Control and Cybernetics

vol. 26 (1997) No. 1

Regularized penalty methods for ill-posed optimal control problems with elliptic equations

Part II: Distributed and boundary control with unbounded control set and state constraints

by

R. Hettich*, A. Kaplan**, R. Tichatschke*

* Universitiit Trier, FB IV (Math.), D-54286 Trier, Germany

** TH Darmstadt, FB Mathematik, Schlol3gartenstral3e 7, D-64289 Darmstadt, Germany

Abstract: We investigate the application of Prox-Regularization to ill-posed control problems for systems governed by elliptic equations. Stable variants of Penalty Methods are obtained by means of One- and Multi-Step Regularization of the penalized problems. Convergence of the resulting methods is proved in the case of distributed control as well as boundary control with Dirichlet and Neumann conditions. In the first part of this paper the case of distributed control under Dirichlet boundary conditions and a bounded set of admissible controls has been considered.

Keywords: distributed and boundary control, elliptic equations, prox-regularization, penalty methods.

1. The case of unbounded set of controls

The main contents of this paper, i.e. description of the probleins, method and statements, is autonomous. However, in the proofs we often make use of the results described in Part I, the preceding paper in this issue. References to formulas, sections and statements from Part I are denoteded with "prime" (').

In this section we deal again with Problem 2 (see Part I of the paper) but the set of admissible controls may now be unbounded.

1.1. Problem 2 without state constraints

Let $\Omega \subset \mathbb{R}^n$ be an open domain with boundary Γ of the class C^2, U_{ad} be a closed, convex subset of $L_2(\Omega)$ and

$$Ay = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial y}{\partial x_j}\right) + a_0 y$$

be a strongly elliptic operator with coefficients $a_{ij} \in C^2(\bar\Omega)$, $a_0 \in C^2(\bar\Omega)$ and $a_0 \geq 0$ on Ω.

We introduce the Hilbert space

$$Y = \{y \mid y \in H_0^1(\Omega),\ Ay \in L_2(\Omega)\}$$

with the norm $\|\cdot\|_Y$ endowed with $\|y\|_Y = \|Ay\|$ ($\|\cdot\|$ is the norm in $L_2(\Omega)$).

Denoting by $y(u)$ the unique solution of the Dirichlet problem

$$Ay = f + u \text{ in } \Omega,\quad y = 0 \text{ on } \Gamma,$$

with $f \in L_2(\Omega)$ a given function and $u \in U_{ad}$, we recall Problem 2:

$$\textit{minimize } J(u) = \int_\Omega (Cy(u) - \kappa_d)^2 d\Omega \textit{ subject to } u \in U_{ad},\ y(u) \in G.$$

Here $G \subset Y$ is a closed, convex set, $C \in \mathcal{L}(H_0^1(\Omega), L_2(\Omega))$ and $\kappa_d \in L_2(\Omega)$.

In the sequel we suppose that the set U^* of optimal controls is non-empty.

In Part I two numerical methods have been considered for solving this problem in case U_{ad} is bounded. Now, convergence of Method 2 will be investigated for an unbounded set U_{ad}.

To this end we recall the description of Method 2. Let $X = Y \times L_2(\Omega)$ be a Hilbert space with the norm $\|(y,u)\|_X = (\|Ay\|^2 + \|u\|^2)^{\frac{1}{2}}$.

As previously, the abbreviations $z = (y,u)$, $z^* = (y^*,u^*)$, $z^{i,s} = (y^{i,s}, u^{i,s})$ etc. will be used for elements in X.

Method 2 *Let $u^{1,0} \in U_{ad}$ and positive sequences $\{r_i\}$, $\{\epsilon_i\}$ and $\{\delta_i\}$ be given with*

$$\sup_i r_i < 1,\ \sup_i \epsilon_i < 1 \textit{ and } \lim_{i\to+\infty} r_i = \lim_{i\to+\infty} \epsilon_i = 0.$$

Step i: *Given $u^{i-1} \in U_{ad}$.*

(a) *Set $u^{i,0} = u^{i-1}$, $s = 1$.*

(b) *Given $u^{i,s-1}$, let*

$$\begin{aligned} w_{i,s}(z) &= \int_\Omega (Cy - \kappa_d)^2 d\Omega + \frac{1}{r_i}\int_\Omega (Ay - f - u)^2 d\Omega \\ &+ \int_\Omega (u - u^{i,s-1})^2 d\Omega, \end{aligned} \tag{1}$$

and

$$z^{i,s} = \text{argmin}\{-w_{i,s}(z) : z \in G \times U_{ad}\} - \quad (2)$$

The point $z^{i,s} \in G \times U_{ad}$ *is generated such that*

IIV'il! $_{i,s}$ (i •^{8}) - V"iITi,s(i•^{8}) IIx / E -

(c) *If* $\|u^{i,s} - u^{i,s-1}\| > \delta_i$, *set* $s := s + 1$ *and repeat (b).*
Otherwise, set $u^i = u^{i,s}$, $s(i) = s$, *and continue with Step* $(i + 1)$.

