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Regularized penalty methods for ill-posed optimal control problems with elliptic equations

Part I: Distributed control with bounded control set and state constraints

by

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Abstract: We investigate the application of Prox-Regularization to ill-posed convex control problems governed by elliptic equations. Stable variants of Penalty Methods are obtained by means of One- and Multi-Step Regularization of the penalized problems. Convergence of the resulting methods IR proved in the case of distributed control (with a bounded set of admissible controls) and Dirichlet boundary conditions. More general types will be considered in the second part of this paper.

Keywords: distributed control, elliptic equations, prox-regularization, penalty methods.

1. Introduction

A large number of interesting physical and technical problems give rise to optimal control models where the state of the system is governed by partial differential equations. The fundamental monograph of Lions (1968) gives an excellent introduction into the mathematics of these models for various types of differential equations, boundary conditions, and control. In this paper, we deal with problems whose states are described by second order elliptic equations. A general class may be specified as follows. Let D C IR^n be an open domain with boundary Γ of the class C^2, D = D U Γ. Then, with coefficients

$$\% \ E\ C^2(D),\ i,j = 1,\dots,n,\ a_0\ E\ C^2(D) \qquad (1)$$

such that for all x E n, E R^n and a constant c > 0

$$a_0(x) > 0 \text{ and } \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c\sum_{i=1}^{n} \xi_i^2, \tag{2}$$

consider the elliptic second order differential operator A defined by

$$Ay := - L_{i,J=1} \frac{8}{OXi} \left(a\, ij \frac{oy}{OXl} \right) + aoy. \tag{3}$$

Further let H denote a Hilbert space and

U_{ad} C H a non-empty, closed, convex set, (4)

the set of admissible controls.

In case of distributed control, the state of the system is governed by

$$\begin{array}{ll} Ay = f + \text{Dov}, & \text{in !1} \\ By = 0 & \text{on r} \end{array} \tag{5}$$

with u E U_{ad} C H = L2(D), f E L2(!1), Do E Z(L2(!1), L2(!1)) a linear continuous operator, and B a boundary operator (of Dirichlet or Neumann type, for instance).

In case of boundary control, the state is described by

$$\begin{array}{ll} Ay = f & \text{in n} \\ By = g + D_1 u & \text{on r} \end{array} \tag{6}$$

with f E L2(!1), g E L2(f), v, E U_{ad} C H = L2(f), D1 E Z(L2(r),L2(f)) and B as above.

Let us assume now that, for v, E U_{ad}, $y(v)$ is uniquely determined by (5) or (6) respectively and that $y(v)$ E V, V a Hilbert space. For instance, in case of (5) with B the trace operator, an appropriate choice could be V = HJ(D),

Given now U_{ad} and a state equation as above, the problem is to minimize the functional

$$J(u) = \text{IICy(v,) - K,dllt} + ((Nv,, u))H \tag{7}$$

subject to v, E U_{ad}, where C E $l(V, Ji)$, li a Hilbert space, K_d E li a given element, and N E l(H, H) a positive semi-definite operator. Throughout the paper, II, 11s, ((·, ·))s denote norm and scalar product in space S, the subscript being omitted, i.e. II. II, ((·, ·)), in case of S = L2(!1), For later reference we define

Problem 1 *With a control set* U_{ad}, *a boundary problem, and an objective* J *given by (4), (5) or (6), and (7), respectively,*

minimize $J(u)$ *subject to* u E U_{ad}. (8)

In addition, a state constraint

$$y(v) \in G \subset V, \; G \text{ closed, convex} \tag{9}$$

may be considered

(cf. e,g. Bergounioux, 1989, Casas, 1986, and for problems governed by semi-linear equations, Bonnans and Casas, 1995, Casas, 1993).

We note that the assumption on the smoothness of the data and the type of the boundary conditions may differ essentially from the above (for instance $a_{i,j}$, $a_0 \in L^{\infty}(\Omega)$ instead of (1), or a non-homogeneous boundary condition in (5) etc.).

Particularly in numerical contributions it is common to assume the operator N to be positive definite, implying Problem 1 to be well-posed. A frequent choice is $N = \lambda I$, $\lambda > 0$. We believe that in many practical cases $N \equiv 0$ corresponds to a natural and relevant model, too. It concentrates on the primary aim of the process expressed by the first term in the objective (7) and leaves restrictions on the cost of control to the constraints defining U_{ad}.

For $N = 0$, however, the problem is likely to become ill-posed and more difficult to handle. For bounded sets U_{ad}, a solution still exists (cf. Lions, 1968) but it may be non-unique unless rather unreasonable assumptions are imposed (e.g. that B and C be injective). In case of an unbounded U_{ad} it may occur that the set of optimal controls is empty or unbounded.

In this paper, in order to deal with the ill-posed case, we consider penalty methods stabilized by means of iterative prox-regularization. In this, we follow a general approach developed in Kaplan, Tichatschke (1994). To be applicable to our optimal control Problem 1 a number of substantial modifications and supplements are necessary due to the following circumstances:

- There are serious difficulties in estimating the closeness between the solutions of the original and discretized problems;
- The objective J depends in an implicit way on the control u;
- In general, it is impossible to uniformly estimate the Lagrange multipliers of the discretized problems as there are no suitable regularity conditions available.

For technical reasons, the paper is split in two parts. Part I extensively and exemplarily deals with the case of distributed control, Dirichlet boundary conditions, and bounded U_{ad}. In Section 2 this (Problem 2) is specified in detail together with some of its properties and some notation. In Section 3 the numerical methods are given together with the main result (Theorem 1) of this paper guaranteeing weak convergence of the iterates to a solution under appropriate assumptions on the parameters controlling the methods. As mentioned above, a penalty technique (cf Bergounioux, 1992) is combined with iterative prox-regularization. Two variants are considered: In Method 1 the penalty and the regularization term are adapted synchronously whereas in Method 2 proximal iterations with fixed penalty term continue as long as reasonable progress is achieved (cf. step (c) in Method 2 below). A further significant peculiarity of our approach is that regularization is accomplished only with respect to the control variable but not the state variable. Such a partial regularization (in a subspace of the space of variables) for non-separable problems has been considered first in Kaplan, Tichatschke (1994). Section 4 is devoted to the rather laborious proof of Theorem 1. In Section 5 of Part I, we demonstrate the effect of regularization by means of a simple example.

In Part II the results are extended to more general problems admitting unbounded Uad (Section 1), Neumann boundary conditions and/or boundary control (Section 2).

2 Statement of Problem 2

In the whole of Part I we will deal with the following instance of Problem 1 (cf. Bergounioux, 1992, 1993, Butkovski, 1969, Casas, 1986, Fortin, Glowinski, 1982, Gruver, Sachs, 1984, Hoffmann, Krabs, 1983, Lions, 1983): With D C R^n an open and bounded domain, r its boundary, and D = D U r, let

$$H = \mathcal{H} = L_2(\Omega),\ V = H_0^1(\Omega) \tag{10}$$

and A a second order, elliptic operator (3) with coefficients a_0, aij satisfying (1) and (2). The inequality ao(x) > 0 may be relaxed to a_0(x) 0 on D. For v, E U_ad we consider the Dirichlet problem

$$S1(v): \begin{array}{ll} Ay = f + v, & \text{in D}, \\ y = Q & \text{on f}, \end{array} \tag{11}$$

with given f E L2(D) and Uad a bounded, closed, and convex subset of L2(D).

