] h_+(x-\tfrac{L}{2})(\delta_-)^2 + \left[c\,\frac{1+\upsilon}{2\sqrt{\upsilon}} - 3\frac{\mathcal{M}}{\psi} - \frac{\varepsilon_0}{\psi}\right] h_-(x-\tfrac{L}{2})(\delta_-)^2 dx.
\end{aligned}
$$

If

$$
3\,\mathcal{M} + \varepsilon_0 \leq c\,\psi\,\frac{1-\upsilon}{2\,\sqrt{\upsilon}} = \frac{c}{2}\,\psi\left(\frac{1}{\sqrt{\upsilon}} - \sqrt{\upsilon}\right) \tag{24}
$$

we have

$$
\frac{\mathcal{M}}{\psi} \leq c\,\frac{1-\upsilon}{2\,\sqrt{\upsilon}}
$$

and this yields

$$\begin{aligned}
&\mathcal{E}'(t) \\
&\le -\psi \frac{c}{2} \left[\frac{1+\upsilon}{2\sqrt{\upsilon}} - \frac{1-\upsilon}{2\sqrt{\upsilon}} \right] \int_0^L h_+(x - \tfrac{L}{2}) |\delta_+(t,\, x)|^2 + h_-(x - \tfrac{L}{2}) |\delta_-(t,\, x)|^2 \, dx \\
&= -c\psi\sqrt{\upsilon}\mathcal{E}(t).
\end{aligned}$$

By Gronwall's inequality this implies $\mathcal{E}(t) \le \exp(-c\,\psi\,\sqrt{\upsilon}\,t)\,\mathcal{E}(0)$, so it remains to check the assumption on the boundary term. Below we will derive sufficient conditions that are stated in (26).

Note that (24) is equivalent to

$$3\mathcal{M}L + \varepsilon_0 L \le \frac{c}{2} \psi\, L \frac{1-\upsilon}{\sqrt{\upsilon}} = \frac{c}{2}\, \psi L \left(\frac{1}{\sqrt{\upsilon}} - \sqrt{\upsilon} \right). \tag{25}$$

For the right-hand-side of (25) we have the upper bound

$$\frac{c}{2} \sup_{\psi > 0} \sup_{\upsilon \in (\tanh^2(\psi\, L), 1]} \psi L \left(\frac{1}{\sqrt{\upsilon}} - \sqrt{\upsilon} \right) = \frac{c}{2} \sup_{z > 0} z \left(\frac{1}{\tanh(z)} - \tanh(z) \right) = \frac{c}{2}.$$

(Note that the value $\frac{c}{2}$ is attained in the limit $z \to 0$.)

So, we obtain an upper bound for the values of $\mathcal{M}L$, for which we can guarantee that the system is exponentially stable, namely

$$3\mathcal{M}L + \varepsilon_0 L < \frac{c}{2}.$$

This bound is useful in the application in gas dynamics, where c and d are related to the sound speed that is quite large, see Banda, Herty and Klar (2006).

The condition on the boundary terms

$$\begin{aligned}
&h_+(-\tfrac{L}{2})\, \lambda_+(\delta(0))\, \tfrac{k_0^2 \delta_-^2(0)}{2} + h_-(-\tfrac{L}{2})\, \lambda_-(\delta(0))\, \tfrac{\delta_-^2(0)}{2} \\
&-h_+(\tfrac{L}{2})\, \lambda_+(\delta(L))\, \tfrac{\delta_+^2(L)}{2} - h_-(\tfrac{L}{2})\, \lambda_-(\delta(L))\, \tfrac{k_L^2 \delta_+^2(L)}{2} \\
&\le 0
\end{aligned}$$

is satisfied if

$$h_+(-\tfrac{L}{2})\, \lambda_+(\delta(0))\, k_0^2 \le -h_-(-\tfrac{L}{2})\, \lambda_-(\delta(0))$$

and

$$-h_-(\tfrac{L}{2})\, \lambda_-(\delta(L))\, k_L^2 \le h_+(\tfrac{L}{2})\, \lambda_+(\delta(L)).$$