In this subsection the case without state constraints $G = Y$ is considered.

We start with the proof of an auxiliary estimate which is essential for the analysis of convergence of the method in case of unbounded U_{ad}. Let z^* be an optimal process of Problem 2.

Choose $c_0 \geq \|Cy^* - A_d\|$ and put

$$P_{i,s} > |z^* - z^{i,s}|, \quad (3)$$

with $|\cdot|$ a new norm in X introduced by $|z|^2 = \|Ay - u\|^2 + \|u\|^2$.

Of course, inequality (3) makes sense for certain (i, s) only if $s(k) < \infty$ for $k < i$ (up to now we are not sure whether this is true).

Note that $y^{1,0}$ does not occur in the method, and formally one can suppose that $y^{1,0} = y^*$ i.e., $p_{1,0}$ may be defined by

$$P_{1,0} > \|u^{1,0} - u^*\|.$$

Denote $J_i(z) = \int_\Omega (Cy - A_d)^2 d\Omega + r_i \int_\Omega (Ay - f - u)^2 d\Omega$.

Proposition 1 *There exists a constant* c_3 *such that for the points* $z^{i,s}$ *defined in (2), with* $G = Y$, *the estimate*

$$J_i(z^*) - J_i(z^{i,s}) < c_3(c_0 + P_{i,s-1})^2 (1 + \sqrt{r_i}(c_0 + P_{i,s-1}))^2 r_i \quad (4)$$

is true.

Proof: Due to (41')-(44'), we have for $s \geq 1$ and $K_{i,s}$ defined by (43')

$$r_i \int_\Omega (Ag^{i,s} - f - u^{i,s})^2 d\Omega + K_{i,s}(z^{i,s}) < c_5 + p_{i,s-1},$$

from which it follows that

$$\|Ag^{i,s} - f - v^{i,s}\| < (c_0 + P_{i,s-1})/\sqrt{r_i}, \quad (5)$$

$$\|u^{i,s} - u^{i,s-1}\| < c_0 + P_{i,s-1}, \quad (6)$$

$$\|Cg^{i,s} - A_d\| < c_0 + P_{i,s-1}. \quad (7)$$

But (3) and (6) together with

$$\|v^{i,s} - u^*\| \quad \|v^{i,s-1} - u^*\| + \|u^{i,s} - u^{i,s-1}\|$$

lead to

$$\|\bar v^{i,s} - v_*\| < c_0 + 2\rho_{i,s-1}. \tag{8}$$

Let $\hat y^{i,s}$ be a solution of the problem

$$Ay = f + \bar u^{i,s} \text{ in } D, \quad y = 0 \text{ on } \Gamma. \tag{9}$$

Then, with regard to (5), we obtain

$$\|A\bar y^{i,s} - A\hat y^{i,s}\| < (c_0 + \rho_{i,s-1})\sqrt{r_i},$$

which means that

$$\|(\bar y^{i,s}, \bar u^{i,s}) - (\hat y^{i,s}, \bar u^{i,s})\|_X < (c_0 + \rho_{i,s-1})\sqrt{r_i}. \tag{10}$$

For $\bar p^{i,s}$ defined by (52'), due to (7), the estimate

$$\|\bar p^{i,s}\|_{H^2(\Omega)} < c_7(c_0 + \rho_{i,s-1}) \tag{11}$$

is true.

Denote $\bar q^{i,s} = \frac{1}{r_i}(A\bar y^{i,s} - f - \bar u^{i,s})$. Using the relations (55'), (5) and (11), we infer

$$\begin{aligned} \|\bar q^{i,s}\| &\le \|\bar p^{i,s}\| + \sqrt{r_i}\|\bar p^{i,s}\|\|\sqrt{r_i}\,\bar q^{i,s}\| \\ &< c_7(c_0 + \rho_{i,s-1})(1 + \sqrt{r_i}(c_0 + \rho_{i,s-1})). \end{aligned}$$

Thus,

$$\|A\bar y^{i,s} - f - \bar u^{i,s}\| < c_7(c_0 + \rho_{i,s-1})(1 + \sqrt{r_i}(c_0 + \rho_{i,s-1}))r_i$$

and

$$\|(\bar y^{i,s}, \bar u^{i,s}) - (\hat y^{i,s}, \bar u^{i,s})\|_X < c_7(c_0 + \rho_{i,s-1})(1 + \sqrt{r_i}(c_0 + \rho_{i,s-1}))r_i.$$