Assuming r belongs to the class C^2, it is well-known (cf. e.g. Aubin, 1972, Theorem 7.1.1) that, given v, E L2(D), S1(v,) has a unique solution

$$y(u) \in H^2(\Omega) \cap H_0^1(\Omega) \tag{12}$$

such that the mapping

$$Tu := y(u) \ \ (T: \ L_2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)) \tag{13}$$

is well defined. By Y we denote the space of functions

$$Y = \{YIY \text{ E H5(D}, Ay \text{ E L2(D)}\}. \tag{14}$$

Employing the Cauchy-Schwarz-Inequality and (1), it gets immediate that by

$$((y,z))_Y := ((Ay,Az)), \quad \|y\|_Y := \|Ay\| \tag{15}$$

a scalar product and a norm are given on Y (recall that((,,·)) and ‖·‖ denote the scalar product and the norm in L2(D)). Moreover, we have

Proposition 1 *Y is a Hilbert space with scalar prodv,ct and norm given by (15), Algebraically, Y coincides with the set offunctions* H^2(D) n HJ(D)

To establish the second part of Proposition 1, let y E Y, Thus, $f := Ay$ E L$_2$(D), and, of course, y solves

$$Ay = f \text{ in D}, \; y = 0 \text{ on } \mathbf{r},$$

implying y E H^2(D) n HJ(D), The opposite implication is obvious,

Further, we define

$$X = Y \times L_2(\Omega) \tag{16}$$

with norm

$$\text{II}(y, v,) \text{llx} = (\text{IIYllir} + \text{lv,ll}^2)^{\frac{1}{2}} = (\text{IIAYll}^2 + \text{llull}^2)^{\frac{1}{2}}. \tag{17}$$

Using (L$_2$(D))' = L$_2$(D) one gets $X' = Y'$ x L$_2$(D) for the dual, We have (Lions, 1968):

Proposition 2 *Under the assumptions on S1 (v,), every u E U_{ad} v,niquely defines a pair (y(u),v,)* EX.

Finally, to specify the objective functional, let (see (10))

$$C \text{ E l(HJ(D),L}_2\text{(D))} = l(V, 1i), \; \text{"d E L}_2\text{(D)}. \tag{18}$$

Then the problem considered in Part I of the paper is

Problem 2 *Minimize*

$$J(u) := \int_\Omega (Cy(u) - \kappa_d)^2 d\Omega \tag{19}$$

s11,bject to v, E U_{ad} and a state constraint

y(v,) E G C Y, G closed and convex,

where we assume that there exists a ii, E U_{ad} such that y(u) E int G (in Y).

Recall that y(v,) is the v,nique solution of system (11).

Proposition 3 *(Lions, 1968)*

In case U_{ad} is bounded, the set U^* of optimal controls for Problem 2 is a non-empty, closed, convex subset of Uad.

3. Penalty methods with prox-regularization for Problem 2

In this section, we formulate two methods for solving Problem 2 and state their main properties. Proofs will be given in the next section. In the spirit of Lions (1968), a penalty method for solving Problem 2 could proceed as follows:

Given a sequence $\{r_i\}$, $r_i > 0$, $\lim_{i\to\infty} r_i = 0$, define the functionals

$$J_i(y,u) = \int_\Omega (Cy - \kappa_d)^2 d\Omega + \frac{1}{r_i}\int_\Omega (Ay - f - u)^2 d\Omega \tag{20}$$

and compute a sequence of minimal points $z^i = (y^i, u^i)$ of J_i w.r.t. $G \times U_{ad}$.

Then the question is whether z^i converges to an optimal process for Problem 2.

In case of strictly convex functionals J of type (7) with $\mathcal{H} = H = L_2(\Omega)$, $N = I$ and $C : V \to \mathcal{H}$ an embedding operator, a positive answer to this question is given in Bergounioux (1992). For possible ill-posed problems similar results cannot be expected. Therefore, to enforce strict convexity of the auxiliary problems, we use regularization by means of the proximal mapping (Kaplan, Tichatschke, 1994, Rockafellar, 1976) employing an additional "regularization term" w.r.t. variable u (note that in (20), y and u are considered to be independent variables).

Method 1 *(One-step regularization)*

Let positive sequences $\{r_i\}$, $\{\epsilon_i\}$, *with* $\lim_{i\to\infty} r_i = \lim_{i\to\infty}\epsilon_i = 0$, $\sup_i r_i < 1$, $\sup_i \epsilon_i < 1$ *and* $u^0 \in U_{ad}$ *be chosen.*

Step i. *Given* $u^{i-1} \in U_{ad}$, *let, with* J_i *defined by (20),*

$$\Psi_i(y,u) = J_i(y,u) + \int_\Omega (u - u^{i-1})^2 d\Omega \tag{21}$$

and

$$(\bar y^i, \bar u^i) = \operatorname{argmin}\{\Psi_i(y,u) : (y,u) \in G \times U_{ad}\}, \tag{22}$$

Compute an approximation $(y^i, u^i) \in G \times U_{ad}$ *of* $(\bar y^i, \bar u^i)$ *such that*

$$\|\nabla\Psi_i(y^i, u^i) - \nabla\Psi_i(\bar y^i, \bar u^i)\|_{X'} \le \epsilon_i. \tag{23}$$

Here, $\nabla\Psi_i \in X'$ denotes the Gateaux-derivative of Ψ_i which is easily seen to be given by

$$\nabla\Psi_i(y,u)(\eta, v) = 2\int_\Omega \left[(Cy - \kappa_d)C\eta + \frac{1}{r_i}(Ay - f - u)(A\eta - v) + (u - u^{i-1})v\right] d\Omega \tag{24}$$

for $(\eta, v) \in X$. Note that for $(\bar y^i, \bar u^i)$ being a minimum according to (22) it is necessary that

$$\nabla\Psi_i(\bar y^i, \bar u^i)(y - \bar y^i, u - \bar u^i) \ge 0 \text{ for all } (y,u) \in G \times U_{ad}. \tag{25}$$

Method 2 *(Multi-step regularization)*

Let $\{r_i\}$, $\{\epsilon_i\}$, $u^0 \in U_{ad}$ *be as in Method 1, and* $\{\delta_i\}$ *a third positive sequence (not necessarily tending to 0).*

Step i. *Given* $u^{i-1} \in U_{ad}$,

(a) *set* $u^{i,0} = u^{i-1}$, $s = 1$;

(b) *given* $u^{i,s-1}$, *let (with* J_i *defined by (20))*

$$W_{i,s}(y,u) = J_i(y,u) + \int_\Omega (u - u^{i,s-1})^2 \, d\Omega \tag{26}$$

and

$$(\bar{y}^{i,s}, \bar{u}^{i,s}) = \arg\min\{W_{i,s}(y,u) \mid (y,u) \in G \times U_{ad}\}. \tag{27}$$

Compute an approximation $(y^{i,s}, u^{i,s}) \in G \times U_{ad}$ *of* $(\bar{y}^{i,s}, \bar{u}^{i,s})$ *such that*

$$\|\nabla W_{i,s}(y^{i,s}, u^{i,s}) - \nabla W_{i,s}(\bar{y}^{i,s}, \bar{u}^{i,s})\|_{X'} \le \epsilon_i. \tag{28}$$

(c) *If* $\|u^{i,s} - u^{i,s-1}\| > \delta_i$, *set* $s := s+1$ *and repeat (b).* *Otherwise, set* $u^i = u^{i,s}$, $s(i) = s$, *and continue with Step* $(i+1)$.

