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Hausdorff convergence of domains and their boundaries for shape optimal design

by

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Abstract: A question, which arises frequently in shape optimal design, is the convergence of domains. If the objective function is defined by using the solution of a PDE with boundary conditions, then also the convergence of the boundary is of importance. In this paper a criterion for a set of domains is defined, such that from $\Omega_n \longrightarrow \Omega$ follows $\Gamma_n \longrightarrow \Gamma$ if one is restricting to this set of domains. Moreover it is proved that this criterion is sharp, meaning that if $\Omega_n \longrightarrow \Omega \Longrightarrow \Gamma_n \longrightarrow \Gamma$ holds for any sequence of this set, then this criterion has to be fulfilled. A similar criterion for the convergence of the Lebesgue measure of the boundaries $\mu(\Gamma_n) \longrightarrow \mu(\Gamma)$ is given.

Keywords: shape optimization, Hausdorff metric, Hausdorff convergence, hyperspace.

1. Introduction

In shape optimal design one usually has a set $\mathcal{O}$ of admissible domains, where one wants to find a domain $\Omega^* \in \mathcal{O}$ which for a cost functional J satisfies

$$J(\Omega^*) \leq J(\Omega) \text{ for all } \Omega \in \mathcal{O}.$$

If the cost functional J is defined by using the solution of some partial differential equation on Ω with boundary conditions in $\Gamma := \partial\Omega$, discontinuities in the cost functional may occur if $\Omega_n \longrightarrow \Omega$ but $\Gamma_n \not\longrightarrow \Gamma$.

Denote for the rest of this article $\Gamma := \partial\Omega$ the boundary of the domain Ω and $\Gamma_n := \partial\Omega_n$ the boundaries of the domains Ω_n, respectively, where a domain means here a nonempty compact subset of $\mathbb{R}^N$. For reasons of clearness, sometimes $\partial\Omega$ is used instead of Γ ($\partial\Omega_n$ instead of Γ_n, respectively).

Consider the example of a membrane on $\mathring{\Omega}_n$ with Dirichlet boundary conditions on Γ_n. One can see that the limit of the solutions $\Omega_n := \left[-1, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, 1\right]$

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behaves still like there were a boundary in $\{0\}$, while the real solution of $\mathring{\Omega}$ with $\Omega = [-1,1]$ has no boundary condition in 0, see Fig. 1.

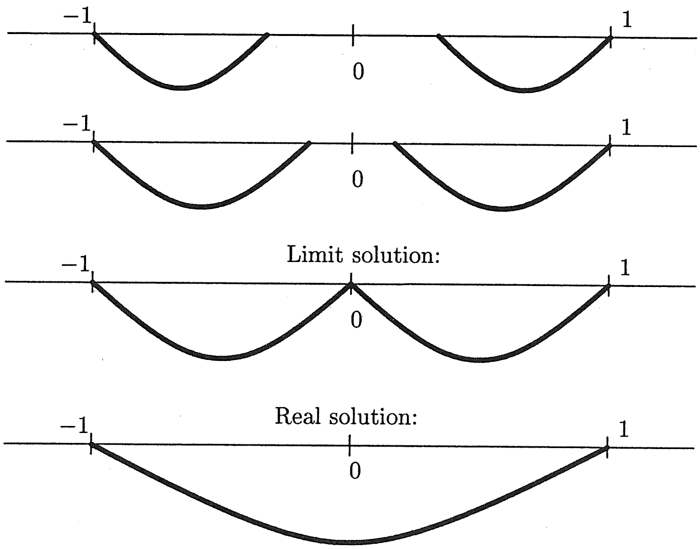

Figure 1. Convergence of the membrane problem for $\Omega_n = \left[-1,-\frac{1}{n}\right] \cup \left[\frac{1}{n},1\right]$

One frequently used convergence criterion in domain optimization is the convergence in the Hausdorff metric which has nice compactness properties for classes of closed sets. Unfortunately, the solution of a PDE may not converge to the solution for the limit domain when $\Omega_n \longrightarrow \Omega$ but $\Gamma_n \not\longrightarrow \Gamma$, as one can see in the previous example. Various restrictions on the set of domains, like, for example, a cone property or Lipschitz boundaries have been used in order to avoid difficulties of this kind, see, for example, Bucur and Zolesio (1994a), (1994b), (1994c), (1995), Pironneau (1984), Haslinger and Neittaanmäki (1988). It is well known that $\Omega_n \longrightarrow \Omega \Longrightarrow \Gamma_n \longrightarrow \Gamma$ holds in Hausdorff sense for domains with Lipschitz boundary (see Example 3.2) or cone property (see Chenais, 1975, for definition), both with given constants. The motivation is now to find more general classes of sets such that $\Omega_n \longrightarrow \Omega \Longrightarrow \Gamma_n \longrightarrow \Gamma$ holds.

Consider now the membrane example on the sets $\Omega_n := \left[-1, 1-\frac{1}{n}\right] \cup \left[1-\frac{1}{2n}, 1\right]$. Note that these sets do not have a Lipschitz boundary nor satisfy a cone property with given constants. But one can notice heuristically that the boundaries Γ_n converge to $\{-1,1\}$, which is the boundary of the limit domain $[-1,1]$, and also that the solutions of the membrane problems on Ω_n converge to the limit solution, see Fig. 2.

