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Exact controllability of a string to rest with a moving boundary*

by

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Dedicated to Günter Leugering on the occasion of His 65th birthday

Abstract: We consider the problem of steering a finite string to the zero state in finite time from a given initial state by controlling the state at one boundary point while the other boundary point moves. As a possible application we have in mind the optimal control of a mining elevator, where the length of the string changes during the transportation process. During the transportation process, oscillations of the elevator-cable can occur that can be damped in this way.

We present an exact controllability result for Dirichlet boundary control at the fixed end of the string that states that there exist exact controls for which the oscillations vanish after finite time. For the result we assume that the movements are Lipschitz continuous with a Lipschitz constant, whose absolute value is smaller than the wave speed. In the result, we present the minimal time, for which exact controllability holds, this time depending on the movement of the boundary point. Our results are based upon travelling wave solutions. We present a representation of the set of successful controls that steer the system to rest after finite time as the solution set of two point-wise equalities. This allows for a transformation of the optimal control problem to a form where no partial differential equation appears. This representation enables interesting insights into the structure of the successful controls. For example, exact bang-bang controls can only exist if the initial state is a simple function and the initial velocity is zero.

Keywords: pde constrained optimization, optimal control of pdes, optimal boundary control, wave equation, analytic solution, exact controllability, moving boundaries, mining elevator

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1. Introduction

1.1. The problem outline

The movement of the cable of a mining elevator can be modeled by the wave equation. This is our motivation for considering a string of finite length that is governed by the wave equation. The feedback control of a dual-cable mining elevator has recently been considered in Wang, Pi and Krstic (2018), Wang et al. (2018). Related control problems for a nonlinear moving string are considered in He, Ge and Huang (2015). Here we study a control problem for an elevator with one cable that is modeled as a linear string. The string is controlled through the boundary values at one end of the string (one–point Dirichlet control). The boundary control of the wave equation has been studied by many authors and results concerning exact controllability are well known. The method of moments is an important tool for the analysis of this system (see, e.g., Russell, 1967; Lions, 1988; Krabs, 1982, 1992; Avdonin and Ivanov, 1995, and the references therein). Also the controllability of the discretized problems and the relation between the optimal controls for the continuous and the discrete case have been investigated, see Zuazua (2004). The stabilization of the wave equation on 1-d networks has been studied in Valein and Zuazua (2009). In Gugat and Sigalotti (2010), switching boundary feedback stabilization for stars of vibrating strings has been considered. The approximation of Dirichlet boundary control problems for the wave equation on curved domains is investigated in Gugat and Sokolowski (2013).

A related problem of one–point time optimal control has been solved in Malanowski (1969), where the control functions are assumed to have a second derivative whose norm is constrained. The vibrating string with one moving boundary point has also been considered in Gugat (2007) for the case of homogeneous Dirichlet boundary conditions. In Gugat (2007) the movement is used as a control and the aim is to find movements such that at the terminal time the string has the same length as at the beginning and the terminal energy is minimal. It turns out that with the movement as a control it is not always possible to reduce the energy. Also in Gugat (2008), a vibrating string with one moving boundary point has been studied with a Neumann velocity–feedback control that acts at the moving boundary point. Similarly as for the string with fixed length, also in the case of a string with a moving boundary point for a certain feedback parameter the energy vanishes after finite time.

The well-posedness of the wave equation in a non-cylindrical, time periodical domain has been studied in Truchi and Zolesio (1988). In Bardos and Chen (1981), distributed control of the wave equation in a domain with moving boundary has been considered.

In this paper, in contrast to Gugat (2008), we consider a string where the feedback acts at the fixed boundary point. We consider the exact controllability of the system. Our main interest is to study the structure of the controls that steer the cable to a position of rest at the terminal time where the movement of the elevator terminates. We give an explicit representation of the successful controls in terms of the given initial data for a system where one boundary point is not fixed, but can be moved as the time proceeds and at the moving boundary point we have homogeneous Dirichlet boundary conditions and at the fixed boundary point we have a Dirichlet control. The controllability for a wave equation with a boundary point, for which the movement is given by a linear function has been studied in Cui and Song (2017). In this paper, we consider a more general class of movements that is described by Lipschitz continuous functions.

From a given initial state, where the position and the integral of the velocity are given by a Lebesgue–integrable function, the system is controlled to the zero state in a given finite time.

To guarantee that this control problem is solvable for all initial states, the control time has to be greater than or equal to two times the time that a wave needs to travel from one end of the string to the other (the characteristic time). In our case, this time depends upon the movements of the boundaries. In Theorem 1 we give an exact controllability result where the initial states that can be steered to zero with boundary controls from the spaces L^p $(p \in [1, \infty])$ are characterized: These are the initial states where the initial position and the integral of the initial velocity are functions in the spaces L^p on the space interval.

The requirement that the target state be reached in the given terminal time does not determine a unique solution. So, we can choose from the set of successful controls a point that minimizes our objective function, which is the L^p–norm of the controls.

In Gugat (2002), Gugat and Leugering (2002), and Gugat, Leugering and Sklyar (2005) we have studied the related problem with fixed boundaries to steer the system from the zero state to a given terminal state in such a way that the L^p-norm $(p \in [2, \infty])$ of the control functions is minimized. In these papers, the method of moments and Fourier–series have been used in the proofs. In Gugat (2005), the L^1–case is analyzed with the method of characteristics that we also use in the present paper. Note that in contrast to the L^1–case, for $p \in (1, \infty)$ the corresponding optimal controls are uniquely determined.

This paper has the following structure: We define the optimal control problem and some important auxiliary variables, for example the characteristic time. Then, the problem is transformed and reformulated in terms of the Riemann invariants. For this purpose, we use the d'Alembert solution of the wave equation. After the introduction of auxiliary functions as variables in the optimization problem, the exact controllability result of Theorem 1 can be proved.

In Theorem 2, we give the solution of the initial boundary value problem with oscillating boundaries in explicit form. With this travelling waves representation of the solution, we can characterize the set of admissible control by two pointwise linear equalities that are stated in Theorem 4. This allows for transforming the optimal control problem to a representation, in which no partial differential equation appears.