For $W_{i,s}$ (24) holds with $u^{i-1,s}$ instead of u^{i-1} and the necessary condition (25) applies accordingly.

To simplify the convergence analysis, we do not include a penalization of the state constraints. Such kind of penalization was used, for instance, in Neittaanmäki, Tiba (1995).

Proposition 4 *The functional* $W_{i,s}$ *(also* W_i*) is quadratic, continuous on* X, *and strongly convex, i.e. for some* $\alpha > 0$ *we have for all* $z = (y,u)$, $\bar{z} = (\bar{y}, \bar{u}) \in X$

$$W_{i,s}(\bar{z}) - W_{i,s}(z) \ge \langle \nabla W_{i,s}(z), \bar{z} - z \rangle + \alpha \|\bar{z} - z\|^2.$$

Proposition 4 ensures that the optimization problems (22), (27) become well-posed. Of course, the stopping rules (23) and (28) are not practicable. But, due to the strong convexity of $W_{i,s}$ and W_i, one can use (in order to satisfy (23) or (28)) in principle, any method which enables to define a point $(y,u) \in G \times U_{ad}$ such that

$$W_{i,s}(y,u) \le \inf_{(y,u) \in G \times U_{ad}} W_{i,s}(y,u) + \nu, \text{ with given } \nu > 0,$$

(analogously, for W_i).

For instance, if $U_{ad} \subset L_\infty(\Omega)$ and U_{ad}, G are given by means of pointwise constraints, the usual discretization approach (for instance, FEM) can be combined with (finite) conjugate direction methods or simple gradient projection methods which possess a suitable estimate of the convergence rate w.r.t. the values of the objective functional (see Kaplan, Tichatschke, 1994).

Remark 1 *Note that Method 1 can be considered as a special case of Method 2 by taking* δ_i *"sufficiently large", for instance*

$$\delta_i > d_0 := \sup\{\|u - v\| : u, v \in U_{ad}\}, \tag{29}$$

Method 2 allows for a fixed J_i *a more accurate minimization. This gives rise to the hope that in general, to obtain a certain accuracy, the value of the penalty parameter* f *can be kept smaller than in Method 1. Therefore, the numerical behavior of Method 2 can be expected to be much better.*

As mentioned above, penalization of the state equation permits to handle y and u as independent variables. But, of course, this complicates the discretization process: For instance, when applying finite element methods, elements with order higher than one have to be used. Concerning the application of high order finite element approximations to optimal control problems see Lasiecka (1995) and Hendrickson (1995).

Following the approach developed in Kaplan, Tichatschke (1994) for some problems in elasticity theory, for the control problems considered here a direct application (without penalization) of the multi-step regularization coupled with finite element methods is possible.

We start the convergence analysis with two auxiliary estimates which are in the essence consequences of the properties of the operator A and the boundedness of U_{ad}.

Proposition 5 *Let* (y^*, u^*) *be an optimal process of Problem 2. With an arbitrarily chosen* $u^{i,s-1} \in U_{ad}$ *let* $(y^{i,s}, u^{i,s})$ *and* $(\bar{y}^{i,s}, \bar{u}^{i,s})$ *be as in Substep (b) of Method 2. Then there exist constants* d_1 *and* d_2*, independent of* $u^{i,s-1}$, $\{\varepsilon_i\}$, $\{r_i\}$, i *and* $s \geq 1$*, such that*

$$J_i(y^*, u^*) - J_i\left(\bar{y}^{i,s}, \bar{u}^{i,s}\right) < d_1 r_i \tag{30}$$

and

$$\|(\bar{y}^{i,s}, \bar{u}^{i,s}) - (y^*, u^*)\| \leq d_2, \tag{31}$$

where $\|\cdot\|$ *is a norm on* $X = Y \times L_2(D)$ *defined by*

$$\|(y, u)\|^2 = \|Ay - Bu\|^2 + \|u\|^2. \tag{32}$$

Theorem 1 *Assume that the sequences* $\{r_i\}$, $\{\varepsilon_i\}$, *and* $\{\delta_i\}$ *in Method 2 satisfy the conditions*

$$\sup_i \varepsilon_i < 1, \quad \sup_i r_i < 1, \quad \sum_{i=1}^{\infty} \sqrt{r_i} < \infty, \quad \sum_{i=1}^{\infty} \varepsilon_i < \infty \tag{33}$$

and, with d_1, d_2 *from (30), (31),*

$$\frac{1}{2d_2}\left(d_1 r_i - \left(\delta_i - \frac{\varepsilon_i}{2}\right)^2\right) + \frac{3}{2}\varepsilon_i < 0, \quad \delta_i > \frac{3}{2}\varepsilon_i. \tag{34}$$

Then, for every choice of $v^0 \in U_{ad}$, *Method 2 is well-defined, especially* $s(i) < \infty$ *for every* i, *and* $\{u^{i,s}\}$, $\{y^{i,s}\}$ *converge weakly in* $L_2(D)$, Y, *to* u, $\hat{y}$ *respectively,* $(\hat{y}, u)$ *being an optimal process for Problem 2.*

For Method 1 already (33) is sufficient for weak convergence of $\{v^i\}$ *and* $\{y^i\}$ *to* $\hat{u}$, *and* $\hat{y}$, *respectively.*

In this paper, we do not focus on the explicit calculation of various constants, especially those which are connected with the estimation of solutions of boundary value problems and the norm of certain operators. We note that penalty methods in combination with prox-regularization (w.r.t. the full space) have been applied to convex optimization problems, for instance, in Alart, Lemaire (1991), Auslender, Crouzeix, Fedit (1987), Kaplan, Tichatschke (1996), Lemaire (1989). In contrast with the above, in these contributions Slater's condition with regard to the penalized constraints appears to be substantial.

4. Proofs

In this section we will give the proofs of the propositions and the theorem of the last section.

4.1. The case of $G = Y$, i.e. no state constraints

Proof of Proposition 4 The fact that Ψ_i, $\Psi_{i,s}$ are continuous, quadratic functionals on X is obvious.

To show that $\Psi_{i,s}$ is strongly convex (the proof for Ψ_i is analogous) it is sufficient to show that its quadratic part is a positive definite quadratic form.