Therefore, a feasible process $\tilde z^{i,s}$, which is the nearest to $\bar z^{i,s}$ in the norm $\|\cdot\|_X$, satisfies the inequality

$$\|\bar z^{i,s} - \tilde z^{i,s}\|_X < R_{i,s-1}(r_i)(c_0 + \rho_{i,s-1})r_i, \tag{12}$$

with $R_{i,s-1}(r_i) = c_7(1 + \sqrt{r_i}(c_0 + \rho_{i,s-1}))$. In view of (59') we have

$$\begin{aligned} \|\bar y^{i,s} - \tilde y^{i,s}\| &< c_5 c_7(c_0 + \rho_{i,s-1})(1 + \sqrt{r_i}(c_0 + \rho_{i,s-1}))r_i \\ &= c_8(c_0 + \rho_{i,s-1})(1 + \sqrt{r_i}(c_0 + \rho_{i,s-1}))r_i, \end{aligned} \tag{13}$$

with $c_8 = c_5 c_7$. Due to (7) and (13), it follows that

$$\begin{aligned} \|C\bar y^{i,s} + C\tilde y^{i,s} - 2K_d\| &\le 2\|C\bar y^{i,s} - K_d\| + \|C\bar y^{i,s} - C\tilde y^{i,s}\| \\ &< (c_0 + \rho_{i,s-1})(2 + \|C\|c_8(1 + \sqrt{r_i}(c_0 + \rho_{i,s-1}))r_i). \end{aligned} \tag{14}$$

Now, using

Ji(z*) - Ji(z^i,^s) = Ji(z*) - J_i(z^i,^s) + Ji(z^i,^s) - J_i(z^i,^s)

= ∫_lo (Cy* - Kd)^2 dD - ∫_lo (Cf/'^8 - Kd)^2 dD + ∫_lo (Cf/'^s - Kd)^2 dD

- ∫_lo (Cff^s _ Kd)^2 dD _ 2/r_i ∫_lo (Ag^i,^s _ f _ v^i,^s)^2 dD,

and

$$\|Cy^* - \kappa_d\| \leq \|C\bar{\bar{y}}^{i,s} - \kappa_d\|$$

together with the inequalities *(13)*, *(14)* and $r_i < 1$, we conclude that

Ji(z*) - Ji(i•^3) :;; IICIIIIY^i,^s - g ^i,^s 111c g^i,^s + Cy^i,^s - 2Kdll
< (co + P_i,s - 1)^2 IIC IJca(2 + JJCllca (1 + fo(co + P_i,s - 1))r_i)
x(1 + fo(co + P_i,s - 1))r_i
< (co + P_i,s - 1)^2 IIC llca(2 + IICIJca)(l + fo(co + P_i,s - 1))^2 r_i
= d3(co + P_i,s - 1)2(1 + y'r';(co + P_i,s - 1))^2 r_i

with dB = 11Cllca(2 + IICIJcs). ■

Theorem 1 *Let ρ_i be defined recursively by*

$$\rho_{i+1} = \rho_i + \sqrt{d_3}(c_0 + \rho_i)(1 + \sqrt{r_i}(c_0 + \rho_i))\sqrt{r_i} + \frac{3}{2}\sqrt{3}\epsilon_i, \tag{15}$$

with P1 = P1,o-

Moreover, assume that the sequences {ri} and {E_i} in Method 2 are chosen such that

$$\sup_i E_i < 1, \quad \sup_i r_i < 1, \quad \sum_{i=1}^{\infty} fo < \infty, \quad \sum_{i=1}^{\infty} E_i < \infty \tag{16}$$

and for each i *the relations*

$$\begin{array}{l} \frac{1}{2\rho_i}\left[d_3(c_0+\rho_i)^2(1+\sqrt{r_i}(c_0+\rho_i))^2 r_i - \left(\delta_i - \frac{3}{2}\epsilon_i\right)^2\right] + \frac{3}{2}\sqrt{3}\epsilon_i < 0 \\ \delta_i > \frac{3}{2}\epsilon_i \end{array} \tag{17}$$

and

$$\rho_i\sqrt{r_i} \leq d_4 \tag{18}$$

hold with an arbitrary constant d4. Then, in Method 2 relation s(i) < oo is true for each i; $\{u^{i,s}\}$ *converges weakly to il, in $L_2(D)$ and $\{y^{i,s}\}$ converges weakly to y in Y, with (y, v,) an optimal process for Problem 2.*

Proof: At first we establish the following fact: If $P_i > |z^* - z^{i,0}|$ for an optimal process z^* and for some i, then $s(i) < \infty$ and

$$P_i > |z^* - z^{i,s}| \text{ for } s = 1, \dots, s(i) - 1$$

is true, i.e., relation (3) is valid for $s = 0, \dots, s(i) - 1$ and $P_{i,s} = P_i$. Applying Proposition 1 and Lemma 2' (inequality (68')) with the data $Z = X$, $\Phi = f_i$, $Z_1 = \{z = (y, u) \in X : y = 0\}$, $P : Z \to Z_1$ defined by $Pz = (0, u)$, $a(z, z') = \int_\Omega C y C y' d\Omega + \ldots b(z, z')$, $b(z, z') = \int_\Omega (Ay - v)(Ay' - u') d\Omega$, $l(z) = -J_i(z) + a(z, z)$, $K = X$ (with the norm $\|\cdot\|$), $a^0 = z^{i,0}$,
then the case $s(i) > 1$ leads to