1.2. Notation

For $T > 0$ and $p \in [1, \infty)$ let

$$\|u\|_{p,(0,T)} = \left(\int_0^T |u(t)|^p \, dt \right)^{1/p}$$

and let $L^p(0, T)$ denote the corresponding space of Lebesgue–integrable functions on the interval $(0, T)$, for which the p–norm is finite. For $p = \infty$ let

$$\|u\|_{\infty,(0,T)} = \text{ess} \sup_{t \in (0,\, T)} |u(t)|$$

with the corresponding space $L^\infty(0,\, T)$ of essentially bounded measurable functions.

2. The optimal control problem

At the initial time zero, the string has the length $L > 0$. Let the time $T > 0$ and the wave speed $c > 0$ be given. Let $p \in [1, \infty]$ be given. Let $y_0 \in L^p(0,\, L)$ and y_1 be given such that the function $x \mapsto \int_0^x y_1(s)\, ds$ is in $L^p(0,\, L)$.

Let a number $\delta_0 > 0$ be given. Let the map $\phi : [0, T] \mapsto [\delta_0,\, \infty)$ be Lipschitz continuous with $\phi(0) = L$ and assume that there exists a number $\delta_1 \in (0,\, c)$ such that for all $a,\, b \in [0,\, T]$ we have

$$|\phi(b) - \phi(a)| \leq (c - \delta_1)\, |b - a|. \tag{1}$$

We consider the problem

$$\mathbf{P} : \qquad \text{minimize } \|u\|_{p,(0,T)} \text{ subject to } u \in L^p(0, T) \text{ and} \tag{2}$$

$$y(x, 0) = y_0(x),\ y_t(x, 0) = y_1(x),\ x \in (0,\, L) \tag{3}$$

$$y(0,\, t) = u(t),\ y(\phi(t), t) = 0,\ t \in (0, T) \tag{4}$$

$$y_{tt}(x, t) = c^2 y_{xx}(x, t),\ (x, t) \in \Omega = \{(x, t) :\ t \in (0, T),\ x \in (0,\ \phi(t))\} \tag{5}$$

$$y(x, T) = 0,\ y_t(x, T) = 0,\ x \in (0, \phi(T)). \tag{6}$$

To show that $\mathbf{P}$ has a solution, we first have to study the exact controllability properties of the system.

3. Definition of the characteristic times

Define

$$t_0 = 0.$$

For $t \geq 0$, define the Lipschitz continuous function

$$\psi_1(t) = \phi(t) - ct. \tag{7}$$

Since for all a, $b \in [0, T]$ with $a < b$ we have the inequality

$$\begin{aligned} \psi_1(b) - \psi_1(a) &= \phi(b) - \phi(a) - c\,(b-a) \\ &\leq |\phi(b) - \phi(a)| - c\,(b-a) \\ &\leq [(c-\delta_1) - c]\,(b-a) \\ &= -\delta_1\,(b-a) < 0. \end{aligned}$$

Hence, ψ_1 is strictly decreasing on $[0, \infty)$ and thus invertible. Moreover, ψ_1^{-1} is also Lipschitz continuous. Since $\psi_1(0) = L$, we have $\psi_1^{-1}(L) = 0$. Define the characteristic time

$$t_1 = \psi_1^{-1}(0)$$

that a characteristic curve starting at time $t_0 = 0$ at $x = 0$ needs to travel to the other end of the string, i.e. to the point $x = \phi(t_1)$. (Note that $0 + ct_1 = \phi(t_1)$.) Assumption (1) implies that

$$t_1 > \frac{1}{2}\,\frac{L}{c}.$$

This can be seen as follows. Since

$$|\phi(t) - \phi(0)| \leq (c - \delta_1)\,t$$

we have

$$\psi_1(t) = \phi(t) - c\,t \geq \phi(0) - (c-\delta_1)\,t - ct = L - (2c - \delta_1)t.$$

Hence, $\psi_1(t_1) = 0$ implies that $0 > L - 2\,c\,t_1$, an so $t_1 > \frac{L}{2\,c}$.

We also define the Lipschitz continuous function

$$\psi_2(t) = \phi(t) + ct. \tag{8}$$

Assumption (1) implies that we have

$$\begin{aligned} \psi_2(b) - \psi_2(a) &= \phi(b) - \phi(a) + c\,(b-a) \\ &\geq c\,(b-a) - |\phi(b) - \phi(a)| \\ &\geq c\,(b-a) - (c-\delta_1)\,(b-a) \\ &= \delta_1\,(b-a) > 0. \end{aligned}$$

Hence, ψ_2 is strictly increasing on $[0, \infty)$ and thus invertible. Moreover, the inverse function ψ_1^{-1} is also Lipschitz continuous. Note that for all $t \geq 0$ we have

$$-\psi_1(t) \leq \psi_2(t) \text{ and } \psi_2(t) = \psi_1(t) + 2ct = -\psi_1(t) + 2\phi(t).$$

The characteristic time when a characteristic curve starting at time $t = t_1$ from the end $\phi(t_1)$ of the string arrives at the end zero is given by

$$t_2 = \frac{1}{c}\,\psi_2(\psi_1^{-1}(0)).$$

Assumption (1) implies that

$$t_2 > \frac{L}{c}.$$

This can be seen as follows. We have

$$\psi_2(t) = \phi(t) + c\,t \geq ct + \phi(0) - (c - \delta_1)\,t = L + \delta_1\,t.$$

Hence, $c\,t_2 = \psi_2(t_1)$ implies $c\,t_2 > L$, so that $t_2 > \frac{L}{c}$.

4. Transformation of the problem

In order to come closer to a solution of Problem **P**, we transform it to a form that we can solve. For this purpose, we write the solution of the wave equation in the form

$$y(x,t) = [\alpha(x+ct) + \beta(x-ct)]/2 \tag{9}$$

which means that we describe our solution in terms of the Riemann invariants or, in other words, as the sum of travelling waves. For an introduction to linear hyperbolic systems, see LeVeque (1999). The *end conditions* (6) yield the equations

$$\alpha(x+c\,T)+\beta(x-c\,T) = 0,\ \alpha'(x+c\,T)-\beta'(x-c\,T) = 0,\ x \in (0,\,\phi(T)) \tag{10}$$

where the derivatives are in the sense of distributions. This is equivalent to

$$\alpha(x) = -\beta(x-2c\,T),\ \alpha'(x) = \beta'(x-2c\,T),\ x \in (c\,T, \phi(T)+c\,T). \tag{11}$$