An elementary calculation gives

$$\Psi_{i,s}(y, u) = Q_1(y, u) + Q_2(y, u) + L(y, u)$$

with an affine linear part L,

$$Q_1(y,u) = \int_\Omega (Cy)^2 d\Omega + \left(\frac{1}{r_i} - 1\right) \int_\Omega (Ay - u)^2 d\Omega$$

a positive semi-definite quadratic form (recall that $r_i < 1$) and

$$Q_2(y, u) = \int_\Omega (Ay - u)^2 d\Omega + \int_\Omega u^2 d\Omega.$$

We are done if we can show that Q_2 is positive definite. One can calculate (see (17))

$$\begin{aligned} Q_2(y,u) &= \int_\Omega \left(\tfrac{1}{2}Ay - u\right)^2 d\Omega + \tfrac{1}{4}(\|Ay\|^2 + \|u\|^2) + \tfrac{1}{4}\|u\|^2 \\ &\geq \frac{1}{c}\|(y, u)\|_X^2. \end{aligned} \qquad (35)$$

Thus, Proposition 4 is proved, 11111

From the above proof it follows that Q2 defines another norm $|\cdot|$ on X:

$$|(y,u)|^2 = Q_2(y,u) = \|Ay - u\|^2 + \|u\|^2. \tag{36}$$

From (35) and a simple estimation we have

1
311(y,v.)llx² :S l(y,v.)I² :S 3ll(y,v.)llx²• (37)

For shortness, in the sequel, the abbreviations

z = (y ,v.), z* = (y *,v,*), zi,s = (tf'8,ui,s) etc. (38)

will be used for elements in X = Y x L2(D).

To prove Proposition 5 we need

Lemma 1 *Let* zi,s, zi,s *be as in Method* 2, i.e. *(see* (28))

$$\left\|\nabla\Psi_{i,s}(z^{i,s}) - \nabla\Psi_{i,s}\left(\bar{z}^{i,s}\right)\right\|_{X'} \leq \epsilon_i.$$

Then,

llz,r s - z,r s llx :S 3/2 Ei, (39)

The proof is immediate from (35) and the fact that Wi,s - Q2 is a convex, quadratic functional.

We emphasize that Lemma 1 is a consequence only of the properties of Wi,s and (28) and otherwise independent of the method.

Proof of Proposition 5: Recall that in Proposition 5 ui,s-1 may be any point in Uad and note that all the constants ci in this proof are independent from {Ei}, {ri}, i and s 2: 1. For zi,s = (ii•S, iJi,s), the unique minimal point of wi,s on Y x Uad, let

lf'i s := 1/ri (Agi,s - f - v.i,s) . (40)

Using (24) and the optimality condition (25) for Wi,s instead of Wi we obtain due to the independence of *y*, *v*. that

/,.n (Cff,s - Kd) (Cy - Cff,8) *dD* + ;.a qi,s (Ay - Aif s) *dD* 2: 0 (41)

for all y E Y and

/.a (ii,i,s _ U,s- 1) (*v* _ ii,i,s) *dD* _ *Jn* if's (*v* _ ii,i,s) *dD* 2: 0 (42)

for all v, E Uad•

Furthermore, with

$$K_{i,s}(z) := \int_\Omega (Cy - K_d)^2 \, d\Omega + \int_\Omega (u - v^{i,s-1})^2 \, d\Omega, \tag{43}$$

in view of the gradient inequality for convex functions, we obtain for all $y \in Y$, $v \in L_2(\Omega)$ the inequality

$$\begin{aligned} K_{i,s}(z) - K_{i,s}(z^{i,s}) \geq\ & 2\int_\Omega (Cy^{i,s} - K_d)(Cy - Cy^{i,s}) \, d\Omega \\ & + 2\int_\Omega (v^{i,s} - u^{i,s-1})(u - u^{i,s}) \, d\Omega. \end{aligned} \tag{44}$$

Taking $y = y^*$, $u = u^*$ in (41)-(44) and observing that $Ay^* - f - u^* = 0$, we find

$$K_{i,s}(z^*) \geq \frac{2}{r_i} \int_\Omega \left(A\bar{y}^{i,s} - f - \bar{u}^{i,s}\right)^2 d\Omega. \tag{45}$$

Thus,

$$\left(\frac{1}{2} K_{i,s}(z^*)\right)^{\frac{1}{2}} \geq \left(\frac{r_i}{2} K_{i,s}(z^*)\right)^{\frac{1}{2}} \geq \left\|A\bar{y}^{i,s} - f\right\| - \left\|\bar{u}^{i,s}\right\|. \tag{46}$$

Together with the boundedness of U_{ad}, $\sup_i r_i < 1$, $\sup_i \varepsilon_i < 1$, and (39) this shows that there exists a constant c_1 such that

$$\left\|\bar{y}^{i,s}\right\|_Y < c_1, \; \left\|y^{i,s}\right\|_Y < c_1. \tag{47}$$

Now, $y^{i,s}$ and $\bar{y}^{i,s}$ solve the boundary value problems

$$Ay = Ay^{i,s}, \; y|_\Gamma = 0 \text{ and } Ay = A\bar{y}^{i,s}, \; y|_\Gamma = 0.$$

Therefore, a standard result from the theory of elliptic operators (see, for instance, Aubin, 1972) gives the estimate $\|y^{i,s}\|_{H^2(\Omega)} \leq \text{const} \cdot \|Ay^{i,s}\|$, or, due to (47), the existence of c_2 such that

$$\left\|\bar{y}^{i,s}\right\|_{H^2(\Omega)} < c_2 \text{ and, analogously, } \left\|y^{i,s}\right\|_{H^2(\Omega)} < c_2. \tag{48}$$

Let $\hat{z}^{i,s} = (\hat{y}^{i,s}, \hat{u}^{i,s})$ be a feasible point (i.e. $\hat{u}^{i,s} \in U_{ad}$, $\hat{y}^{i,s} = T\hat{u}^{i,s}$) with minimal distance (with regard to $\|\cdot\|_X$) to $\bar{z}^{i,s}$, i.e.

$$\hat{z}^{i,s} = \operatorname{argmin}\{\|\bar{z}^{i,s} - z\|_X : z \text{ feasible}\}, \tag{49}$$

Now we estimate $\|\hat{z}^{i,s} - \bar{z}^{i,s}\|_X$. Inequality (45) shows that, due to the boundedness of U_{ad}, there exists a constant c_3 such that

$$\|r_i \bar{q}^{i,s}\| = \|A\bar{y}^{i,s} - f - \bar{u}^{i,s}\| < c_3 \sqrt{r_i}. \tag{50}$$

Let $\tilde{y}^{i,s}$ be the solution of $Ay = f + \bar{u}^{i,s}$, $y|_\Gamma = 0$. Then

$$\|A\bar{y}^{i,s} - A\tilde{y}^{i,s}\| = \|A\bar{y}^{i,s} - f - \bar{u}^{i,s}\| < c_3\sqrt{r_i},$$

and hence,

$$\|(\bar{y}^{i,s}, \bar{u}^{i,s}) - (\tilde{y}^{i,s}, \bar{u}^{i,s})\|_X < c_3\sqrt{r_i}.$$

By the definition (49) of $\bar{\bar{z}}^{i,s}$ this yields

$$\left\|\bar{z}^{i,s} - \bar{\bar{z}}^{i,s}\right\|_X < c_3\sqrt{r_i}. \tag{51}$$

Next, with C^* the adjoint operator to C and A^* the formally adjoint to A (see Section 6.2.7 in Aubin, 1972), let $\bar{p}^{i,s}$ be a solution of the problem

$$A^*\bar{p}^{i,s} = C^*(C\bar{y}^{i,s} - \kappa_d),\ \bar{p}^{i,s}|_\Gamma = 0. \tag{52}$$