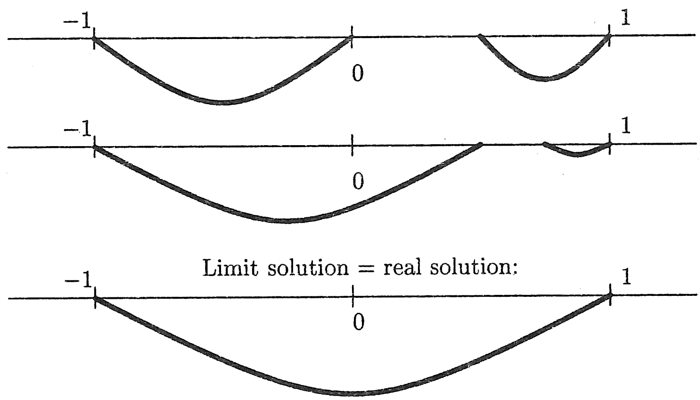

Figure 2. Convergence of the membrane problem for $\Omega_n = \left[-1, 1-\frac{1}{n}\right] \cup \left[1-\frac{1}{2n}, 1\right]$

Consider now the following variation equations, which may result from boundary value problems:

$$\text{Find } u_n \in V(\Omega_n) \text{ such that } a_n(u_n, v) = l_n(v)\ \forall v \in V(\Omega_n),$$

and analogously for the domain Ω:

$$\text{Find } u \in V(\Omega) \text{ such that } a(u, v) = l(v)\ \forall v \in V(\Omega),$$

where a_n, a are continuous, possibly symmetric V-elliptic bilinear forms on the Hilbert spaces $V(\Omega_n)$, $V(\Omega)$, respectively. Analogously, l_n and l are continuous linear functionals on $V(\Omega_n)$ and $V(\Omega_n)$, while a_n, l_n, a and l depend themselves on Ω_n, Ω.

The right hand side $l_n(v)$ of variation equations arising from boundary value problems of Neumann type usually contains a term $\int_{\Gamma_n} fv\,d\Gamma$, $\int_\Gamma fv\,d\Gamma$ for $l(v)$, respectively.

Let us assume that there is a unique solution for the variation equations above (see the appropriate literature in this point). Consider now a series of domains $\Omega_n \longrightarrow \Omega$ (in Hausdorff sense, see Section 2). If $\mu(\Gamma) > 0$ then difficulties arise in evaluation of the term $\int_\Gamma fv\,d\Gamma$. Such a case for $\Omega_n \longrightarrow \Omega$, $\Gamma_n \longrightarrow \Gamma$ and $\mu(\Gamma_n) = 0$ for all $n \in \mathbb{N}$, but $\mu(\Gamma) > 0$ is described in Section 4.

Furthermore, consider the sequence $\Omega_n := \left\{\frac{i}{2^n} \,\middle|\, i = 0, \ldots, 2^n\right\}$. Of course, $\Gamma_n = \Omega_n$ and one can see intuitively that $\Omega_n = \Gamma_n$ converges for $n \to \infty$ to $[0, 1]$ in Hausdorff sense (this example will be treated later in a more exact way). But the Lebesgue measure of Γ_n is always zero and does not converge to the Lebesgue measure of $[0, 1]$ which is 1. One may expect that the volume of Ω_n will not be preserved, but this example shows that also the sequence of boundaries Γ_n may not preserve volume 0 for its limit if $\Omega_n \longrightarrow \Omega$!

2. Preliminaries

Now the basic ideas and results of Hausdorff metric and Hausdorff convergence will be described. Most of the notations and results here are from Salinetti and Wetts (1979). Like in the reference, the results given there can be extended to more general normed linear spaces of finite dimension.

DEFINITION 2.1 *The* distance *between a point $x \in \mathbb{R}^N$ and a set $\Omega \subset \mathbb{R}^N$ is defined as:*

$$d(x,\Omega) := \begin{cases} \inf_{y\in\Omega}\{\|x-y\|\} & \text{if } \Omega \neq \emptyset \\ \infty & \text{if } \Omega = \emptyset. \end{cases}$$

The Hausdorff distance *between two sets $\Omega_1 \subset \mathbb{R}^N$ and $\Omega_2 \subset \mathbb{R}^N$ is defined as:*

$$d_h(\Omega_1,\Omega_2) := \max\{\sup_{x\in\Omega_1}\{d(x,\Omega_2)\}, \sup_{x\in\Omega_2}\{d(x,\Omega_1)\}\}.$$

Following Salinetti and Wetts (1979) denote by $B_\varepsilon(\Omega) := \{x \in \mathbb{R}^N \mid d(x,\Omega) < \varepsilon\}$ the *ball around Ω with radius ε*. Furthermore, for any set M denote by $2^M := \mathcal{P}(M) := \{N \mid N \subset M\} = \{f \mid f : M \longrightarrow \{0,1\}\}$ the *potential set* of M. If $\mathbf{X}$ is a topological space, then a topological structure on $2^{\mathbf{X}}$ or on a subset of $2^{\mathbf{X}}$ is called a *hyperspace of* $\mathbf{X}$.