Differentiation of the first equation in (11) yields

$$\alpha'(x) = -\beta'(x-2cT),\ x \in (c\,T, \phi(T)+c\,T)$$

hence we have $\alpha'(x) = -\alpha'(x)$ and thus

$$\alpha'(x) = 0,\ x \in (c\,T, \phi(T)+c\,T);\ \ \beta'(x) = 0,\ x \in (-c\,T, \phi(T)-c\,T). \tag{12}$$

So, the first equation in (11) implies that there exists a real constant r such that

$$\alpha(x) = r,\ x \in (c\,T, \phi(T)+c\,T);\ \ \beta(x) = -r,\ x \in (-c\,T, \phi(T)-c\,T). \tag{13}$$

We have shown that if (9) satisfies the end conditions (6), then (13) holds. The reverse statement is obviously true. The *initial conditions* (3) yield the equations

$$y_0(x) = (1/2)\ [\alpha(x)+\beta(x)],\ y_1(x) = (c/2)\ [\alpha'(x)-\beta'(x)],\ x \in (0,L). \tag{14}$$

Hence we have

$$y_0(x) + \frac{1}{c}\int_0^x y_1(s)\,ds = \alpha(x) - k_1,\ x \in (0, L) \tag{15}$$

$$y_0(x) - \frac{1}{c}\int_0^x y_1(s)\,ds = \beta(x) + k_1,\ \ x \in (0, L) \tag{16}$$

for a real constant k_1 that we can choose as zero, which implies

$$\alpha(x) = y_0(x) + \frac{1}{c}\int_0^x y_1(s)\,ds,\ x \in (0, L), \tag{17}$$

$$\beta(x) = y_0(x) - \frac{1}{c}\int_0^x y_1(s)\,ds,\ x \in (0, L). \tag{18}$$

We have shown that if (9) satisfies the initial conditions (3), then with the normalization $k_1 = 0$ (which is equivalent to $\alpha(0) = \beta(0)$) equations (17), (18) hold. The converse also holds: If α, β satisfy (17), (18), the initial conditions (3) are valid for y given by (9).

5. Exact controllability

The considerations in the last section imply the following exact controllability result which also holds for the L^∞–case:

Theorem 1 (Exact Null–Controllability for $p \in [1, \infty]$) *Let $T \geq t_2$ and $p \in [1, \infty]$ be given. The initial boundary–value problem (3)–(5) has a traveling waves solution in the sense (9) that satisfies the end conditions (6) with $u \in L^p(0, T)$, if and only if the initial states y_0, y_1 satisfy the following conditions: $y_0 \in L^p(0, L)$ and $Y_1 \in L^p(0, L)$, where $Y_1(x) = \int_0^x y_1(s)\,ds$, that is $y_1 \in W^{-1,p}(0, L)$.*

*This implies that if for $T \geq t_2$ and $p \in [1, \infty]$ Problem **P** is solvable (in the sense that an optimal control $u \in L^p(0,\, T)$ exists) then y_0 and Y_1 are in $L^p(0, L)$.*

Define the sequence $(s_n)_n$ of numbers inductively by $s_0 = L$ and the recursion

$$s_n = \psi_2(\psi_1^{-1}(-s_{n-1})). \tag{19}$$

*If $T = s_l/c$ ($l \in \{1, 2, ..\}$), $p \in (1, \infty)$ and y_0 and Y_1 are in $L^p(0, L)$, Problem **P** is solvable.*

Proof of one direction: First, we consider the case of $T = t_2$. Assume that y_0 and $Y_1 \in L^p(0, L)$. Choose the function $\beta \in L^p(-cT, L)$ such that $\beta(x) = -r$ for all $x < 0$, in particular such that (13) holds. Since $T \geq t_1$, we have $\psi_1(T) = \phi(T) - c\,T \leq \psi_1(t_1) = 0$ and thus we can choose β such that also (18) holds, for example with $r = 0$. For $x \in (0,\, L)$, let α be defined by (17).

We have

$$y(\phi(t),\, t) = [\alpha(\phi(t) + c\,t) + \beta(\phi(t) - c\,t)]/2.$$

Hence, the boundary condition (4) at $x = \phi(t)$ holds if

$$\alpha(\phi(t) + c\,t) = -\beta(\phi(t) - c\,t)$$

which is equivalent to $\alpha(s) = -\beta(\psi_1(\psi_2^{-1}(s)))$ for $s > L$. With our choice of β, this implies that $\alpha(s) = r$ for $s > \psi_2(\psi_1^{-1}(0)) = \psi_2(t_1) = c\,t_2$. Since $T \geq t_2$ we have $c\,T > c\,t_2 - \phi(t_2) = -\psi_1(t_2)$. Thus, $\psi_1^{-1}(-c\,T) > t_2$. This implies that

$$\psi_2(\psi_1^{-1}(-c\,T)) > \psi_2(t_2) = c\,t_2 + \phi(t_2).$$

Now, the equation $\alpha(s) = r$ holds for

$$s \in (c\,t_2,\ \psi_2(\psi_1^{-1}(-c\,T)) \supset (c\,t_2,\ c\,t_2 + \phi(t_2)).$$

Hence (13) holds. For $t \in (0,\, T)$ define the control

$$u(t) = y(0, t) = [\alpha(ct) + \beta(-ct)]/2.$$

Then the solution y given by (9) satisfies the initial conditions (3), the boundary conditions (4) and the end conditions (6). Moreover, u is in $L^p(0, T)$. Thus, we have shown the assertion for the case $T = t_2$.

If $T > t_2$ we can continue the solution with the control $u(t) = 0$ for $t > t_2$ and it remains $y(x,\, t) = y_t(x,\, t) = 0$ for all $t > t_2$.

The proof of the converse is given later in Section 7.