Because A^* is again an elliptic operator of second order with coefficients in $C^2(\bar{\Omega})$, we have $\bar{p}^{i,s} \in H^2(\Omega) \cap H_0^1(\Omega)$ and, due to (48),

$$\|\bar{p}^{i,s}\|_{H^2(\Omega)} < c_4. \tag{53}$$

From

$$\begin{aligned}\int_\Omega \bar{p}^{i,s} A(y - \bar{y}^{i,s})d\Omega &= \int_\Omega (y - \bar{y}^{i,s})A^*\bar{p}^{i,s}d\Omega \\ &= \int_\Omega (y - \bar{y}^{i,s})C^*(C\bar{y}^{i,s} - \kappa_d)d\Omega \\ &= \int_\Omega (C\bar{y}^{i,s} - \kappa_d)(Cy - C\bar{y}^{i,s})d\Omega\end{aligned} \tag{54}$$

and (41), (42) we get for $(y, u) \in Y \times U_{ad}$

$$\int_\Omega (\bar{p}^{i,s} + \bar\lambda^{i,s})(Ay - A\bar{y}^{i,s})d\Omega + \int_\Omega (u - \bar{u}^{i,s})(\bar{u}^{i,s} - u^{i,s-1} - \bar\lambda^{i,s})d\Omega \geq 0.$$

Choosing $u = \bar{u}^{i,s}$, $y = T\left(\bar{u}^{i,s} - \frac{\bar\lambda^{i,s}}{\|\bar\lambda^{i,s}\|}\right)$, T being the operator defined by (13) (thus $Ay = \bar{u}^{i,s} - \frac{\bar\lambda^{i,s}}{\|\bar\lambda^{i,s}\|} + f$), this gives

$$-\int_\Omega (\bar{p}^{i,s} + \bar\lambda^{i,s})\left(-\frac{\bar\lambda^{i,s}}{\|\bar\lambda^{i,s}\|} + r_i\bar\lambda^{i,s}\right)d\Omega \geq 0. \tag{55}$$

Together with (50), (53), and $r_i < 1$, we obtain with the aid of Cauchy-Schwarz-Inequality

$$\begin{aligned}\|\bar\lambda^{i,s}\| &\leq -\int_\Omega \bar{p}^{i,s}\left(r_i\bar\lambda^{i,s} - \frac{\bar\lambda^{i,s}}{\|\bar\lambda^{i,s}\|}\right)d\Omega - r_i\|\bar\lambda^{i,s}\|^2 \\ &\leq \|\bar{p}^{i,s}\|\,\|\bar\lambda^{i,s}\|\left(r_i + \frac{1}{\|\bar\lambda^{i,s}\|}\right) \\ &= \|\bar{p}^{i,s}\| + \sqrt{r_i}\|\bar{p}^{i,s}\|\,\|\sqrt{r_i}\bar\lambda^{i,s}\| \\ &< c_4 + c_4c_3 \equiv c_5.\end{aligned} \tag{56}$$

Therefore, we get the improved bounds

$$\|A\bar{y}^{i,s} - f - \bar{u}^{i,s}\| < c_5 r_i, \tag{57}$$

and, from this, according to the derivation of (51),

$$\left\|\bar{z}^{i,s} - \bar{\bar{z}}^{i,s}\right\|_X < c_5 r_i. \tag{58}$$

With an argument analogous to that used to derive (48), one can find

$$\|y\| \le c_6 \|y\|_Y \text{ for all } y \in Y. \tag{59}$$

Therefore, (58) gives

$$\|\bar{y}^{i,s} - \bar{\bar{y}}^{i,s}\| < c_5 c_5 r_i; \tag{60}$$

Furthermore, using (with J_i according to (20))

$$J_i(z^*) = \int_\Omega (Cy^* - \kappa_d)^2 d\Omega \le \int_\Omega \left(C\bar{y}^{i,s} - \kappa_d\right)^2 d\Omega = J_i\left(\bar{z}^{i,s}\right)$$

we obtain with (47) and (60)

$$J_i(z^*) - J_i(\bar{\bar{z}}^{i,s}) = J_i(z^*) - J_i(\bar{z}^{i,s}) + J_i(\bar{z}^{i,s}) - J_i(\bar{\bar{z}}^{i,s})$$

$$\le \int_\Omega (C\bar{y}^{i,s} - \kappa_d)^2 d\Omega - \int_\Omega (C\bar{\bar{y}}^{i,s} - \kappa_d)^2 d\Omega$$

$$= \int_\Omega (C\bar{y}^{i,s} - C\bar{\bar{y}}^{i,s}) \left(C\bar{y}^{i,s} + C\bar{\bar{y}}^{i,s} - 2\kappa_d\right) d\Omega$$

$$\le \|C\| \left(\|C\| \|\bar{y}^{i,s} + \bar{\bar{y}}^{i,s}\| + 2\|\kappa_d\|\right) \|\bar{y}^{i,s} - \bar{\bar{y}}^{i,s}\|$$

$$< \|C\| \left(\|C\| \left(c_1 c_5 + \sup_{v \in U_{ad}} \|Tv\|\right) + 2\|\kappa_d\|\right) \cdot c_5 c_5 \cdot r_i,$$

proving (30).

The existence of a constant d_2 such that (31) holds, is an easy consequence of (37), (47) and the boundedness of U_{ad}. Thus Proposition 5 is proved. ■

To prove Theorem 1, we need one more lemma.

Let Z be a Hilbert space, Z_1 a subspace and $P : Z \to Z_1$ the orthogonal projection operator. Let $a(\cdot,\cdot)$ be a continuous, symmetric, positive semi-definite bilinear form on $Z \times Z$ and l a linear, continuous functional on Z. With $K \subset Z$ convex and closed, consider the problem

$$\text{minimize } \varphi(z) := a(z, z) - l(z) \text{ subject to } z \in K. \tag{61}$$

Let $b(\cdot,\cdot)$ be a second symmetric bilinear form on $Z \times Z$ such that

$$0 \le b(z, z) \le a(z, z) \text{ for } z \in Z \tag{62}$$

and

$$b(z,z) + \|Pz\|^2 \geq \beta\|z\|^2 \text{ for all } z \in Z \tag{63}$$

with some $\beta > 0$. By

$$|z|^2 = b(z,z) + \|Pz\|^2 \tag{64}$$

another norm is defined on Z equivalent to $\|\cdot\|_Z$ according to the obvious relation

$$(M+1)\|z\|^2 \geq |z|^2 \geq \beta\|z\|_Z^2 \tag{65}$$

with $M \geq \sup_{z \neq 0} \frac{b(z,z)}{\|z\|^2}$.