The following proposition gives the basic results for the Hausdorff distance, the interested reader is referred to Salinetti and Wetts (1979) and Alt (1992) for further details:

PROPOSITION 2.1

1. *It holds that $d(x,\Omega) = d(x,\overline{\Omega})$. Furthermore, if $\Omega \neq \emptyset$ is closed, then there exists $y \in \Omega$ such that $0 \leq d(x,\Omega) = d(x,y) < \infty$.*
2. *d_h is nonnegative, symmetric and satisfies the triangle inequality on $2^{\mathbb{R}^N} \times 2^{\mathbb{R}^N}$. If Ω_1, Ω_2 is closed, then from $d_h(\Omega_1,\Omega_2) = 0$ follows $\Omega_1 = \Omega_2$. Furthermore d_h is a metric on $\{\Omega \subset \mathbb{R}^N \mid \Omega \neq \emptyset$ and Ω compact$\}$ and is called* Hausdorff metric.
3. *For $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ the Hausdorff distance can be expressed as:*
   $d_h(\Omega_1,\Omega_2) = \inf\{\varepsilon > 0 \mid \Omega_1 \subset B_\varepsilon(\Omega_2)$ *and* $\Omega_2 \subset B_\varepsilon(\Omega_1)\}$

Using the Hausdorff metric also convergence for sequences of sets can be defined, which is a well established concept:

DEFINITION 2.2 *Let (Ω_n) be a series in $\{\Omega \subset \mathbb{R}^N \mid \Omega \neq \emptyset$ and Ω* compact$\}$. Ω_n converges in Hausdorff sense *to $\Omega \in \{\Omega \subset \mathbb{R}^N \mid \Omega \neq \emptyset$ and Ω* compact$\}$ $:\Longleftrightarrow$

$$\Omega_n \xrightarrow{h} \Omega :\Longleftrightarrow \Omega = \text{h-lim}\,\Omega_n :\Longleftrightarrow d_h(\Omega,\Omega_n) \longrightarrow 0$$

The following proposition gives mainly the criteria for verifying this convergence. It summarizes results of Pironneau (1984) and Salinetti and Wetts (1979), which will be used in the following sections. More details can also be found in Hausdorff (1962) and Mosco (1969):

PROPOSITION 2.2

1. *Let* $\Omega, (\Omega_n) \in \{\Omega \subset \mathbb{R}^N \mid \Omega \text{ closed}\}$. *Then* $\Omega_n \xrightarrow{h} \Omega$ *if and only if one of the following conditions holds:*
   - $\Omega = \emptyset$ *and there exists* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$, $\Omega_n = \emptyset$.
   - *For all* $\varepsilon > 0$ *there exists a* $n_\varepsilon \in \mathbb{N}$ *such that for all* $n > n_\varepsilon$
   $$\emptyset \neq \Omega \subset B_\varepsilon(\Omega_n) \textit{ and } \emptyset \neq \Omega_n \subset B_\varepsilon(\Omega).$$
2. *If a sequence* $(\Omega_n) \subset \{\Omega \subset \mathbb{R}^N \mid \Omega \textit{ closed, bounded}\}$ *has a Hausdorff limes* Ω, *it can be written as:*
   $$\Omega = \{x \in \mathbb{R}^N \mid \exists((x_n) \subset \mathbb{R}^N \textit{ sequence}) \, (\forall (n \in \mathbb{N}) \; x_n \in \Omega_n) \textit{ and } x_n \longrightarrow x\}.$$
3. *Let* $(\Omega_n^1), \Omega^1, (\Omega_n^2), \Omega^2 \in \{\Omega \subset \mathbb{R}^N \mid \Omega \textit{ closed, bounded}\}$ *with* $\Omega_n^1 \xrightarrow{h} \Omega^1$ *and* $\Omega_n^2 \xrightarrow{h} \Omega^2$. *Then it holds that:*
   - $(\forall (n \in \mathbb{N}) \; \Omega_n^1 \subset \Omega_n^2) \Longrightarrow \Omega^1 \subset \Omega^2$
   - $\Omega_n^1 \cup \Omega_n^2 \xrightarrow{h} \Omega^1 \cup \Omega^2$.

3. Boundary convergence

DEFINITION 3.1 *Let* $\Upsilon \subset \mathbb{R}^N$.

- *Denote by* $\mathcal{H}(\mathbb{R}^N) := \{\Omega \subset \mathbb{R}^N \mid \Omega \neq \emptyset, \textit{ closed, bounded}\}$, *together with the Hausdorff metric* $d_h(.,.)$, *the* Hausdorff space on $\mathbb{R}^N$.
- *Denote by* $\mathcal{H}(\Upsilon) := \{\Omega \subset \Upsilon \mid \Omega \neq \emptyset, \textit{ closed, bounded}\}$, *together with the Hausdorff metric* $d_h(.,.)$, *the* Hausdorff space on Υ.

From Beer (1993), Theorem 3.2.4, one has immediately:

COROLLARY 3.1 *Let* $\Upsilon \subset \mathbb{R}^N$. *It holds that:*

- $\mathcal{H}(\mathbb{R}^N)$ *is complete.*
- $\mathcal{H}(\Upsilon)$ *is compact in* $\mathcal{H}(\mathbb{R}^N) \Longleftrightarrow \Upsilon$ *is compact in* $\mathbb{R}^N$.