REMARK 1 *For the case of $p \in [2, \infty]$ with $\phi(t) \equiv L$ ($t \geq 0$) Theorem 1 is already proven in Gugat, Leugering and Sklyar (2005) using Fourier series. Note, however, that in Gugat, Leugering and Sklyar (2005) the initial state is the zero state, which is controlled in the time T to the target state $(y_0,\, y_1)$.*

REMARK 2 *The construction in the proof of Theorem 1 implies that the time t_2 is the minimal time, for which exact controllability is possible for all initial states. This can be seen as follows. Consider the initial state $y_0(x) = x$, $y_1(x) = 0$. Then, for $x \in (0,\, L)$ we have $\alpha(x) = \beta(x) = x$. Suppose that the system satisfies the end conditions (6). Then there exists a real constant r such that $\alpha(x) = r$ for $x \in (c\,T,\ \phi(T) + c\,T)$ and $\beta(x) = -r$ for $x \in (-c\,T,\ \phi(T) - c\,T)$. This is only possible if $\psi_1(T) = \phi(T) - c\,T \leq 0$, which is equivalent to $T \geq \psi_1^{-1}(0) = t_1$. Moreover, the boundary condition at $x = \phi(t)$ implies $\alpha(s) = -\beta(\psi_1(\psi_2^{-1}(s)))$ for $s > L$. For $s > c\,T$ this implies $\beta(\psi_1(\psi_2^{-1}(s))) = -r$. Hence, we have $\psi_1(\psi_2^{-1}(cT) \leq 0$ which is equivalent to $\psi_2^{-1}(cT) \geq \psi_1^{-1}(0) = t_1$. Therefore, we have $T \geq \frac{1}{c}\psi_2(t_1) = t_2$.*

6. Solution of the initial boundary value problem

In this section we give a representation for the travelling waves solution (9) of the initial boundary value problem (3), (4), (5). The initial conditions (3) yield the values of α and β on the interval $[0,\, L] = [0, \psi_2(0)]$ that are given in (17), (18).

We can write the boundary conditions (4) for the left–hand end of the string in the following way:

$$u(t) = y(0,t) = \frac{1}{2}[\alpha(ct) + \beta(-ct)].$$

Thus, we obtain the equation

$$\beta(-c\,t) = 2u(t) - \alpha(c\,t). \tag{20}$$

We can write the boundary conditions (4) for the right–hand end of the string in the following way:

$$0 = y(\phi(t),t) = \frac{1}{2}[\alpha(\phi(t) + ct) + \beta(\phi(t) - ct)].$$

Thus, we obtain the equation

$$\alpha(\psi_2(t)) = -\beta(\psi_1(t)). \tag{21}$$

For $t \in [0, t_1)$ we have $\psi_1(t) \in [0,\, L]$. The values of the function α are known on the interval $(0,\, L)$. This implies that the values of $\alpha(c\,t)$ are known for $t \in (0,\, L/c)$. So, equation (20) yields the values of β on the interval $(-L,\, 0)$. Due to (21) we can write

$$\beta(s) = -\alpha(\psi_2(\psi_1^{-1}(s))), \quad s \in (\psi_2(\psi_1^{-1}(L)),\, \psi_2(\psi_1^{-1}(0))) = (L,\, c\,t_2). \tag{22}$$

Since the function β is known on the interval $[-L, L]$, the values of $\beta(\psi_1(t))$ are known for $t \in (\psi_1^{-1}(L),\, \psi_1^{-1}(-L)) = (0,\, \psi_1^{-1}(-L))$. Equation (21) yields the values of α on the interval $(\psi_2(\psi_1^{-1}(L)), \psi_2(\psi_1^{-1}(-L))) = (L,\, \psi_2(\psi_1^{-1}(-L)))$. Due to (20), we can write

$$\beta(-s) = 2\,u(s/c) - \alpha(s), \quad s \in (L,\, \psi_2(\psi_1^{-1}(-L))). \tag{23}$$

Now we proceed in an inductive definition for the values of α and β on the intervals that are necessary to define a travelling waves solution (9) on the set Ω. Assume that for a number $n \in \{1, 2, ...\}$, α is known on the interval $(s_{n-1},\, s_n)$ with s_n as defined in (19). Then, equation (20) yields the values of the function β on the interval

$$(-s_n,\, -s_{n-1}).$$

If β is known on the interval $(-s_n,\, -s_{n-1})$, the values of the function $t \mapsto \beta(\psi_1(t))$ are known for $t \in (\psi_1^{-1}(-s_{n-1}),\, \psi_1^{-1}(-s_n))$. So, equation (21) yields the values of the function α on the interval

$$(\psi_2(\psi_1^{-1}(-s_{n-1})),\, \psi_2(\psi_1^{-1}(-s_n))) = (s_n,\; s_{n+1}).$$

Note that the definition of ψ_2 implies $\psi_2(t) \geq \delta_0 + c\,t$ since $\phi(t) \geq \delta_0$. Moreover, we have $\psi_1(s_n/c) = \phi(s_n/c) - s_n \geq -s_n$. Hence $\psi_1^{-1}(-s_n) \geq s_n/c$ and we obtain

$$s_{n+1} = \psi_2(\psi_1^{-1}(-s_n)) \geq \delta_0 + c\,\psi_1^{-1}(-s_n) \geq \delta_0 + s_n.$$

In particular, we have $\lim_{n\to\infty} s_n = \infty$.

So we see that in this inductive way, we can construct the values of the solution $y(x,t)$ for all $(x,t) \in \Omega$, with the set Ω defined in (5).

Now we want to obtain a more explicit representation of the solution.

Theorem 2 (Solution of the initial boundary value problem)
Consider the initial boundary value problem (3), (4), (5). A travelling waves solution of the form (9) can be computed in the following way:

For $s \geq -L$, define the strictly increasing and bi-Lipschitz continuous function function

$$h(s) = \psi_2(\psi_1^{-1}(-s)). \tag{24}$$

Note that $h(-L) = L$ and $s_{j+1} = h(s_j)$. We use the following notation: h^0 is the identity, i.e. $h^0(s) = s$, $h^1(s) = h(s)$, $h^2(s) = h(h(s))$, $h^3(s) = h(h(h(s)))$, and so forth. Then for all $j \in \{0, 1, 2, ...\}$ we have $s_j = h^{j+1}(-L)$. For $s \in (0,\, L)$ the function α is given by (17) and β is given by (18). For all $s > -L$ we have

$$\beta(-s) = -\alpha(h(s)). \tag{25}$$

In particular, this determines the values of α on $(L,\, h(0))$. Let a natural number $j \geq 1$ be given and $s > 0$. If $h^{j-1}(s) < c\,T$, we have

$$\alpha(h^j(s)) = \alpha(s) - \left[\sum_{k=0}^{j-1} 2\,u\left(\tfrac{1}{c}\,h^k(s)\right)\right]. \tag{26}$$

This implies $\alpha \in L^p(0,\, \phi(T) + c\,T)$.