Lemma 2 *For each $a^0 \in Z$ and*

$$a^1 = \arg\min\{\varphi(z) + \|Pz - Pa^0\|^2 : z \in K\}$$

the inequalites

$$|a^1 - z|^2 - |a^0 - z|^2 \leq -\|Pa^1 - Pa^0\|^2 + \varphi(z) - \varphi(a^1) \tag{66}$$

and

$$|a^1 - z| \leq |a^0 - z| + \eta(z) \tag{67}$$

hold for each $z \in K$, where

$$\eta(z) = \begin{cases} 0 & \text{if } \varphi(z) \leq \varphi(a^1) \\ (\varphi(z) - \varphi(a^1))^{1/2} & \text{otherwise} \end{cases}.$$

If, moreover, $\|Pa^1 - Pa^0\|_Z \geq \delta > \eta(z)$*, then*

$$|a^1 - z| \leq |a^0 - z| + \frac{\eta^2(z) - \delta^2}{2|a^0 - z|}. \tag{68}$$

Proof: $\|Pz\|_Z \leq \|z\|_Z$ shows the boundedness of the bilinear form $((Pz, Pw))_Z$ on the space $Z \times Z$. Due to the optimality of a^1 we have for all $z \in K$

$$2a(a^1, z - a^1) - l(z - a^1) + 2((Pa^1 - Pa^0, Pz - Pa^1))_Z \geq 0. \tag{69}$$

Taking account of (64) a simple calculation shows

$$\begin{aligned} |a^1 - z|^2 - |a^0 - z|^2 &= b(a^1, a^1) - 2b(a^1, z) + 2b(a^0, z) - b(a^0, a^0) \\ &\quad - \|Pa^1 - Pa^0\|^2 - 2((Pa^1 - Pa^0, Pa^1 - Pz))_Z. \end{aligned}$$

Utilizing (69), (62) and the simple inequality

$$2a(a^1, z - a^1) \leq a(z, z) - a(a^1, a^1)$$

a straightforward calculation leads to (66), and (67) follows immediately.

In case $\|Pa^1 - Pa^0\|_Z = \delta - \eta(z)$, (66) gives

$$|a^1 - z|^2 - |a^0 - z|^2 \leq -\delta^2 + \eta^2(z) \leq 0,$$

hence,

$$|a^1 - z| \leq |a^0 - z|$$

and

$$\begin{aligned} 0 &\geq -\delta^2 + \eta^2(z) = (|a^1 - z| + |a^0 - z|)(|a^1 - z| - |a^0 - z|) \\ &\geq 2|a^0 - z|(|a^1 - z| - |a^0 - z|) \end{aligned}$$

proving (68). II

We note that in Lemma 2 closedness of K is only required to ensure the existence of a^1 and could be replaced by the requirement that a^1 exists.

Proof of Theorem 1: From (34), the definition of $s(i)$ in substep (c) of Method 2, and Lemma 1 we conclude

$$\|v^{i,s} - v^{i,s-1}\| = \|u^{i,s} - v^{i,s-1}\| - \|u^{i,s} - \bar{u}^{i,s}\| > \delta_i - \%\varepsilon_i > 0$$

for all $1 \leq s < s(i)$.

Together with inequality (30) in Proposition 5 and $d_1 r_i < (\delta_i - ½\varepsilon_i)^2$ due to the first inequality (34), this implies

$$\|u^{i,s} - u^{i,s-1}\|^2 > J_i(z^*) - J_i(i,s).$$

Let $z^{1,0} = (Tu^{1,0}, v^{1,0})$. Application of Lemma 2 (inequality (68)) with

$$Z = X, \quad Z_1 = \{z = (y, v) \in X : y = 0\},$$

$$\varphi = J_i, \text{ given by (20)},$$

$$a(z, z) = \int_\Omega CyCy\,d\Omega + \frac{2}{r_i} b(z, z) \text{ for } z, z \in X,$$

$$b(z, z) = \int (Ay - v)(Ay - u)\,d\Omega,$$

$$l(z) = -J_i(z) + a(z, z),$$

$$K = Y \times U_{ad}, \quad u = u_i - \frac{3}{2}\varepsilon_i, \text{ and } a^0 = i, s-1$$

together with Proposition 5 gives for $1 \leq s \leq s(i)$,

$$|z^{i,s} - z^*| < |z^{i,s-1} - z^*| + \frac{1}{2d_2}\left(d_1 r_i - (\delta_i - \frac{3}{2}\varepsilon_i)^2\right)$$

and (see (67))

$$|\bar{z}^{i,s(i)} - z^*| < |z^{i,s(i)-1} - z^*| + -$$

Utilizing (37), (39), and (34) we get for $1 \leq s < s(i)$

$$|\bar{z}^{i,s} - z^*| - |z^{i,s-1} - z^*| < \frac{1}{2d_2}\left(d_1 r_i - \left(\delta_i - \frac{3}{2}\epsilon_i\right)\right)^2 + \frac{3}{2\sqrt{3}}\epsilon_i < 0 \quad (70)$$

and

$$|\bar{z}^{i,s(i)} - z^*| - |z^{i,s(i)-1} - z^*| < \qquad + \frac{\sqrt{3}}{2}\epsilon_i, \quad (71)$$

Inequality (70) proves that $s(i) < \infty$, because, as long as $\|v^{i,s} - v^{i,s-1}\| > \delta_i$, the reduction in $|\bar{z}^{i,s} - z^*|$ is better than a nonzero amount independent of s.

Lemma 2.2.2 in Polyak (1987) enables to state that the sequence $\{|z^{i,s} - z^*|\}$ converges if

$$\sum_{i=1}^{\infty} \epsilon_i < \infty \text{ and } \sum_{i=1}^{\infty} \gamma_i < \infty.$$

Due to (37) and (39), $\{|\bar{z}^{i,s} - z^*|\}$ converges to the same limit.

Let $\{z^{i_k,s_k}\}$ with $s_k > 0$ for each k, be a weakly convergent subsequence in X, $\bar{z} = (\bar{y},\bar{u})$ its weak limit. Observing (49), (51) and the convexity and closedness of $\{(y,u) | u \in U_{ad}, y \in Tu\}$ (T given by (13)) in X, we conclude that $\bar{z}$ is feasible.

By the definition of J_i and $J_i(z^*) = J_i(u^*)$, the estimate

$$|\bar{z}^{i,s} - z^*|^2 - |z^{i,s-1} - z^*|^2 \leq J_i(z^*) - J_i(\bar{z}^{i,s})$$

yields

$$|\bar{z}^{i,s} - z^*|^2 - |z^{i,s-1} - z^*|^2 \leq J(u^*) - \int_\Omega (C\bar{y}^{i,s} - \kappa_d)^2 d\Omega. \quad (72)$$

Taking the limit in (72) for the subsequence $\{z^{i_k,s_k}\}$ the weak lower semicontinuity of the functional $\int_\Omega (Cy - \kappa_d)^2 d\Omega$ in Y leads to

$$J(u^*) \geq \int_\Omega (C\bar{y} - \kappa_d)^2 d\Omega.$$

Since $\bar{z}$ is feasible, this proves that $\bar{u}$ is optimal for Problem 2. Finally, Lemma 1 in Opial (1967) proves the weak convergence of both $\{z^{i,s}\}$ and $\{\bar{z}^{i,s}\}$ to $\bar{z}$ in X. ∎

According to Remark 1, Theorem 1 also proves the convergence of Method 1 even without condition (34), which is fulfilled automatically for sufficiently large D_i.