If $\Omega_n \xrightarrow{h} \Omega$ in $\mathcal{H}(\mathbb{R}^N)$, then (Γ_n) can be divergent as the following example shows:

$$\Omega_n := \begin{cases} [-1,1] & \text{if } n \text{ even} \\ [-1,-\frac{1}{n}] \cup [\frac{1}{n},1] & \text{if } n \text{ odd.} \end{cases}$$

One can show that $\Omega_n \xrightarrow{h} [-1,1]$, but the series of boundaries (Γ_n) is divergent and has two cluster points $\{-1,1\}$ and $\{-1,0,1\}$.

The main technical difficulty of this section is contained in the following lemma:

LEMMA 3.1 *Let* $(\Omega_n) \subset \mathcal{H}(\mathbb{R}^N)$, $\Omega \in \mathcal{H}(\mathbb{R}^N)$ *such that* $\Omega_n \xrightarrow{h} \Omega$. *Then* $(\partial\Omega_n)$ *has at least one cluster point* Θ. *Moreover, for each such cluster point* Θ *it holds that:*

$$\partial\Omega \subset \Theta \subset \Omega$$

Proof. For $\varepsilon > 0$ arbitrary, but fixed, there exists, because of $\Omega_n \xrightarrow{h} \Omega$ and proposition 2.2 (1), an n_0 such that for all $n \geq n_0$, $\Omega_n \subset B_\varepsilon(\Omega)$ where $B_\varepsilon(\Omega)$ is bounded. Of course, also $\bigcup_{i=1}^{n_0} \Omega_i$ is closed and bounded so that $\Upsilon := \bigcup_{i=1}^{n_0} \Omega_i \cup \overline{B_\varepsilon(\Omega)}$ is a compact set. By Corollary 3.1 we have now that $\mathcal{H}(\Upsilon)$ is compact and therefore there exists a cluster point Θ of the sequence $(\Gamma_n) \subset \mathcal{H}(\Upsilon) \subset \mathcal{H}(\mathbb{R}^N)$. Then, there exists a subsequence (from now on again denoted by (Ω_n)) such that $\Omega_n \xrightarrow{h} \Omega$ and $\Gamma_n \xrightarrow{h} \Theta$.

Now assume in a second step that there exists an $x \in \partial\Omega$ such that $x \notin \Theta$. Because Θ is closed there has to exist an $\varepsilon > 0$ such that

$$B_\varepsilon(x) \cap \Theta \neq \emptyset. \tag{1}$$

From $\Gamma_n \xrightarrow{h} \Theta$ one has by Proposition 2.2 (1) that there exists an n_0 such that for all $n \geq n_0$

$$\Gamma_n \subset B_{\frac{\varepsilon}{2}}(\Theta). \tag{2}$$

Therefore, one gets with (1) that $B_{\frac{\varepsilon}{2}}(x) \cap \Gamma_n = \emptyset$ for all $n \geq n_0$. Hence, either $B_{\frac{\varepsilon}{2}}(x) \subset \mathbb{R}^N \setminus \Omega_n$ or $B_{\frac{\varepsilon}{2}}(x) \subset \mathring{\Omega}_n$ has to hold. So, one can divide the sequence Ω_n into two subsequences, both converging to Ω and their boundaries to Θ in Hausdorff sense, where at least one is infinite so that one has at least one of the two cases:

1. (Ω_j) such that $B_{\frac{\varepsilon}{2}}(x) \subset \mathbb{R}^N \setminus \Omega_j$:
   Because $\Omega_n \xrightarrow{h} \Omega$ one has by Proposition 2.2 (1) that for some index j_0, $\Omega \subset B_{\frac{\varepsilon}{2}}(\Omega_j)$ has to hold. So, this is a contradiction to $x \in \partial\Omega \subset \Omega$.
2. (Ω_k) such that $B_{\frac{\varepsilon}{2}}(x) \subset \mathring{\Omega}_k$:
   Because $x \in \partial\Omega$ and Ω is closed there exists a $y \in \mathbb{R}^N \setminus \Omega$ such that $d(x, y) \leq \frac{\varepsilon}{2}$, $0 < d(y, \Omega) =: \delta \leq \frac{\varepsilon}{2}$ and one has from before that $y \in \mathring{\Omega}_k$ for all k. But this is a contradiction to $\Omega_k \xrightarrow{h} \Omega$ because, by Proposition 2.2 (1), a k_0 has to exist such that $\Omega_k \subset B_{\frac{\delta}{2}}(\Omega)$ for $k > k_0$.

Hence, x must be also in Θ.

Now in a third step $\Theta \subset \Omega$ follows directly from $\Omega_n \xrightarrow{h} \Omega$, $\Gamma_n \xrightarrow{h} \Theta$ and $\Gamma_n \subset \Omega_n$ for all n by Proposition 2.2 (3). ∎

DEFINITION 3.2 *Let $\Omega \in \mathcal{H}(\mathbb{R}^N)$ and $\delta \geq 0$. Then we define:*

- $g_\Omega(x,\delta) := d(x, \mathbb{R}^N \setminus B_\delta(\Omega))$.
- $g_\Omega(\delta) := \sup_{x\in\Gamma} g_\Omega(x,\delta)$ *is called the boundary complementary capacity of* Ω.