This is equivalent to

$$k_0^2 \le \frac{h_-(-\frac{L}{2})}{h_+(-\frac{L}{2})}\,\frac{|\lambda_-(\delta(0))|}{\lambda_+(\delta(0))} = \frac{h_+(\frac{L}{2})}{h_-(\frac{L}{2})}\,\frac{|\lambda_-(\delta(0))|}{\lambda_+(\delta(0))},\quad k_L^2 \le \frac{h_+(\frac{L}{2})}{h_-(\frac{L}{2})}\,\frac{\lambda_+(\delta(L))}{|\lambda_-(\delta(L)|)}.$$

Sufficient conditions are (with $i \in \{0,\, L\}$)

$$k_i^2 \le \frac{c}{d}\frac{h_+(\frac{L}{2})}{h_-(\frac{L}{2})} = \frac{c}{d}\,\frac{\sqrt{\upsilon}\left(1+\frac{1}{\cosh(\psi\, L)}\right) - \tanh(\psi\, L)}{\sqrt{\upsilon}\left(1+\frac{1}{\cosh(\psi\, L)}\right) + \tanh(\psi\, L)}, \tag{26}$$

where the last equality follows from (6).

Thus, we have shown the following results for the boundary control of the quasilinear system:

Theorem 1 *Assume that the source term $G_\pm$ satisfies (22).*

Assume that the closed loop system (17), (18), (19) has a classical solution on $[0,\, T]$, such that (20) and (21) hold for all $t \in [0,\, T]$, that we have

$$(2\,\varepsilon_0 + 6\,\mathcal{M})\ L < c \tag{27}$$

and the feedback gains $|k_0|$ and $|k_L|$ are sufficiently small (for example, that (26) holds).

Then the closed loop system (17), (18), (19) decays exponentially fast on $[0,\, T]$. If $\psi > 0$ and $\upsilon \in (\tanh^2(\psi\, L), 1]$ are chosen such that (25) holds, the Lyapunov function $\mathcal{E}(t)$ decays exponentially with the rate

$$c\,\psi\,\sqrt{\upsilon}.$$

Remark 2 *In order to make sure that the system is globally well-posed for the time interval $[0,\, \infty)$, also Lyapunov functions for the first and second derivatives can be considered. This yields the exponential decay of solutions with values in $H^2(0, L)$, see Gugat, Leugering and Wang (2017) and Hayat and Shang (2021). Solutions in H^2 are also studied in Bastin and Coron (2016).*

In order to show the exponential decay of an H^1-Lyapunov function, we have to assume that G has partial derivatives that are continuous and an additional assumption for the derivatives of the source term is necessary:

$$|\partial_\pm G_\pm(\delta_+, \delta_-)| \le \mathcal{M}\left(|\delta_+| + |\delta_-|\right). \tag{28}$$

Similarly, in order to show the exponential decay of an H^2-Lyapunov function, we have to assume that G has second order partial derivatives that are continuous and an additional assumption for the second derivatives of the source term is necessary:

$$z^\top \begin{pmatrix} \partial_{++}G_\pm & \partial_{+-}G_\pm \\ \partial_{+-}G_\pm & \partial_{--}G_\pm \end{pmatrix} z \le \mathcal{M}. \tag{29}$$

Remark 3 *Condition (27) requires that L be sufficiently small. Since often in the applications the right hand side c can be chosen proportional to the sound speed, (27) is valid for interesting lengths. The bound ε_0 can often be chosen quite small in the applications, since only small changes in the states occur. Horizontal pipes with anti-fricion coating allow for small values of $\mathcal{M}$.*

Compared with Lemma 5.2. in Gugat and Herty (2011), conditions (27), (24) and (26) have the advantage that they can be verified more easily.

8. Hyperbolic weights for quasilinear systems: instability

In this section we show that for sufficiently large values of L, boundary feedback stabilization of quasilinear hyperbolic systems in general is not possible.

We assume that there exists a number $\mathcal{N} > \varepsilon_0 > 0$ such that for the source term, we have

$$\delta_\pm\, G_\pm(\delta_+, \delta_-) \ge \mathcal{N} |\delta_\pm|^2. \tag{30}$$