4.2. The case of state constraints

Now let a state constraint

y(v,) E G c Y

be given, with G convex, closed and such that for some ii, E Uad

y(u) E int G, (73)

as assumed in Problem 2.

The only relevant modification is in the proof of Proposition 5. We show that inequalities (51), (57), and (58) remain true with modified constants ci. Then the other parts of the proofs of Proposition 5 and Theorem 1 remain unchanged.

We give the changed part of the proof of Proposition 5 in between relations (50) and (58).

Even though f)>s may not satisfy the state constraints we still have, with (i,s = (fi• S,]i,s)' that

llzi,s - (i,s llx < Cs\ffi- (74)

As zi,s E G x Uad, this shows that

$$\min_{v\in G\times U_{ad}} \|\zeta^{i,s} - v\|_X < c_3\sqrt{r_i}. \tag{75}$$

With the abbreviation z(11) := (y(11), n), let, with 11, according to (73),

Tmin := min wE&G IITii, -w11Y, Tmax := max uEUad llz(ii,) - z(v,)llx,

wi,s := arg min vEGxUad ll(i,s -v11x-

In case of f/-8 rf_G, wi,s rf_{z(ii,) + A((i,s - z(ii,)): ,\;::: O}, let

hi,s E {z(ii,) + >-((i,s - z(ii,)) : ,\;::: O} n (8G x Uad),
bi,s E {z(ii,) + >.((i,s - wi,s) : ,\;::: O} n {hi,s + µ(hi,s _ wi,s) : µ;::: O}

(hi,s,]i,s arc uniquely defined by these relations).

Obviously,

$$\frac{\text{ll}(^{i,s} - \text{w}^{i,s}\text{llx}}{\text{lb}^{i,s} - \text{z(ii,)llx}} = \frac{\text{ll}(^{i,s} - \text{h}^{i,s}\text{llx}}{\text{llh}^{i,s} - z(\text{ii,})\text{llx}}.$$

Due to the trivial implication

$$-1-/: \frac{a}{\beta} = \frac{\gamma}{6} = \} \frac{a}{a+\beta} = \frac{\gamma}{,y+o'}$$

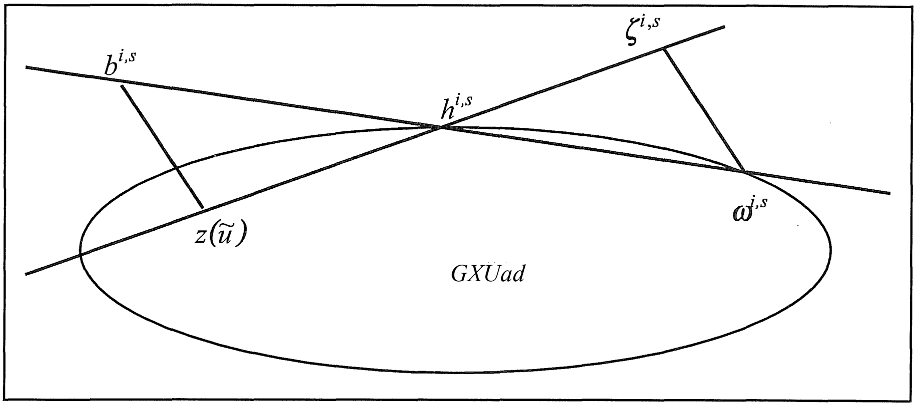

Figure 1.

we obtain together with (74)

$$\|\zeta^{i,s} - h^{i,s}\|_X = \frac{\|z(\tilde u) - \zeta^{i,s}\|_X}{\|b^{i,s} - z(\tilde u)\|_X + \|\zeta^{i,s} - \omega^{i,s}\|_X} \cdot \|\zeta^{i,s} - \omega^{i,s}\|_X \le \frac{r_{max}}{r_{min}} \|\zeta^{i,s} - \omega^{i,s}\|_X \le \frac{r_{max}}{r_{min}} c_3 \gamma r_i, \qquad (76)$$

If $\omega^{i,s} \in G$, $w^{i,s} \in \{z(\tilde u) + \lambda(\zeta^{i,s} - z(\tilde u)) : \lambda \ge 0\}$, this estimate is obvious.

Let $z^{i,s}$ be again the feasible point closest to $\zeta^{i,s}$. Then because $h^{i,s}$ is feasible, (74) and (76) yield

$$\|z^{i,s} - \zeta^{i,s}\|_X \le \|z^{i,s} - h^{i,s}\|_X \le \|z^{i,s} - \zeta^{i,s}\|_X + \|\zeta^{i,s} - h^{i,s}\|_X \le \left(\frac{r_{max}}{r_{min}} + 1\right) c_3 \gamma r_i \qquad (77)$$

corresponding to (51).

With $\zeta^{i,s}$ and $\psi^{i,s}$ given by (52), (40) we again have for all $(y, v) \in G \times U_{ad}$ the inequality

$$\int_\Omega (\zeta^{i,s} + \psi^{i,s})(A y - A\zeta^{i,s})\,d\Omega + \int_\Omega (v - u^{i,s})(u^{i,s} - u^{i,s-1} - \chi\psi^{i,s})\,d\Omega \ge 0. \qquad (78)$$

Let, with $\tilde u$ according to (73), $w^{i,s}$ be given by

$$w^{i,s} = T\left(\tilde u - \gamma_0 \frac{\psi^{i,s}}{\|\psi^{i,s}\|}\right), \qquad (79)$$

where $\gamma_0 > 0$ is chosen to be small number such that $w^{i,s} \in G$ for all (i, s). Such a γ_0 exists because the solution of $S1(v)$ (see (11)) depends continuously on $v \in L_2(\Omega)$ with regard to $\|\cdot\|_Y$.

Taking $v_i = u$ and $y = w^{i,s}$, inequality (78) leads to

$$\begin{aligned} \|q^{i,s}\| \leq & \frac{1}{\chi_0}\left[\|\bar{z}^{i,s}\| \left(\|\tilde{u}^{i,s} - \bar{v}^{i,s}\| + c_0\right) + \|\bar{v}^{i,s}\|\|\tilde{u}^{i,s}\| \right. \\ & \left. + \|u - \bar{v}^{i,s}\|\|v^{i,s} - u^{i,s-1}\|\right], \end{aligned} \tag{80}$$

and, using (50), which is true in this case too, we obtain

$$\|A\bar{z}^{i,s} - f - v^{i,s}\| < c_5 r_i$$

$$\left\|\bar{z}^{i,s} - \bar{\bar{z}}^{i,s}\right\|_X < \left(\frac{\tau_{\max}}{\tau_{\min}} + 1\right) \bar{c}_5 r_i$$

analogously to (57), (58).

5. A simple example

To demonstrate the effect of regularization, let us consider an example:

$$\textit{Minimize} \int_{-1}^{1} (y(0) - 1)^2 dx$$

subject to

$$-y'' = v, \quad y(-1) = y(1) = 0,$$

with

$$\begin{aligned} u \in U_{ad} := & \{v \in L_2(-1,1) : v(x) \leq 0 \ a.e. \text{ on } (0,1), \\ & \int_{-1}^{0} \int_{-1}^{x} v(t)dtdx = 0, \int_{-1}^{0} v(t)dt = 0\}. \end{aligned}$$

The set U_{ad} is a convex and closed subset of $L_2(-1, 1)$, and the above objective corresponds to the choice $Cy = y(0)$, $z_d = 1$. According to (18) we consider the operator C as a mapping from $H_0^1(-1, 1)$ on $L_2(-1, 1)$ and its boundedness is a consequence of the continuous embedding of $H_0^1(-1, 1)$ into $C[-1, 1]$.