REMARK 3.1 *It is convenient to collect the following proper theses of $g_\Omega(\delta)$ which are readily observed:*

- *Because Ω is bounded $g_\Omega(\delta) : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is a well defined function for each $\Omega \in \mathcal{H}(\mathbb{R}^N)$.*
- *$g_\Omega(0) = 0$ because Ω is closed.*
- *g_Ω is strictly increasing, because for $\delta_1 < \delta_2$ it is clear that $B_{\delta_1}(\Omega) \subset B_{\delta_2}(\Omega)$ with $B_{\delta_1}(\Omega) \neq B_{\delta_2}(\Omega)$. Therefore we have that for each $x \in \Omega$ $d(x, \mathbb{R}^N \setminus B_{\delta_1}(\Omega)) < d(x, \mathbb{R}^N \setminus B_{\delta_2}(\Omega))$ from which the assumption follows.*
- *$g_\Omega(\delta) \geq \delta$ because for all $x \in \Omega$ it holds by definition that $d(x, \mathbb{R}^N \setminus B_\delta(\Omega)) \geq \delta$ and because $\Gamma \subset \Omega$ this holds also for $x \in \Gamma$.*
- *$g_\Omega(\delta) = d_h(\Gamma, \partial B_\delta(\Omega))$ because $d(y,\Gamma) = \delta$ for all $y \in \partial B_\delta(\Omega))$. From the previous point we know that $d(x, \mathbb{R}^N \setminus B_\delta(\Omega)) \geq \delta$ and by Proposition 2.1 (1) that there exists a $y \in \mathbb{R}^N \setminus B_\delta(\Omega)$ for which $d(x,y) = d(x, \mathbb{R}^N \setminus B_\delta(\Omega))$, because $\mathbb{R}^N \setminus B_\delta(\Omega)$ is closed. That $y \in \mathbb{R}^N \setminus B_\delta(\Omega)$ has to be in $\partial(\mathbb{R}^N \setminus B_\delta(\Omega))$, otherwise one could easily find one nearer to x. Now the assumption follows directly from the definition of d_h.*

DEFINITION 3.3 *Let $\mathcal{O} \subset \mathcal{H}(\mathbb{R}^N)$. Then we define:*
$g_\mathcal{O}(\delta) := \sup_{\Omega\in\mathcal{O}} g_\Omega(\delta)$ is called the boundary complementary capacity of $\mathcal{O}$.

REMARK 3.2 *Of course also $g_\mathcal{O}(0) = 0$, $g_\mathcal{O}(\delta) \geq \delta$ holds and $g_\mathcal{O}$ is strictly increasing.*

Let now $\mathcal{O} \subset \mathcal{H}(\mathbb{R}^N)$ such that $g_\mathcal{O}$ is continuous from the right in 0.

Now choose a sequence $(\Omega_n) \in \mathcal{O}$ such that $\Omega_n \xrightarrow{h} \Omega$. Because $\mathcal{H}(\mathbb{R}^N)$ is complete it holds that $\Omega \in \mathcal{H}(\mathbb{R}^N)$.

From Lemma 3.1 we have now that there exists a subsequence of (Ω_n) (which again will be denoted by (Ω_n)) such that $\Omega_n \xrightarrow{h} \Omega$ and $\Gamma_n \xrightarrow{h} \Theta$, where $\Gamma \subset \Theta \subset \Omega$. So, select now an $x \in \Theta$. From Proposition 2.2 (2) one knows that there exists a sequence $x_n \longrightarrow x$ with $x_n \in \Gamma_n$ for all $n \in \mathbb{N}$.

Because $g_\mathcal{O}$ is continuous from the right in 0 and the other properties of $g_\mathcal{O}$ hold we have that for each $\varepsilon \geq 0$ there exists a δ with $0 \leq \delta \leq \varepsilon$ such that $g_\mathcal{O}(\delta) \leq \varepsilon$. This means that for all $n \in \mathbb{N}$ it holds that $d(x_n, \mathbb{R}^N \setminus B_\delta(\Omega_n)) \leq \varepsilon$.

Because $\mathbb{R}^N \setminus B_\delta(\Omega_n)$ are closed sets and by Proposition 2.1 (1), for every $n \in \mathbb{N}$ there exists a $y_n \in \mathbb{R}^N \setminus B_\delta(\Omega_n)$ such that $d(y_n, x_n) \leq \varepsilon$, where we have from before that $d(y_n, \Omega_n) \geq \delta$. Because y_n is a sequence in the set $\overline{\bigcup_{n=1}^{\infty} B_\varepsilon(x_n)}$, which is bounded because $x_n \longrightarrow x$, it has a cluster point y. Now it should be shown that also $d(x,y) \leq \varepsilon$ and $d(y,\Omega) \geq \delta$ hold.

Consider now one particular subsequence (which is again denoted by (Ω_n)) for which it holds simultaneously that $\Omega_n \xrightarrow{h} \Omega$, $\Gamma_n \xrightarrow{h} \Theta$, $x_n \longrightarrow x$ and $y_n \longrightarrow y$.