Proof First we demonstrate (25). By (21), for all $s > -L$ we have $\beta(-s) = -\alpha(\psi_2(\psi_1^{-1}(-s))) = -\alpha(h(s))$. Now we give the proof for equation (26) by induction. For $j = 1$ and $s > 0$, by (20) we have

$$\alpha(s) = 2\,u(s/c) - \beta(-s).$$

By (25) this yields

$$\alpha(s) = 2\,u(s/c) + \alpha(h(s)).$$

Thus, equation (26) holds for $j = 1$.

Assume now that for $s > 0$ equation (26) holds. Equation (20) implies for $s > 0$

$$\begin{aligned}\beta(-h^j(s)) &= 2u(h^j(s)/c) - \alpha(h^j(s)) \\ &= 2u(h^j(s)/c) + \left[\sum_{k=0}^{j-1} 2\,u\left(\tfrac{1}{c}\,h^k(s)\right)\right] - \alpha(s).\end{aligned}$$

By (25) this yields

$$-\alpha(h^{j+1}(s)) \quad = \quad \left[\sum_{k=0}^{j} 2\,u\left(\tfrac{1}{c}\,h^k(s)\right)\right] - \alpha(s).$$

Thus, (26) holds for $j+1$. $\square$

7. Proof of Theorem 1

Here we complete the proof of Theorem 1.

Assume that a control $u \in L^p(0,T)$ is given such that the travelling waves solution (9) satisfies the initial conditions (3) and the end conditions (6). The end condition (13) for α implies that the function α is in $L^p(cT,\, \psi_2(T))$.

For $s \in (0,\, L)$ choose a minimal number $j_0(s) \in \{1, 2, 3, ...\}$ such that $h^{j_0(s)}(s) \geq cT$. This number j_0 may depend on s, but the corresponding function $j_0(s)$ attains at most two values on $(0,\, L)$ and is increasing. Hence, the function j_0 is piecewise constant with at most one jump on $(0,\, L)$. Then, (26) implies that

$$\alpha(s) = \alpha(h^{j_0(s)}(s)) + \left[\sum_{k=0}^{j_0(s)-1} 2\,u\left(\tfrac{1}{c}\,h^k(s)\right)\right].$$

The function h is strictly increasing and bi-Lipschitz continuous. Since $u \in L^p(0,T)$ and the function $s \mapsto \alpha(h^{j_0(s)}(s))$ is in $L^p(0,\, L)$ this implies $\alpha \in L^p(0,\, L)$ due to change of variables formula (see, for example, Naumann, 2005). Moreover, (25) implies $\beta \in L^p(0,\, L)$. Thus, α and β are both in $L^p(0,\, L)$. Therefore, equations (17), (18) imply that y_0 is in $L^p(0,L)$ and that Y_1 is in $L^p(0,L)$.

If there exists an optimal control $u \in L^p(0,\, T)$ that solves Problem **P**, the above arguments imply that y_0 is in $L^p(0,L)$ and that Y_1 is in $L^p(0,L)$.

Now assume that $p \in (1,\, \infty)$ and y_0 is in $L^p(0,L)$ and Y_1 is in $L^p(0,L)$. Let ω denote the optimal value of Problem **P**. Then there exists a minimizing sequence $(u_n)_n$ of controls $u_n \in L^p(0,\, T)$ that solve the exact null-controllability problem such that $\lim_{n\to\infty} \|u_n\|_{p,\,(0,\,T)} = \omega$. Theorem 4 below implies that there exists a bounded sequence of real numbers $(r_n)_n$ such that for all $n \in \{1, 2, 3, ...\}$ we have

$$\begin{aligned} r_n + \sum_{k=0}^{l} 2\,u_n\left(\tfrac{1}{c}\,h^k(s)\right) \quad &= \quad y_0(s) + \tfrac{1}{c}\int_0^s y_1(\sigma)\,d\sigma, \\ r_n + \sum_{k=1}^{l} 2\,u_n\left(\tfrac{1}{c}\,h^k(-s)\right) \quad &= \quad -y_0(s) + \tfrac{1}{c}\int_0^s y_1(\sigma)\,d\sigma. \end{aligned}$$

for $s \in (0,\, L)$ almost everywhere. By going to a suitably chosen subsequence, we can assume without restriction that $\lim_{n\to\infty} r_n = r$.

Moreover, since the sequence u_n is bounded and $p \in (1, \infty)$, we can also assume that it is weakly convergent to a control $u \in L^p(0, T)$. Mazur's Theorem implies that there exist convex combinations

$$w_k = \sum_{m=k}^{N(k)} \lambda_m^k u_m$$

with $\sum_{m=k}^{N(k)} \lambda_m^k = 1$ and $\lambda_m^k \geq 0$, $k \leq m \leq N(k)$, such that

$$\lim_{k\to\infty} w_k = u$$

strongly in $L^p(0, T)$. Hence, by going further to a subsequence we can assume that the sequence converges also pointwise almost everywhere and we have

$$\begin{aligned}
\sum_{k=0}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(s)\right) &= \lim_{k\to\infty} \sum_{k=0}^{l} 2 \sum_{m=k}^{N(k)} \lambda_m^k u_m \left(\tfrac{1}{c}\, h^k(s)\right) \\
&= y_0(s) + \tfrac{1}{c} \int_0^s y_1(\sigma)\, d\sigma - r, \\
\sum_{k=0}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(-s)\right) &= \lim_{k\to\infty} \sum_{k=0}^{l} 2 \sum_{m=k}^{N(k)} \lambda_m^k u_m \left(\tfrac{1}{c}\, h^k(s)\right) \\
&= -y_0(s) + \tfrac{1}{c} \int_0^s y_1(\sigma)\, d\sigma - r.
\end{aligned}$$

Hence, control u satisfies (32), (33), and thus steers the system to zero exactly. Moreover, the sequential weak lower semicontinuity of the objective function of Problem **P**, that is, the p-norm, implies that

$$\|u\|_{p,(0,T)} \leq \liminf_{n\to\infty} \|u_n\|_{p,\,(0,T)} = \omega.$$

Hence, u is a solution of Problem **P**. □

Remark 3 *Note that the system is reversible in time in the following sense: If the control $u \in L^p(0, T)$ solves the exact null–controllability problem (3), (4), (5), (6), then the control $u_1(t) = u(T-t) \in L^p(0, T)$ solves the following exact controllability problem:*

$$\begin{aligned}
&y(x,0) = 0,\ y_t(x,0) = 0,\ x \in (0, L) \\
&y(0, t) = u_1(t),\ y(\phi(t), t) = 0,\ t \in (0,T) \\
&y_{tt}(x,t) = c^2 y_{xx}(x,t),\ (x,t) \in \Omega = \{(x,t) :\ t \in (0,T),\ x \in (0,\ \phi(t))\} \\
&y(x,T) = y_0(x),\ y_t(x,T) = y_1(x),\ x \in (0, \phi(T)).
\end{aligned}$$