For a given $v \in U_{ad}$, the function

$$y'(x) = y'(-1) - \int_{-1}^{x} u(t)dt$$

is absolutely continuous, and

$$y(x) = y'(-1)(x + 1) - \int_{-1}^{x}\int_{-1}^{\xi} u(t)dtd\xi$$

solves the boundary value problem. Observing the latter formula, due to the boundary conditions, we conclude for $x = 1$ that

$$y'(-1) = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{\xi} v(t)dtd\xi$$

and for $x = 0$

$$y(0) = y'(-1) - \int_{-1}^{0}\int_{-1}^{t} u(\tau)\,d\tau\,dt$$

Taking into account that $u \in U_{ad}$,

$$\begin{aligned} y'(-1) &= \frac{1}{2}\int_{-1}^{0}\int_{-1}^{t} u(\tau)\,d\tau\,dt + \frac{1}{2}\int_{0}^{1}\int_{-1}^{t} u(\tau)\,d\tau\,dt \\ &= \frac{1}{2}\int_{-1}^{0}\int_{-1}^{t} u(\tau)\,d\tau\,dt + \frac{1}{2}\int_{0}^{1}\int_{0}^{t} u(\tau)\,d\tau\,dt \\ &\le \frac{1}{2}\int_{-1}^{0}\int_{-1}^{t} u(\tau)\,d\tau\,dt \end{aligned}$$

Hence, for all $u \in U_{ad}$, we infer

$$y(0) = y'(-1) - \int_{-1}^{0}\int_{-1}^{t} u(\tau)\,d\tau\,dt \le -\frac{1}{2}\int_{-1}^{0}\int_{-1}^{t} u(\tau)\,d\tau\,dt \le 0,$$

and

$$\int_{-1}^{1}(y(0) - 1)^2\,dx \ge 2.$$

However, it is obvious that the process $(y^{(1)}, u^{(1)}) = (0, 0)$ is feasible for the control problem and gives objective value $\int_{-1}^{1}(y(0) - 1)^2\,dx = 2$, i.e., $(y^{(1)}, u^{(1)})$ is optimal.

It is easily seen that there are other solutions, in particular,

$$y^{(2)}(x) = \begin{cases} -\frac{1}{4\pi^2}(\cos 2\pi x - 1) & \text{for } x \in [-1, 0] \\ 0 & \text{for } x \in (0, 1] \end{cases},$$

$$u^{(2)}(x) = \begin{cases} \cos 2\pi x & \text{for } x \in [-1, 0] \\ 0 & \text{for } x \in (0, 1] \end{cases},$$

and

$$y^{(3)}(x) = \begin{cases} -\frac{x^2}{2} - x - \frac{1}{2} & \text{for } x \in [-1, -\frac{3}{4}] \\ \frac{x^2}{2} + \frac{x}{2} + \frac{1}{16} & \text{for } x \in (-\frac{3}{4}, -\frac{1}{4}] \\ -\frac{x^2}{2} & \text{for } x \in (-\frac{1}{4}, 0] \\ & \text{for } x \in (0, 1] \end{cases},$$

$$u^{(3)}(x) = \begin{cases} +1 & \text{for } x \in [-1, -\frac{3}{4}] \\ -1 & \text{for } x \in (-\frac{3}{4}, -\frac{1}{4}] \\ +1 & \text{for } x \in (-\frac{1}{4}, 0] \\ 0 & \text{for } x \in (0, 1] \end{cases}$$

Moreover, it is not difficult to verify that for arbitrary $a \in R^1$

$$(ay^{(2)}, au^{(2)}) \text{ and } (ay^{(3)}, au^{(3)}),$$

are again solutions, i.e. the optimal set U^* is unbounded.

Now, let us consider the penalized problem: Minimize, with fixed $r_i > 0$,

$$J_i(y,u) = \int_{-1}^{1} (y(0)-1)^2 dx + \frac{1}{r_i}\int_{-1}^{1} (y''+u)^2 dx \quad \text{s.t. } y \in Y,\ u \in U_{ad},$$

where, according to (14), $Y = H_0^1(-1,1) \cap H^2(-1,1)$.

No solution of this problem (if it is solvable) can satisfy the differential equation $-y'' = u$.

Indeed, the first variation of the functional J_i in $(y,u) \in Y \times U_{ad}$ gives for $\eta \in Y$, $v \in U_{ad}$

$$\frac{d}{d\alpha}\Big[\int_{-1}^{1} \{(y(0)+\alpha\eta(0)-1)^2 - (y(0)-1)^2\} dx +$$

$$\frac{1}{r_i}\int_{-1}^{1} \{(y''+\alpha\eta''+u+\alpha v)^2 - (y''+u)^2\} dx\Big]_{\alpha=0}$$

$$= 2\Big[\int_{-1}^{1} (y(0)-1)\eta(0)dx + \frac{1}{r_i}\int_{-1}^{1} (y''+u)(\eta''+v)\Big] dx.$$

If $y''+u = 0$, then (y,u) is a feasible process, hence $y(0)-1 = -1$, and, choosing for instance $\eta(x) = x^2-1$ and $v(x) = 0$, we obtain a non-zero value of the first variation.

However, if $(\hat{y},\hat{u})$ is a solution of the penalized problem, then it is easily seen that for arbitrary a the pairs

$$(\hat{y} + a y^{(2)}, \hat{u} + a u^{(2)}) \quad \text{or} \quad (\hat{y} + a y^{(3)}, \hat{u} + a u^{(3)})$$

are also solutions.

Moreover, approximating u by piecewise constant functions and y by Hermitian cubic splines, one can verify that the approximate penalized problem is in general not uniquely solvable. For instance, if on the interval $[-1,1]$ a grid with step-size $h = 1/4$ is chosen, then, with (y_h, u_h) a solution of the approximate problem, the pair $(\hat{y} + a y^{(3)}, \hat{u} + a u^{(3)})$ is also a solution for arbitrary a (this is true because the Hermite approximation of the function $y^{(3)}$ coincides with $y^{(3)}$).

Hence, the Hessian of the approximate penalty function is not regular.

In Method 2 we deal with regularized penalty problems:

Minimize

$$\Psi_{i,s}(y,u) = \int_{-1}^{1} (y(0)-1)^2 dx + \frac{1}{r_i}\int_{-1}^{1} (y''+u)^2 dx + \|u-u^{i,s-1}\|^2_{L_2(-1,1)}$$

subject to $y \in Y$, $u \in U_{ad}$,

where W_is are strongly convex functions in $X = Y \times \mathcal{L}_2(-1, 1)$ (see (16), (17)). Approximating these problems as mentioned above, we obtain quadratic programming problems with strongly convex objective functions in corresponding finite-dimensional spaces. Hence, fast converging methods can be applied for solving these finite-dimensional problems.

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