By the triangle inequality it holds that

$$d(x,y) \leq d(x,x_n) + d(x_n,y_n) + d(y_n,y)$$

where $d(x,x_n) \longrightarrow 0$, $d(y_n,y) \longrightarrow 0$ and $d(x_n,y_n) \leq \varepsilon$. Therefore, also

$$d(x,y) \leq \varepsilon.$$

Furthermore, as Ω is closed we have by Proposition 2.1 (1), that there exists a $z \in \Omega$ such that $d(y,z) = d(y,\Omega)$. Let now z_n be a sequence such that (see Proposition 2.2 (2)) $z_n \in \Omega_n$ for all $n \in \mathbb{N}$ and $z_n \longrightarrow z$. Then we have by the triangle inequality

$$\delta \leq d(y_n,y) + d(y,z) + d(z,z_n)$$

and as $d(y_n,y) \longrightarrow 0$ and $d(z,z_n) \longrightarrow 0$ we get

$$\delta \leq d(y,z) = d(y,\Omega).$$

So, finally we have that for an arbitrary $x \in \Theta$ and for all $\varepsilon > 0$ there exists a $y \in B_\varepsilon(x)$ for which $0 < \delta \leq d(y,\Omega)$. Therefore x must be a boundary point and we have proved the following:

LEMMA 3.2 *Let $\mathcal{O} \subset \mathcal{H}(\mathbb{R}^N)$ and let $g_{\mathcal{O}}$ be continuous from right in 0. Then it holds for every sequence $(\Omega_n) \subset \mathcal{O}$ that*

$$\Omega_n \xrightarrow{h} \Omega \Longrightarrow \Gamma_n \xrightarrow{h} \Gamma.$$

Additionally to the previous lemma, from the proof above one gets the following compactness criterion as a side result:

LEMMA 3.3 *Let $g : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ be a function continuous from the right in 0. If there exists an $\mathcal{O} \subset \mathcal{H}(\mathbb{R}^N)$, $\mathcal{O} \neq \emptyset$ with $g_{\mathcal{O}} = g$, then $\mathcal{H}(g) := \{\Omega \in \mathcal{H}(\mathbb{R}^N) \mid g_\Omega \leq g\}$ is complete.*

Using the lemmas above one can now prove the main result of this section:

THEOREM 3.1 *Let $\mathcal{O}$ be a compact subset of $\mathcal{H}(\mathbb{R}^N)$. Then it holds that $g_{\mathcal{O}}$ is continuous from right in 0 if and only if for every sequence $(\Omega_n) \subset \mathcal{O}$*

$$\Omega_n \xrightarrow{h} \Omega \Longrightarrow \Gamma_n \xrightarrow{h} \Gamma.$$

Proof.

$\Longrightarrow$: Follows directly from Lemma 3.2.

$\Longleftarrow$: Let $\mathcal{O}$ be a compact subset of $\mathcal{H}(\mathbb{R}^N)$ such that $\Omega_n \xrightarrow{h} \Omega \Longrightarrow \Gamma_n \xrightarrow{h} \Gamma$.

Assume now that $g_{\mathcal{O}}$ is not continuous from the right in 0, which means, as $g_{\mathcal{O}}(0) = 0$ and $g_{\mathcal{O}}$ is strictly increasing, that there exists an $\varepsilon > 0$ such that for all $\delta > 0$ $g_{\mathcal{O}}(\delta) > \varepsilon$. That means there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there exists an $\Omega \in \mathcal{O}$ and $x \in \Gamma$ for which it holds that

$$g_\Omega(x,\delta) = d(x, \mathbb{R}^N \setminus B_\delta(\Omega)) > \varepsilon. \tag{3}$$

Let now $\varepsilon > 0$ be such that (3) holds for all $\delta > 0$.

Select a sequence δ_k with $\delta_k \longrightarrow 0$. Then one can select a sequence (Ω_k), and $x_k \in \Gamma_k$ such that

$$d(x_k, \mathbb{R}^N \setminus B_{\delta_k}(\Omega_k)) > \varepsilon \tag{4}$$

where Γ_k denotes the boundary of Ω_k. By compactness of $\mathcal{O}$ there exists a subsequence of Ω_k (again denoted by Ω_k) such that $\Omega_k \xrightarrow{h} \Omega$ and $x_k \longrightarrow x$. This can be done because (x_k) is in the compact set $\overline{\bigcup_{l=1}^{m_0} \Omega_l \cup B_\nu(\Omega)}$, where ν is such that for all $m > m_0$ $\Omega_m \subset B_\nu(\Omega)$ holds (by Proposition 2.2 (1)).

By Proposition 2.2 (2) $x \in$ h-lim Γ_k holds. If $x \in \Gamma$ then there has to be a $y \in \mathbb{R}^N \setminus \Omega$ such that $d(x,y) < \varepsilon/2$.