$$|z^{i,1} - z^*| < |z^{i,0} - z^*|$$

$$+ \frac{2}{\ldots_i}\left[d_3(c_0 + P_i)^2 (1 + \gamma' r_i (c_0 + P_i))^2 r_i - (\delta_i - \tfrac{3}{4}\varepsilon_i)^2\right]$$

and, in view of (37'), (39'),

$$|z^{i,1} - z^*| - |z^{i,0} - z^*|$$

$$< \frac{1}{2P_i}\left[d_3(c_0 + P_i)^2 (1 + \gamma' r_i (c_0 + P_i))^2 r_i - \left(\delta_i - \frac{3}{2}\varepsilon_i\right)^2\right] + \frac{3}{2}\sqrt{3\varepsilon_i}.$$

Thus, due to (17),

$$|z^{i,1} - z^*| < P_i .$$

In the same way one can successively establish that for $s = 2, 3, \ldots$, $(s < s(i))$ the inequalities

$$|z^{i,s} - z^*| - |z^{i,s-1} - z^*|$$

$$< \frac{\ldots}{2 \ldots_i}\left[d_3(c_0 + P_i)^2 (1 + \gamma' r_i (c_0 + P_i))^2 r_i - (\delta_i - \tfrac{3}{4}\varepsilon_i)^2\right]$$

$$+ \frac{3}{2}\sqrt{3\varepsilon_i} < 0 \tag{19}$$

and

$$|z^{i,s} - z^*| < P_i$$

are satisfied and, moreover, (19) yields $s(i) < \infty$.

For $s = s(i)$ we conclude from Proposition 1 and Lemma 2' that

$$|z^{i,s(i)} - z^*| - |z^{i,s(i)-1} - z^*| < \sqrt{d_3}(c_0 + P_i)(1 + \gamma' r_i (c_0 + P_i))\sqrt{r_i} + \tfrac{3}{4}\sqrt{3\varepsilon_i} \tag{20}$$

hence,

$$|z^{i,s(i)} - z^*| < P_i + \sqrt{d_3}(c_0 + P_i)(1 + \gamma' r_i (c_0 + P_i))\sqrt{r_i} + \tfrac{3}{4}\sqrt{3\varepsilon_i}\,.$$

Now, (15) implies that lz^{i+1},^{D}- z*I < Pi+l·

Continuing in a similar manner, we obtain *s(i)* < oo for all i and

lz^{i},^{s} - z * I < P$_{i}$ for each i and 0:::; *s* < *s(i).*

However, due to Lemma 2.2.2 in Polyak (1987), the conditions (16), (18) together with (15) ensure that P$_{i}$ -+ p < oo monotonously. Thus,

lz^{i},^{s} - z*I < p for all (i, *s).*

Replacing in (20) P$_{i}$ by p, we obtain

lz^{i},^{s}(i) - z * 1 - 1z^{i},^{s}(i)- 1 - z * I< yds(co+p)(1+-*Jr:'(co+p)*.)-Jr:'+ v'3E i ·(21)

Taking into account (19), (16) and (21), convergence of the sequence {1z^{i},^{s} - z*I} can be stated from Lemma 2.2.2 mentioned above.

Now, in order to complete this analysis, nothing else has to be done than to use the arguments made for bounded U$_{ad}$ (see the proof of Theorem 1 in Part I).

II

The conditions (15)-(18), reflecting the choice of the controlling parameters, are compatible. In particular, they can be chosen as follows:

(i) take {r a and {E;}, satisfying (16);

(ii) choose r1 :::; r_ such that P1\fTl:::; d4;

(iii) knowing El, p1 and r1, calculate s1 according to (17) and p2 via (15)-;

(iv) define r2 :::; r; such that p2/ri,:::; d4 etc.

To state convergence for Method 1 (described in Part I) condition (17) in the theorem above is superfluous.

In Hettich, Kaplan, Tichatschke (1994) there is a different result concerning the choice of *ri* in Method 2 for unbounded Ua$_{d}$, in particular, instead of the conditions

L fo, < oo and P$_{i}$ v'r;:' :::; d4 forall i,
i=l

the assumption

r/$^{L v}$ < oo with an arbitrary chosen v E (0, 4]
i=1

was made.

1.2. Problem 2 with state constraints

Now, we consider Problem 2 with state constraints, i.e., *G/= Y,* and the control set Ua$_{d}$ may be unbounded. Let us assume that the condition

Tu E int G(in Y) for some ii, E U$_{ad}$ (22)

is fulfilled. As before, let co and Pi,s be constants such that

ꝏ 2 [[Cy* - Kd[[

and

Pi,s > [z* - zi,s [, s = 0,. , . , s(i) - 1.

Now, observe that zi,s = (yi,s, ui,s) E G X Uad.