Moreover, if we have a control u_1 that steers the initial state (y_0, y_1) to $(0, 0)$ and a control u_2 that steers the initial state $(0,0)$ to (z_0, z_1) at time T, due to the linearity of the system the control $u = u_1 + u_2$ steers the initial state (y_0, y_1) to the terminal state (z_0, z_1). This implies the following exact controllability result:

THEOREM 3 (EXACT CONTROLLABILITY FOR $p \in [1, \infty]$) *Let $T \geq t_2$ and $p \in [1, \infty]$ be given. Let an initial state $(y_0, y_1) \in L^p(0, L) \times W^{-1,p}(0, L)$ and a terminal state $(z_0, z_1) \in L^p(0, L) \times W^{-1,p}(0, L)$ be given. Then there exists $u \in L^p(0, T)$ that solves the exact controllability problem*

$$y(x, 0) = y_0(x),\ y_t(x, 0) = y_1(x),\ x \in (0,\ L) \tag{27}$$

$$y(0,\ t) = u(t),\ y(\phi(t), t) = 0,\ t \in (0, T) \tag{28}$$

$$y_{tt}(x, t) = c^2 y_{xx}(x, t),\ (x, t) \in \Omega = \{(x, t) :\ t \in (0, T),\ x \in (0,\ \phi(t))\} \tag{29}$$

$$y(x, T) = z_0(x),\ y_t(x, T) = z_1(x),\ x \in (0, \phi(T)). \tag{30}$$

8. The successful controls

Now we give a characterization of the successful controls that steer the system to a position of rest at the terminal time $T = s_l/c$, where l is a natural number and s_l is defined as in (19). Equation (13) implies that $\alpha(x) = r$ for all $x \in (s_l,\ \psi_2(s_l/c))$. Moreover, we have $\beta(x) = -r$ for all $x \in (-s_l,\ \psi_1(s_l/c))$. Due to (25) this yields $\alpha(h(s)) = -\beta(-s) = r$ for all $s \in (-\psi_1(s_l/c),\ s_l)$.

Hence, we have $\alpha(s) = r$ for all $s \in (h(-\psi_1(s_l/c)),\ s_{l+1})$. Note that the definition of h implies that we have $\psi_2(s_l/c) = h(-\psi_1(s_l/c))$.

Thus, we have $\alpha(s) = r$ for all $s \in (s_l,\ s_{l+1})$.

This is equivalent to $\alpha(h^{l+1}(s)) = r$ for all $s \in (h^{-l-1}(s_l),\ h^{-l-1}(s_{l+1})) = (h^{-1}(s_0),\ s_0) = (-L,\ L)$. Due to (26) we have

$$\alpha(s) = \alpha(h^{l+1}(s)) + \left[\sum_{k=0}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(s)\right)\right].$$

Thus, the successful controls can be characterized by the equation

$$\sum_{k=0}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(s)\right) = \alpha(s) - r \tag{31}$$

for all $s \in (-L,\ L)$, where for $s \in (0,\ L)$, the function $\alpha(s)$ is given by (17) and due to (20), for $s \in (-L,\ 0)$ we can define $\alpha(s) = 2u(s/c) - \beta(-s)$, where β is given by (18).

Thus, for $s \in (0,\ L)$, we have (31) and for $s \in (-L, 0)$ we have

$$\sum_{k=0}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(s)\right) = \alpha(s) - r = 2u(s/c) - \beta(-s) - r.$$

Hence, for $s \in (-L, 0)$ we have

$$\sum_{k=1}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(s)\right) = -\beta(-s) - r.$$

The representation of the exact controls that we have found is stated in the following theorem.

Theorem 4 (Characterization of the successful controls) *Let a natural number $l \geq 1$ and $T = s_l/c$ be given. The set of controls $u \in L^p(0, T)$, for which the solution of the initial boundary value problem (3), (4), (5) satisfies the end conditions (6), is the solution set of the equations*

$$r + \sum_{k=0}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(s)\right) = y_0(s) + \tfrac{1}{c} \int_0^s y_1(\sigma)\, d\sigma \tag{32}$$

and

$$r + \sum_{k=1}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(-s)\right) = -y_0(s) + \tfrac{1}{c} \int_0^s y_1(\sigma)\, d\sigma. \tag{33}$$

for $s \in (0, L)$ almost everywhere, where r is a real number.

If y_1 is given by a measurable function, a constant control $u = u_0$ is only successful if and only if $y_0 = u_0$ is constant and $y_1 = 0$.

Proof We have demonstrated the first part of the assertion above. The last assertion follows since for a constant control $u = u_0$ the system (32), (33) is equivalent to the equations

$$\begin{aligned} y_0(s) &= u_0, && (34)\\ \tfrac{2}{c} \int_0^s y_1(\sigma)\, d\sigma &= 2\, r + 2u_0 + \sum_{k=1}^{l} 4\, u_0 = 2\, r + (4\, l + 2)\, u_0. && (35) \end{aligned}$$

This implies that y_0 is constant, $y_1 = 0$ and $r = -(2\, l + 1)\, u_0$. □

Remark 4 *Theorem 4 implies that Problem* **P** *is equivalent to the problem*

$$P' : \qquad \text{minimize } \|u\|_{p,(0,T)}^p \text{ subject to} \tag{36}$$

$u \in L^p(0, T)$, $r \in (-\infty, \infty)$ and for s almost everywhere in $(0, L)$

$$\begin{aligned} r + \sum_{k=0}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(s)\right) &= y_0(s) + \tfrac{1}{c} \int_0^s y_1(\sigma)\, d\sigma, && (37)\\ r + \sum_{k=1}^{l} 2\, u\left(\tfrac{1}{c}\, h^k(-s)\right) &= -y_0(s) + \tfrac{1}{c} \int_0^s y_1(\sigma)\, d\sigma. && (38) \end{aligned}$$

The solution of Problem P' can easily be approximated numerically by a suitable discretization. This only requires the approximation of integrals, since in the formulation P', no partial differential equation appears.