Let now k_0 be such that for all $k > k_0$ $d(x_k, x) < \varepsilon/2$. Then by the triangle inequality it holds that

$$d(x_k, y) \leq d(x_k, x) + d(x,y) < \varepsilon. \tag{5}$$

Therefore, one can choose for $k > k_0$ a sequence y_k such that $y_k \in \Omega_k$ and $d(y_k, y) \leq \delta_k$. (If $y \in \Omega_k$ choose $y_k := y$. If $y \notin \Omega_k$ then by (4) and (5) y has to be in $B_{\delta_k}(\Omega_k)$, which means that $d(y, \Omega_k) < \delta_k$. Therefore $y_k \in \Omega_k$ can be selected such that $d(y, y_k) \leq \delta_k$.)

Hence we have constructed a sequence $y_k \longrightarrow y$ with $y_k \in \Omega_k$ and by Proposition 2.2 (2) it holds that $y \in \Omega$, which contradicts the choice of $y \in \mathbb{R}^N \setminus \Omega$. Therefore $x \in \Gamma$ cannot be true. ■

EXAMPLE 3.1

- *Let us consider the example mentioned in the introduction where the domains Ω_n can be seen in Fig. 3:*
$$\mathcal{O}_1 := \left\{ \Omega_n := \left[-1, -\frac{1}{n} \right] \cup \left[\frac{1}{n}, 1 \right] \right\} \cup \{[-1,1]\}$$
Because for each $\delta > 0$ one can find an $n \in \mathbb{N}$ such that $B_\delta(\Omega_n) =]-(1+\delta), 1+\delta[$ by selecting n such that $\frac{1}{n} < \delta$. Then $\partial B_\delta(\Omega_n) = \{-(1+\delta), 1+\delta\}$ and $\partial\Omega_n = \{-1, -\frac{1}{n}, \frac{1}{n}, 1\}$. So $g_\Omega(\delta) = \left|1 + \delta - \frac{1}{n}\right| > 1$ because $\frac{1}{n} < \delta$. Therefore, $g_{\mathcal{O}_1}$ is not continuous from the right in 0.

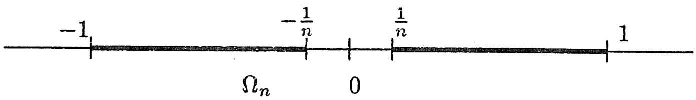

Figure 3. $\Omega_n = \left[-1, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, 1\right]$

- *On the other hand we may consider the following example, see Fig. 4:*
$$\mathcal{O}_2 := \left\{\Omega_n := \left[-1, 1 - \frac{1}{n}\right] \cup \left[1 - \frac{1}{2n}, 1\right]\right\} \cup \{[-1,1]\}$$
Note that for this $\mathcal{O}_2$ the Lipschitz condition for the boundary or the cone property with given constants does not hold (the definition of the Lipschitz

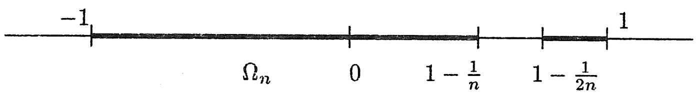

Figure 4. $\Omega_n = \left[-1, 1-\frac{1}{n}\right] \cup \left[1 - \frac{1}{2n}, 1\right]$

boundary can be found in Example 3.2, for the cone property see Chenais, 1975). For every $\delta > 0$ one has
$$B_\delta(\Omega_n) = \begin{cases}]-(1+\delta), 1+\delta[& \text{if } \delta > \frac{1}{4n} \\ ]-(1+\delta), 1-\frac{1}{n}+\delta[\cup]1-\frac{1}{2n}-\delta, 1+\delta[& \text{if } \delta \leq \frac{1}{4n} \end{cases}$$
and
$$\partial B_\delta(\Omega_n) = \begin{cases} \{-(1+\delta), 1+\delta\} & \text{if } \delta > \frac{1}{4n} \\ \{-(1+\delta), 1-\frac{1}{n}+\delta, 1-\frac{1}{2n}-\delta, 1+\delta\} & \text{if } \delta \leq \frac{1}{4n} \end{cases}$$
where
$$\partial\Omega_n = \left\{-1, 1-\frac{1}{n}, 1-\frac{1}{2n}, 1\right\}.$$
Therefore
$$g_{\Omega_n} = \begin{cases} \frac{1}{n}+\delta & \text{if } \delta > \frac{1}{4n} \\ \delta & \text{if } \delta \leq \frac{1}{4n} \end{cases}$$
and $g_{\Omega_n}(\delta) \leq 4\delta + \delta = 5\delta$. So $g_{\mathcal{O}_2}(\delta)$ is continuous from the right in 0. Especially for (Ω_n), for which $\Omega_n \xrightarrow{h} [-1,1]$ holds, this means by Lemma 3.2 that $\partial\Omega_n \xrightarrow{h} \partial[-1,1]$.

So, we see that if a hole vanishes, then it has to move close to a not vanishing boundary, to make sure that from $\Omega_n \xrightarrow{h} \Omega$ follows $\Gamma_n \xrightarrow{h} \Gamma$.