Obviously, in this case the relations (5)-(8) remain true. For Jf,s, defined by (52'), estimate (11) is valid too, and in the same manner as for the case of bounded Uad (see Sect. 4,2' in Part I) we verify the estimate

Ill i,s II 10 -1 [lli5 i,s II(llv, - v, i,s II + 10) + r:;'lli5 i,s 1l1vGi' s II
+llv. - v. i,s l1lfv. i,s - v. i,s-1 lll.

This relation together with (5), (6), (8), (11) and r; < 1 leads to

[[q i,s II 10 1 [ffp i,s II([lii. - u*II + flu* - u i,s II + 10) + r:;'lli ',8 III r:;'t ,s II
+(Ifii, - v.* II + IIv,* - v, i,s II) ffu i,s - u i,s-1 II]
< 10 1 h(co + Pi,s-1)([[u - u*II+ Co+ 2pi,s-1 + 10) + c7(co + Pi,s-1)²
+(co+ Pi,s- 1)([[u - v,*11 +Co+ 2pi,s-1)].

Setting

Cg = max{1, Co 1([[ii, - u*II + 10)}, G0 = 10 1 [2c7C9 + C7+ 2cg],

we get

$$\|\bar{q}^{i,s}\| < c_{10}(c_0 + \rho_{i,s-1})^2.$$

Therefore,

[[A;i/ ,8 - f - U i,8 [[< c10(co+ Pi,s- 1)² r;,

and we infer that

$$\|\bar{z}^{i,s} - \zeta^{i,s}\|_X < c_{10}(c_0 + \rho_{i,s-1})^2 r_i. \tag{23}$$

However, as mentioned before, in case G # Y the point (i,s = (ifs, u i,s) defined by (9) may be not feasible. In this situation, due tor; < 1, (5) and (8), we obtain

[[Ag i,s -Ay*[1 - I[u* - u i,s 1l [[Ag i,s - f - u i,s 1l<Co+ Pi,s- 1,

[[Ag i,s - Ay*[[< 2ꝏ + 3pi,s- 1,

hence,

[[z i,s - z* [[x < 4(Co + Pi,s- 1)· (24)

Using the inequality

$$\|\zeta^{i,s} - z(u)\|_X \leq \|\zeta^{i,s} - z^{i,s}\|_X + \|z^{i,s} - z^*\|_X + \|z^* - z(v)\|_X,$$

together with (10) and (24), we conclude that

$$\|\zeta^{i,s} - z(v)\|_X < \|z^* - z(v)\|_X + 5(c_0 + p_{i,s-1}) \cdot \tag{25}$$

Now, by virtue of the choice of $w^{i,s}$ and $h^{i,s}$ (Sect. 4.2'), from (76'), (23) and (25) it follows (see the figure in Sect. 4.2') that

$$\|\zeta^{i,s} - h^{i,s}\|_X < \frac{1}{\tau_{min}}[\|z^* - z(u)\|_X + 5(c_0 + p_{i,s-1})]c_0(c_0 + p_{i,s-1})^2 r_i .$$

Let $\bar{z}^{i,s}$ be the feasible point closest to $z^{i,s}$ according to the norm $\|\cdot\|_X$. Proceeding as in (77'), we obtain

$$\|\bar{z}^{i,s} - z^{i,s}\|_X$$
$$< \frac{1}{\tau_{min}}[\|z^* - z(u)\|_X + (5 + \tau_{min})(c_0 + p_{i,s-1})]c_0(c_0 + p_{i,s-1})^2 r_i . \tag{26}$$

For arbitrary $v \in (0, 1)$ let

$$R_{i,s-1}(h) = \frac{c_0}{\tau_{min}}[\|z^* - z(v)\|_X + (5 + \tau_{min})(c_0 + p_{i,s-1})](c_0 + p_{i,s-1})r_i^v ,$$

then equality (26) may be rewritten in the form

$$\|\bar{z}^{i,s} - z^{i,s}\|_X < R_{i,s-1}(\tau_i)(c_0 + p_{i,s-1})r_i^{1-v} . \tag{27}$$

To finish the analysis of that case, we use inequality (27) in the same way as (12) and follow the proofs of Proposition 1 and Theorem 1.

Nevertheless, to obtain weak convergence of $\{z^{i,s}\}$, instead of the former condition $\sum_{i=1}^{\infty} \sqrt{r_i} < \infty$, we have now to require that $\sum_{i=1}^{\infty} r_i^{(1-v)} < \infty$. Moreover, the recurrent formula for p_i proves to be more complicated than (15).

2 Further applications of Method 2

In order to reduce the description, here we only consider the case that U_{ad} is bounded and there are no state constraints. A modification of the corresponding results to the case of unbounded sets U_{ad} and state constraints can be done analogously as in Section 1 for Problem 2.

For the problems studied here it is obvious that the optimal set U^* is non-empty.