The representation of the admissible controls with two point-wise equality constraints is based upon the traveling waves solution. In this paper we have adapted this approach to the case of the vibrating string with a moving boundary *point. For the case of a vibrating string with fixed length, the representation of the admissible controls as the solution set of a sequence of moment equations (that is the method of moments) is well known. However, it is not clear how this approach can be adapted to the case with a moving boundary.*

REMARK 5 *If y_1 is given by a measurable function, the representation in the form of Problem P' implies that $\mathbf{P}$ can only have* bang-bang *controls as a solution if y_0 only attains a finite number of values and y_1 is zero.*

REMARK 6 *For $p = \infty$ the representation P' has interesting consequences. In particular, it allows for showing the existence of an optimal control. Note that in the case of $p = \infty$, in general, the optimal controls are not uniquely determined. Due to the representation P', we can give an explicit representation of one of the optimal controls: Let $y_0 \in L^\infty(0,\, L)$ and y_1 be given such that the function $x \mapsto Y_1(x) = \int_0^x y_1(s)\, ds$ is in $L^\infty(0,\, L)$. For the case of $p = \infty$, the objective function of the optimal control problem $\mathbf{P}$ is $\|u\|_{\infty,\,(0,\,T)}$. The representation P' allows for deriving a lower bound for the optimal value ω of Problem $\mathbf{P}$. Let $u \in L^\infty(0,\, T)$ be a control that solves the exact null–controllability problem (3), (4), (5), (6). Then, (32) and (33) imply that there exists a real number r such that $\|u\|_{\infty,\,(0,\,T)}$*

$$\geq \frac{1}{2} \max \left\{ \frac{1}{l+1} \left\| y_0 + \tfrac{1}{c}\, Y_1 - r \right\|_{\infty,\,(0,\,T)},\, \frac{1}{l} \left\| -y_0(s) + \tfrac{1}{c}\, Y_1 - r \right\|_{\infty,\,(0,\,T)} \right\}$$

$=: R(r)$. Hence, we obtain the lower bound

$$\omega \geq \min_r R(r).$$

Choose the number r_0 such that $R(r_0) = \min_r R(r)$. Define u in such a way that for all $k \in \{1, ..., l\}$ we have

$$u\left(\tfrac{1}{c}\, h^k(s)\right) = u\left(\tfrac{1}{c} s\right),$$

and for all $k \in \{2, ..., l\}$ we have

$$u\left(\tfrac{1}{c}\, h^k(-s)\right) = u\left(\tfrac{1}{c} h(-s)\right),$$

and for $s \in (0,\, L)$ almost everywhere

$$u\left(\tfrac{1}{c} s\right) = \frac{1}{2\,(l+1)} \left[y_0(s) + \tfrac{1}{c} \int_0^s y_1(\sigma)\, d\sigma - r_0 \right]$$

and

$$u\left(\tfrac{1}{c} h(-s)\right) = \frac{1}{2\,l} \left[-y_0(s) + \tfrac{1}{c} \int_0^s y_1(\sigma)\, d\sigma - r_0 \right].$$

Then, (32) and (33) hold with $r = r_0$. Since $\|u\|_{\infty,\,(0,\,T)} \leq R(r_0) \leq \omega$, we have constructed u in such a way that it is a solution of $\mathbf{P}$ for $T = s_l/c$. In particular, this implies the existence of an optimal control that solves the optimal control problem $\mathbf{P}$ for the case of $p = \infty$. Hence, the last statement in Theorem 1 also applies to the case of $p = \infty$.

Remark 7 *Similar arguments as in Remark 6 also apply to the case where* $p \in (1, \infty)$. *In fact, the definition of the optimal control* u *is precisely as in Remark 6, but the real number* r_0 *is chosen in such a way that* $\|u\|_{p,(0,T)}$ *is minimal.*

9. Examples

In this section we present some examples. We start with the case where ϕ is constant, that is, the end L of the string is fixed.

Example 1: Assume that $\phi(t) = L$, then $\psi_1(t) = L - ct$, $\psi_2(t) = L + ct$, $\psi_1^{-1}(t) = (L - t)/c$. Hence

$$h(s) = \psi_2(\psi_1^{-1}(-s)) = s + 2\,L.$$

Moreover, we have $t_1 = L/c$, $t_2 = 2\,L/c$.

Example 2: In this example we consider a movement with constant speed A. This leads to a change in the exact controllability time t_2. Let a real number A with $|A| < c$ be given and define $\phi(t) = A\,t + L$. The assumption $\phi(T) > 0$ implies that

$$A > -L/T.$$

Then

$$\psi_1(t) = (A - c)\,t + L,\ \psi_2(t) = (A + c)\,t + L.$$

We have $\psi_1^{-1}(s) = \frac{s-L}{A-c}$. Define

$$\gamma = \frac{c + A}{c - A} \text{ and } \eta = \frac{2\,c\,L}{c - A}.$$

Then, $h(s) = \gamma s + \eta$,

$$t_1 = \frac{L}{c - A}, \qquad t_2 = \frac{2\,L}{c - A}.$$

Example 3: Let

$$\phi(t) = L - t^2 \left(t - \frac{3}{2}T\right).$$

Then, $\phi(0) = L$, $\phi'(0) = 0$, $\phi(T) = L + \frac{1}{2}T^3$ and $\phi'(T) = 0$. Note that $\phi(T/2) = L + \frac{1}{4}T^3$. We have $\phi'(t) = 3t\,(T - t) \leq \frac{3}{4}T^2$. Assume that $c > \frac{3}{4}T^2$. Then, (1) holds and

$$\psi_1(t) = L - t^2(t - \frac{3}{2}T) - c\,t = -t^3 + \tfrac{3}{2}\,T\,t^2 - c\,t + L$$

is strictly decreasing. Thus, $\psi_1(t) = z$ implies

$$t^3 - \tfrac{3}{2}\, T\, t^2 + c\, t + z - L = 0. \tag{39}$$

Hence, for $z \in [L + \frac{1}{2}\, T^3 - c\, T,\, L]$, the number $\psi^{-1}(z)$ is the unique solution in $[0,\, T]$ of (39). With the notation $d = z - L \geq 0$ and

$$F(z) = 2\sqrt{3}\sqrt{54\,(2\,d^2 + 2\,d\,c\,T - d\,T^3) - 9c^2\,T^2 + 16\,c^3} - 36\,d - 18\,cT + 9\,T^3$$

it can be stated in explicit form as

$$\psi^{-1}(z) = t = \frac{T}{2} + \frac{(F(z))^{1/3}}{2 \cdot 3^{2/3}} - \frac{2\,(3\,c - \frac{9\,T^2}{4})}{3^{4/3}\,(F(z))^{1/3}}.$$

Thus, we see that in this case an explicit representation of

$$h(s) = \psi_2(\psi_1^{-1}(-s)) = c\,\psi_1^{-1}(-s) + \phi(\psi_1^{-1}(-s))$$

can be stated. In particular, this allows for the computation of $t_1 = \psi^{-1}(0)$ and the minimal time, for which exact controllability holds, $t_2 = \frac{1}{c}\, h(0)$.