Then there is

$$\begin{aligned}
\mathcal{E}'(t) \quad \ge \quad & -\tfrac{\psi}{2} \int_0^L \left[\frac{1+\upsilon}{2\sqrt{\upsilon}}\right] h_+(x - \tfrac{L}{2})\, \lambda_+(\delta)\, \delta_+^2 + \left[\frac{1-\upsilon}{2\sqrt{\upsilon}}\right] h_-(x-\tfrac{L}{2}) \lambda_+(\delta)\, \delta_+^2 \\
+ \quad & \left[\frac{1-\upsilon}{2\sqrt{\upsilon}}\right] h_+(x-\tfrac{L}{2})\, |\lambda_-(\delta)|\, \delta_-^2 + \left[\frac{1+\upsilon}{2\sqrt{\upsilon}}\right] h_-(x-\tfrac{L}{2})\, |\lambda_-(\delta)|\, \delta_-^2 \, dx \\
+ \quad & \int_0^L h_+(x-\tfrac{L}{2})\, \partial_x \lambda_+(\delta)\, \tfrac{\delta_+^2}{2} + h_-(x-\tfrac{L}{2})\, \partial_x\lambda_-(\delta)\, \tfrac{\delta_-^2}{2}\, dx \\
+ \quad & \int_0^L \mathcal{N}\, h_+(x-\tfrac{L}{2})\, \delta_+^2 + \mathcal{N}\, h_-(x-\tfrac{L}{2})\, \delta_-^2\, dx \\
+ \quad & \left[-h_+(x-\tfrac{L}{2})\, \lambda_+(\delta)\, \tfrac{\delta_+^2}{2} - h_-(x-\tfrac{L}{2})\, \lambda_-(\delta)\, \tfrac{\delta_-^2}{2} \Big|_{x=0}^{x=L} \right].
\end{aligned}$$

With the choice of $\upsilon = 1$, due to Lemma 1 and (20), we obtain the inequality

$$\begin{aligned}
\mathcal{E}'(t) \geq & -\tfrac{\psi}{2} \int_0^L \left[c\, \frac{1+\upsilon}{2\sqrt{\upsilon}} - \frac{\mathcal{N}}{\psi} + \frac{\varepsilon_0}{\psi} \right] h_+(x - \tfrac{L}{2})\, \delta_+^2 + \left[c\, \frac{1-\upsilon}{2\sqrt{\upsilon}} \right] h_-(x - \tfrac{L}{2})\, \delta_+^2 \\
& + \left[c\, \frac{1-\upsilon}{2\sqrt{\upsilon}} \right] h_+(x - \tfrac{L}{2})\, (\delta_-)^2 + \left[c\, \frac{1+\upsilon}{2\sqrt{\upsilon}} - \frac{\mathcal{N}}{\psi} + \frac{\varepsilon_0}{\psi} \right] h_-(x - \tfrac{L}{2})\, (\delta_-)^2 \, dx \\
& + \left[-h_+(x - \tfrac{L}{2})\, \lambda_+(\delta)\, \tfrac{\delta_+^2}{2} - h_-(x - \tfrac{L}{2})\, \lambda_-(\delta)\, \tfrac{\delta_-^2}{2} \Big|_{x=0}^{x=L} \right] \\
\geq & \; \tfrac{\psi}{2} \int_0^L \left[\frac{\mathcal{N}}{\psi} - \frac{\varepsilon_0}{\psi} - c\, \frac{1+\upsilon}{2\sqrt{\upsilon}} \right] h_+(x - \tfrac{L}{2})\, \delta_+^2 + \\
& \quad \left[\frac{\mathcal{N}}{\psi} - \frac{\varepsilon_0}{\psi} - c\, \frac{1+\upsilon}{2\sqrt{\upsilon}} \right] h_-(x - \tfrac{L}{2})\, (\delta_-)^2 \, dx \\
& + \left[-h_+(x - \tfrac{L}{2})\, \lambda_+(\delta)\, \tfrac{\delta_+^2}{2} - h_-(x - \tfrac{L}{2})\, \lambda_-(\delta)\, \tfrac{\delta_-^2}{2} \Big|_{x=0}^{x=L} \right]. \qquad (31)
\end{aligned}$$

For $\psi > 0$ we study the inequality

$$-\tfrac{\psi}{2} \left[c\, \frac{1+\upsilon}{2\sqrt{\upsilon}} - \frac{\mathcal{N}}{\psi} + \frac{\varepsilon_0}{\psi} \right] > 0. \qquad (32)$$

Let $\eta > 0$ be given. With the choice of $\psi = \frac{\eta}{L}$, due to $\upsilon = 1$, (32) yields

$$\tfrac{\eta}{2\,L} \left[\frac{L}{\eta} (\mathcal{N} - \varepsilon_0) - c \right] > 0. \qquad (33)$$