2.1. Distributed control problems with Neumann conditions

Now, we deal with

Problem 3

$$\text{minimize } J(u) = \int_\Omega (Cy(u) - z_d)^2 \, d\Omega, \text{ subject to } u \in U_{ad}, \tag{28}$$

where $y(u)$ is the unique solution of the Neumann problem

$$Ay = f + u \text{ in } \Omega, \quad \frac{\partial y}{\partial \nu_A} = 0 \text{ on } \Gamma. \tag{29}$$

Here we suppose that $C \in \mathcal{L}(H^1(\Omega), L_2(\Omega))$, U_{ad} is a closed, convex and bounded subset of $L_2(\Omega)$ and make the same assumptions as in Section 1 concerning A, z_d and f (additionally, $a_0(x) > 0$ on $\bar{\Omega}$). By $\frac{\partial}{\partial \nu_A}$ the normal derivative associated with A is denoted.

In this case it is convenient to choose

$$Y = \left\{ y \in H^1(\Omega) : \; Ay \in L_2(\Omega, \frac{\partial y}{\partial \nu_A} \in L_2(\Gamma) \right\}, \tag{30}$$

$$\|y\|_Y = \left(\|Ay\|^2 + \left\| \frac{\partial y}{\partial \nu_A} \right\|^2_{L_2(\Gamma)} \right)^{1/2} \tag{31}$$

and $X = Y \times L_2(\Omega)$ with the norm $\|(y,u)\|_X = (\|y\|_Y^2 + \|u\|^2)^{1/2}$.

On X we introduce also the equivalent norm $|\cdot|$:

$$|z| = \left(\|Ay - u\|^2 + \|u\|^2 + \left\| \frac{\partial y}{\partial \nu_A} \right\|^2_{L_2(\Gamma)} \right)^{\frac{1}{2}} \quad \text{for } z = (y; u).$$

It is easy to show, analogously to (37'), that

$$\|z\|_X^2 \leq |z|^2 \leq 3 \|z\|_X^2.$$

Setting

$$\Psi_{i,s}(y, u) = \int_\Omega (Cy - z_d)^2 \, d\Omega$$

$$+ \frac{2}{\chi_i} \left(\int_\Omega (Ay - f - u)^2 \, d\Omega + \int_\Gamma \left(\frac{\partial y}{\partial \nu_A} \right)^2 d\Gamma \right) + \int_\Omega (u - u^{i,s-1})^2 \, d\Omega,$$

the description of Method 2 for Problem 3 is formally the same as in Section 3' or 1.

Again, existence and uniqueness of a solution $\bar{z}^{i,s} = (\bar{y}^{i,s}, \bar{u}^{i,s})$ of the auxiliary problem in Step (b) of Method 2 follows from the strong convexity of $\Psi_{i,s}$ at X,

Here we also obtain that

$$\|\bar{z}^{i,s} - z^{i,s}\|_X \le \frac{3}{2}\epsilon_i.$$

Similarly to (45') one can conclude for each i and s = 1, ... , s(i) that

$$K_{i,s}(z^*) \geq \frac{2}{r_i}\left(\int_\Omega (A\bar{y}^{i,s} - f - \bar{u}^{i,s})^2 d\Omega + \int_\Gamma \left(\frac{\partial \bar{y}^{i,s}}{\partial v_A}\right)^2 d\Gamma\right),$$

with $K_{i,s}$ defined by (43') and z^* an optimal process of Problem 3.

Due to the boundedness of U_{ad} and $\{r_i\}$, this leads to

$$\|\bar{y}^{i,s}\|_Y < c_1, \ \|y^{i,s}\|_Y < c_1, \tag{32}$$

$$\left(\int_\Omega (A\bar{y}^{i,s} - f - \bar{u}^{i,s})^2 d\Omega + \int_\Gamma \left(\frac{\partial \bar{y}^{i,s}}{\partial v_A}\right)^2 d\Gamma\right)^{\frac{1}{2}} < c_2\sqrt{r_i}. \tag{33}$$

(It should be noted that we here start with a new count of the constants).

For the pair $(\hat{y}^{i,s}, \bar{u}^{i,s})$ with

$$A\hat{y}^{i,s} = f + \bar{u}^{i,s} \text{ in } \Omega, \quad \frac{\partial \hat{y}^{i,s}}{\partial v_A} = 0 \text{ on } \Gamma,$$

we obtain from (33)

$$\|(\bar{y}^{i,s}, \bar{u}^{i,s}) - (\hat{y}^{i,s}, \bar{u}^{i,s})\|_X < c_2\sqrt{r_i}.$$

Hence, the feasible point $\hat{z}^{i,s} = (\hat{y}^{i,s}, \bar{u}^{i,s})$, closest to $\bar{z}^{i,s}$ in the norm $\|\cdot\|_X$, observes

$$\|\hat{z}^{i,s} - \bar{z}^{i,s}\|_X < c_2\sqrt{r_i}, \text{ for each } i \text{ and } s = 1, \dots, s(i).$$