10. Conclusions

In this paper we have studied a vibrating string with one moving boundary and Dirichlet control at the other boundary. We have characterized the exact controllability. It turns out that the time, for which exact controllability is possible depends on the movement of the string.

We have presented an explicit representation of the set of successful controls that steer the string to a position of rest in finite time T. This makes it possible to reformulate the optimal control problem as a minimal norm problem subject of two pointwise equality constraints. This formulation can be used for the numerical computation of the optimal controls. It also allows for giving an explicit representation of the optimal controls that we have presented for the L^∞-case. A similar analysis is possible for Neumann controls. This will be the subject of future research. Another important topic is the influence of delay in the control loop. For the vibrating string with fixed length this has been investigated, for example, in Gugat and Tucsnak (2011). Up to now, similar analysis for the case of moving boundary is not available.

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References

Avdonin, S. A. and Ivanov, S. S. (1995) *Families of Exponentials.* Cambridge University Press.

Bardos, C. and Chen, G. (1981) Control and stabilization for the wave equation. III: Domain with moving boundary. *SIAM J. Control Optimization* 19: 123-138.

Cui, L. and Song, L. (2014) Controllability for a Wave Equation with Moving Boundary. *Journal of Applied Mathematics*, Article ID 827698.

Delfour, M.C. and Zolesio, J.P. (2001) *Shapes and Geometries: Analysis, Differential Calculus and Optimization.* SIAM, Philadelphia.

Gugat, M. (2002) Analytic solutions of L^∞–optimal control problems for the wave equation. *Journal of Optimization Theory and Applications*, 114: 397–421.

Gugat, M. (2003) Boundary Controllability between Sub– and Supercritical Flow. *SIAM J. Control and Optimization*, 42: 1056–1070.

Gugat, M. (2005) L^1-optimal boundary control of a string to rest in finite time *Recent Advances in Optimization. Proceedings of the 12th French-German-Spanish Conference on Optimization. Lecture Notes in Economics and Mathematical Systems.* Springer-Verlag.

Gugat, M. and Leugering, G. (2002) Solutions of L^p–norm–minimal control problems for the wave equation. *Computational and Applied Mathematics*, 21: 227–244.

Gugat, M., Leugering, G. and Sklyar, G. (2005) L^p-optimal boundary control for the wave equation. *SIAM J. Control and Optimization* 44, 49-74.

Gugat, M. (2007) Optimal Energy Control in Finite Time by varying the Length of the String. *SIAM J. Control and Optimization* 46, 1705–1725.

Gugat, M. (2008) Optimal boundary feedback stabilization of a string with moving boundary. *IMA Journal of Mathematical Control and Information* 25, 111–121.

Gugat, M. and Sigalotti, M. (2010) Stars of vibrating strings: Switching boundary feedback stabilization. *Networks and Heterogeneous Media* 5, 299–314.

Gugat, M. and Sokolowski, J. (2013) A note on the approximation of Dirichlet boundary control problems for the wave equation on curved domains. *Applicable Analysis* 92, 2200-2214.

Gugat, M. and Tucsnak, M. (2011) An example for the switching delay feedback stabilization of an infinite dimensional system: The boundary stabilization of a string *Systems and Control Letters* 60, 226-233.

He, W., Ge, S.S. and Huang, D. (2015) Modeling and Vibration Control for a Nonlinear Moving String With Output Constraint. *IEEE/ASME Transactions on Mechatronics* 20, 1886-1897.

Krabs, W. (1982) Optimal control of processes governed by partial differential equations part ii: Vibrations. *Zeitschrift fuer Operations Research*, 26:63–86.

Krabs, W. (1992) *On Moment Theory and Controllability of One–Dimensional Vibrating Systems and Heating Processes. Lecture Notes in Control and Information Science* **173**, Springer–Verlag, Heidelberg.

LeVeque, R. J. (1999) *Numerical Methods for Conservation Laws.* Birkhaeuser, Basel.

Lions, J. L. (1988) Exact controllability, stabilization and perturbations of distributed systems. *SIAM Review*, 30: 1–68.

Malanowski, K. (1969) On time–optimal control of a vibrating string (Polish). *Arch. Automat. Telemech.*, 14: 33–44.

Naumann, J. (2005) Transformation of Lebesgue Measure and Integral by Lipschitz mappings, *Preprint Nr 2005* - 8. Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät II, Institut für Mathematik.

Russell, D. L. (1967) Nonharmonic Fourier Series in the Control Theory of Distributed Parameter Systems. *Journal of Mathematical Analysis and Applications*, 18, 542–560.

Truchi, C. and Zolesio, J. P. (1988) Wave equation in time periodical domain. *Stabilization of flexible structures, Proc. ComCon Workshop*, Montpellier, France, 1987, 282-294.

Valein, J. and Zuazua, E. (2009) Stabilization of the Wave Equation on 1-d Networks. *SIAM J. Control and Optimization* 48, 2771-2797.

Wang, J., Pi, Y. and Krstic, M. (2018) Balancing and suppression of oscillations of tension and cage in dual-cable mining elevators. *Automatica* 98, 223–238.

Wang, J., Tang, S.-X., Pi, Y. and Krstic, M. (2018) Exponential regulation of the anti-collocatedly disturbed cage in a wave PDE-modeled ascending cable elevator. *Automatica* 95, 122–136.

Zuazua, E. (2004) Optimal and Approximate Control of Finite-Difference Approximation Schemes for the 1D Wave equation. *Rend. Mat. Appl.* 24, 201–237.