Inequality (33) is equivalent to

$$(\mathcal{N} - \varepsilon_0) L > \eta\, c. \qquad (34)$$

The last term in (31) is greater than or equal to zero if

$$h_+(-\tfrac{L}{2})\, \lambda_+(\delta(0))\, k_0^2 \geq -h_-(-\tfrac{L}{2})\, \lambda_-(\delta(0))$$

and

$$-h_-(\tfrac{L}{2})\, \lambda_-(\delta(L))\, k_L^2 \geq h_+(\tfrac{L}{2})\, \lambda_+(\delta(L)).$$

This is equivalent to

$$k_0^2 \geq \frac{h_-(-\frac{L}{2})}{h_+(-\frac{L}{2})}\, \frac{|\lambda_-(\delta(0))|}{\lambda_+(\delta(0))} = \frac{h_+(\frac{L}{2})}{h_-(\frac{L}{2})}\, \frac{|\lambda_-(\delta(0))|}{\lambda_+(\delta(0))}, \;\; k_L^2 \geq \frac{h_+(\frac{L}{2})}{h_-(\frac{L}{2})}\, \frac{\lambda_+(\delta(L))}{|\lambda_-(\delta(L)|)}.$$

Sufficient conditions are (with $i \in \{0,\, L\}$):

$$k_i^2 \geq \frac{d}{c}\frac{h_+(\frac{L}{2})}{h_-(\frac{L}{2})} = \frac{d}{c}\frac{\sqrt{\upsilon}\left(1+\frac{1}{\cosh(\psi\, L)}\right) - \tanh(\psi\, L)}{\sqrt{\upsilon}\left(1+\frac{1}{\cosh(\psi\, L)}\right) + \tanh(\psi\, L)}, \tag{35}$$

where the last equation follows from (6). Note that we have

$$\lim_{L\to\infty,\, \upsilon\in(\tanh^2(\psi\, L),1]} \frac{\sqrt{\upsilon}\left(1+\frac{1}{\cosh(\psi\, L)}\right) - \tanh(\psi\, L)}{\sqrt{\upsilon}\left(1+\frac{1}{\cosh(\psi\, L)}\right) + \tanh(\psi\, L)} = 0.$$

In particular, for $\psi = \frac{\eta}{L}$ and $\upsilon = 1$, inequality (35) yields the sufficient condition

$$k_i^2 \geq \frac{d}{c}\exp\left(-\eta\right).$$

Hence, for all $k_0 \neq 0$ and $k_L \neq 0$, if L is sufficiently large, we obtain the inequality

$$\mathcal{E}'(t) \geq 0.$$

Thus, $\mathcal{E}(t)$ does not decrease and so the system is not asymptotically stable. Hence, we have shown the following proposition about the instability of the quasilinear system for large values of L:

Theorem 2 *Assume that the closed loop system (17), (18), (19) has a classical solution on* $[0,\, T]$ *such that (20) and (21) hold for all* $t \in [0,\, T]$ *and the feedback gains satisfy* $|k_0| \neq 0$ *and* $|k_L| \neq 0$. *We assume that there exists a number* $\mathcal{N} > \varepsilon_0 > 0$ *such that the source term satisfies (30).*

Then, if L *is sufficiently large, the closed loop system (17), (18), (19) does not decay on* $[0,\, T]$.

To be precise, if $\psi > 0$ *and* $\upsilon = 1$ *are chosen such that (32) and (35) hold, the Lyapunov function* $\mathcal{E}(t)$ *(see 23) does not decrease on* $[0, T]$.

A sufficient condition for instability with a real parameter $\eta > 0$ *is*

$$(\mathcal{N} - \varepsilon_0)\, L > \eta\, c,\; |k_\iota|^2 \geq \frac{d}{c}\exp(-\eta),\; (\iota \in \{0,\, L\}). \tag{36}$$

Remark 4 *In the sufficient condition (36), the left-hand side of the first inequality grows linearly with* L, *similarly as in (27).*

9. Conclusions

We have introduced hyperbolic weights for quadratic Lyapunov functions. We have shown that in certain cases they yield larger stability regions than exponential weights. We have studied this in the context of stabilization problems. We expect the hyperbolic weights to be also useful in the analysis of the synchronization of observers, see Gugat, Giesselmann and Kunkel (2021). It would be very interesting to investigate whether the hyperbolic weights are also suitable for the analysis of networked systems, see, for example, Gugat and Gerster (2019).

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