Now we show that there exists a constant c_3 such that for each $y \in Y$

$$\|y\| \leq c_3\|y\|_Y. \tag{34}$$

It is well-known that the problem

$$A\mu = \eta \text{ in } \Omega, \quad \frac{\partial \mu}{\partial v_A} = \varphi \text{ on } \Gamma,$$

with $\eta \in L_2(\Omega)$, $\varphi \in L_2(\Gamma)$ has a unique solution $\mu \in H^1(\Omega)$. Moreover, the estimate

$$\|\mu\|_{H^1(\Omega)} \leq c_4(\|\eta\| + \|\phi\|_{L_2(\Gamma)}) \tag{35}$$

is true with a constant c_4 independent of η and φ. Indeed, using the equality

$$\int_\Omega \left(\sum_{i,j} a_{ij} \frac{\partial \mu}{\partial x_i} \frac{\partial \mu}{\partial x_j} + a_0 \mu^2 \right) d\Omega = \int_\Omega \eta \mu \, d\Omega + \int_\Gamma \mu \frac{\partial \mu}{\partial \nu_A} \, d\Gamma$$

together with the estimate for the traces of the functions in $H^1(\Omega)$

$$\|\gamma w\|_{L_2(\Gamma)} \le c_5 \|w\|_{H^1(\Omega)} \quad \text{for all } w \in H^1(\Omega)$$

(see, for instance, Necas, 1967), we obtain

$$\|\mu\|^2_{H^1(\Omega)} \le c_6 \left(\|\eta\| \|\mu\| + \|\gamma \mu\|_{L_2(\Gamma)} \left\| \frac{\partial \mu}{\partial \nu_A} \right\|_{L_2(\Gamma)} \right)$$

$$\le c_6 \left(\|\eta\| \|\mu\|_{H^1(\Omega)} + c_5 \|\mu\|_{H^1(\Gamma)} \left\| \frac{\partial \mu}{\partial \nu_A} \right\|_{L_2(\Omega)} \right)$$

and finally

$$\|\mu\|_{H^1(\Omega)} \le c_5 (\|\eta\| + c_5 \|\varphi\|_{L_2(\Gamma)}),$$

Therefore, estimate (35) is fulfilled if $c_4 = c_5 \max\{1, c_5\}$ is chosen. Inequality (34) follows from (35) with $c_3 = c_4$ if we put

$$\eta = Ay \text{ and } \varphi = \frac{\partial y}{\partial \nu_A}.$$

Thus, (32) and (34) yield

$$\|\hat p^{i,s}\| < c_7, \quad \|\hat y^{i,s}\| < c_7.$$

Finally, due to (33) and (34),

$$\|\hat p^{i,s} - \bar p^{i,s}\| < c_8 \sqrt{r_i}.$$

Now, similarly to (57'), one can conclude that for each i and $s = 1, \ldots, s(i)$

$$\left(\int_\Omega (A g^{i,s} - f - \hat u^{i,s})^2 d\Omega + \int_\Gamma \left(\frac{\partial \hat y^{i,s}}{\partial \nu_A} \right)^2 d\Gamma \right)^{\frac{1}{2}} < c_9 r_i,$$

In this case we use $p^{i,s}$ defined by

$$A^* p^{i,s} = C^*(C y^{i,s} - K_d), \quad \frac{\partial p^{i,s}}{\partial \nu_{A^*}} = 0 \text{ on } \Gamma,$$

and the approximate Lagrange multipliers are

$$\lambda_1^{i,s} = \frac{1}{r_i} (A g^{i,s} - f - u^{i,s}), \quad \lambda_2^{i,s} = \frac{1}{r_i} \frac{\partial y^{i,s}}{\partial \nu_A}.$$

Thus, following the proofs of Proposition 5' and Theorem 1', we can state

Theorem 2 *Assume that* Uad *is a bov,nded set in* L_2(f2).. *Let the positive sequences* $\{r_i\}$, {Ei} *and* {oi} *be chosen such that*

svp$_i$Ei < 1, sv,p$_i r_i$ < 1, $\sum_{i=1}^{\infty}$ -/r;. < oo, $\sum_{i=1}^{\infty}$ Ei < oo (36)

and

-1- [d1r! - (0: - E-) 2] + V3E· " < 0' 0" > E- "'
2d$_2$ 2 2 2

with positive constants d_1, d_2 *defined analogously to those in Theorem 1' (i. e., they ensv,re the validity of the relat?:ons (30') and {31 ') with*

,h(y, u) = / Jo (Cy - K,d)^2 df2 + 2/r_i (/ Jo (Ay - f - v,)^{2}dD + / lr (.!!.:.dr) EJvA)

and with corresponding sequences $\{z^{i,s}\}$, $\{z^{i,s}\}$).

Then, starting with an arbitrary element v,^{1,0} E Uad, *Method 2 is well-de.fined for Problem 3, i. e., s(i) < oo for each* i; *the sequence* $\{v^{i,s}\}$ *converges weakly in* $L_2(D)$ *to* il *and* $\{y^{i,s}\}$ *converges weakly in* Y *(given by {30)) toy, (y, u) an optimal process for Problem 3.*

Concerning Method 1 described in Part I, already (36) is sufficient for weak convergence of $\{u^{i,s}\}$ to *il* and $\{y^{i,s}\}$ to *y.*

2.2. Comments on the solution of boundary control problems

At first we consider

Problem 4

m,l,nl,ml,ze / Jo (Cy(v,) - K,d)^2 df2 *subject to* v, E Uad, (37)

where y(v,) is the v,nique solution of the Dirichlet problem

$Ay = f$ inn, $y =$ v, on r, (38)

with C E $l(L_2(D), L_2(D))$, Uad *a convex, closed and bonnded snbset of* L_2(f), f E L_2(f2)$_1$ K,d E $L_2(D)$ *and* A *de.fined as in Section 1.*

In this case, Method 2 is applicable with the following regularized penalty function

$$\Psi_{i,s}(y,u) = \int_\Omega (Cy - \kappa_d)^2 d\Omega + \frac{1}{r_i}\left(\int_\Omega (Ay - f)^2 d\Omega + \int_\Gamma (y-u)^2 d\Gamma\right)$$
$$+ \int_\Gamma (u - u^{i,s-1})^2 d\Gamma.$$

To this end we choose the space $Y = \{y \in L_2(\Omega) : Ay \in L_2(\Omega), \gamma y \in L_2(\Gamma)\}$ with the norm $\|\cdot\|_Y$ defined by

$$\|y\|_Y = \left(\|Ay\|^2 + \|\gamma y\|^2_{L_2(\Gamma)}\right)^{1/2}$$

and put $X = Y \times L_2(\Gamma)$. For $z = (y, v)$ in X two norms $\|\cdot\|_X$ and $|\cdot|$ are introduced by

$$\|z\|_X = \left(\|y\|_Y^2 + \|v\|^2_{L_2(\Gamma)}\right)^{1/2},$$

$$|z| = \left(\|Ay\|^2 + \|v\|^2_{L_2(\Gamma)} + \|\gamma y - v\|^2_{L_2(\Gamma)}\right)^{1/2},$$

and again we derive that

$$\tfrac{1}{2}\|z\|_X^2 \le |z|^2 \le 3\|z\|_X^2.$$

Finally, let us consider

Problem 5

$$\textit{minimize} \int_\Omega (Cy(v) - y_d)^2 \, d\Omega \quad \textit{subject to } v \in U_{ad}, \tag{39}$$

with $y(v)$ *the unique solution of the Neumann problem*

$$Ay = f \ \textit{ in } \Omega, \quad \frac{\partial y}{\partial \nu_A} = v \ \textit{ on } \Gamma, \tag{40}$$

$C \in \mathcal{L}(H^1(\Omega), L_2(\Omega))$ *and the same assumptions w.r.t.* A, U_{ad}, f *and* y_d *as for Problem* 4, *but* $a_0(x) > 0$ *on* Ω.

Here we construct the regularized penalty function

$$\Psi_{i,s}(y, v) = \int_\Omega (Cy - y_d)^2 \, d\Omega + \frac{1}{r_i}\left(\int_\Omega (Ay - f)^2 \, d\Omega + \int_\Gamma \left(\frac{\partial y}{\partial \nu_A} - v\right)^2 d\Gamma\right) + \int_\Gamma (v - v^{i,s-1})^2 \, d\Gamma$$

and define Y by means of (30), (31) and $X = Y \times L_2(\Gamma)$.

The norms

$$|z| = \left(\|Ay\|^2 + \|v\|^2_{L_2(\Gamma)} + \left\|\frac{\partial y}{\partial \nu_A} - v\right\|^2_{L_2(\Gamma)}\right)^{1/2}$$

and

$$\|z\|_X = \left(\|y\|_Y^2 + \|v\|^2_{L_2(\Gamma)}\right)^{1/2},$$

considered in X are equivalent.

After this preparation similar statements on convergence of Method 2 can be established for these problems, using the technique developed in Sections 4' and 1.1. They differ from the Theorems 1' and 2 only in the choice of the constants d_1 and d_2. In case of Method 1 convergence can be stated only under assumption (33').

References

HETTICH, R., KAPLAN, A. and TICHATSCHKE, R. (1994) Regularized penalty methods for optimal control of ill-posed elliptic systems, in: Schwerpunktprogramm der Deutschen Forschungsgemeinschaft, "Anwendungsbezogene Optimierung und Steuerung", Report No., 522.

NECAS, J. (1967) *Les Methodes Directes en Theorie des Equations Elliptiqv,es.* Masson, Paris.

POLYAK, B.T. (1987) *Introduction to Optimization.* Optimization Software, Inc. Puhl. Division, New York.