Because domains with Lipschitz boundary play an important role in the theory of partial differential equations, we consider them in the next example:

EXAMPLE 3.2 *$\Omega \subset \mathbb{R}^N$ has a Lipschitz boundary $:\Longleftrightarrow$ there exist constants $l > 0$, $\nu > 0$ such that for all $x = (x_1, \ldots, x_N) \in \partial\Omega$ there exists a local coordinate system (rotated with respect the original) and a function*

$$\varphi_x : K_\nu^{1,\ldots,N-1}(x) \longrightarrow \mathbb{R}\ y_{1,\ldots,N-1} \longrightarrow \varphi_x(y_{1,\ldots,N-1})$$

which is Lipschitz continuous with constant l, such that for all $y_{1,\ldots,N-1} \in K_\nu^{1,\ldots,N-1}(x)$ it holds that:

- $(y_{1,\ldots,N-1}, \varphi_x(y_{1,\ldots,N-1})) \in \partial\Omega$
- $y_{1,\ldots,N-1} \in K_\nu^{1,\ldots,N-1}(x)$ *and* $\varphi_x(y_{1,\ldots,N-1}) < y_N < \varphi_x(y_{1,\ldots,N-1}) + l\nu\sqrt{N-1} \Longrightarrow y \in \Omega$
- $y_{1,\ldots,N-1} \in K_\nu^{1,\ldots,N-1}(x)$ *and* $\varphi_x(y_{1,\ldots,N-1}) - l\nu\sqrt{N-1} < y_N < \varphi_x(y_{1,\ldots,N-1}) \Longrightarrow y \notin \overline{\Omega}$

where $K_\nu^{1,\ldots,N-1}(x)$ denotes the $N-1$ dimensional cube with center in $(x_1, \ldots \ldots, x_{N-1})$ and side length 2ν.

Although it is well known, it should be demonstrated here that $\mathcal{H}(l,\nu)$, which denotes the sets in $\mathcal{H}(\mathbb{R}^N)$ satisfying the Lipschitz condition with constants l and ν, has the property that $\Omega_n \longrightarrow \Omega \Longrightarrow \Gamma_n \longrightarrow \Gamma$ for all sequences $(\Omega_n) \subset \mathcal{H}(l,\nu)$.

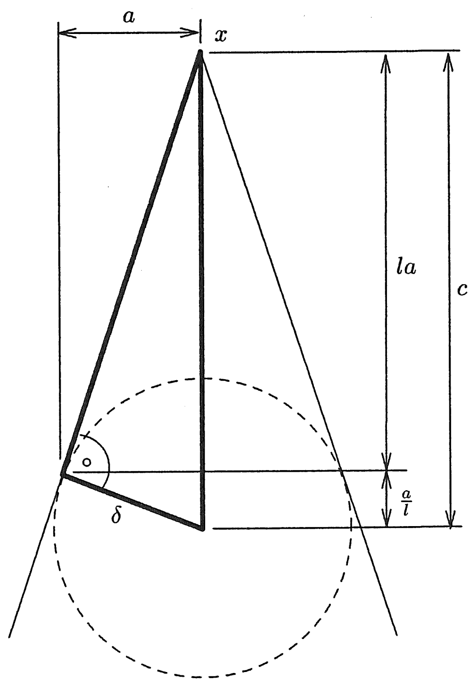

Figure 5. Cone disjoint with respect to Ω

Let now δ be sufficiently small and let $\Omega \in \mathcal{H}(\mathbb{R}^N)$ have a Lipschitz boundary with constants l and ν. Let furthermore $x \in \Gamma$ be arbitrary, but fixed. Then there exists a coordinate system such that Γ can be expressed as a Lipschitz continuous function with constant l.

Because of the Lipschitz condition the boundary $(x_1, \ldots, x_{N-1}, \varphi_x(x_1, \ldots \ldots, x_{N-1}))$ has to be outside of the open cone with vertex x, axes $(0, \ldots, 0, -1)$ and slope l and $-l$ respectively (see Fig. 5).

So, one can calculate $c = la + \frac{a}{l} = a(l + \frac{1}{l})$. Therefore, $a = c\frac{l}{1+l^2}$ and with the usual formulas for triangles $\delta^2 = c\frac{a}{l} = \frac{c^2}{l^2+1}$. From that, it follows that $c = \delta\sqrt{l^2+1}$. This means that for δ sufficiently small ($\delta < \nu$, $\delta(\sqrt{l^2+1}+1) < l\nu\sqrt{N-1}$) the distance between $(x_1, \ldots, x_{N-1}, x_n - \delta\sqrt{l^2+1})$ and Ω is greater than or equal to δ, but $d(x, (x_1, \ldots, x_{N-1}, x_n - \delta\sqrt{l^2+1})) = \delta\sqrt{l^2+1}$. It follows that $g_\Omega(\delta) \leq \delta\sqrt{l^2+1}$ for all $\Omega \in \mathcal{H}(l,\nu)$. Therefore, $g_{\mathcal{H}(l,\nu)}(\delta) \leq \delta\sqrt{l^2+1}$ holds, which means, because of $g_{\mathcal{H}(l,\nu)}(0) = 0$ and the property that this function is strictly increasing, that it is continuous from the right at 0. So, by Lemma 3.2 we have for every sequence $(\Omega_n) \subset \mathcal{H}(l,\nu)$ the implication $\Omega_n \longrightarrow \Omega \Longrightarrow \Gamma_n \longrightarrow \Gamma$.

3.1. Conclusion of boundary convergence

REMARK 3.3 *As $g_{\mathcal{O}}(0) = 0$, $g_{\mathcal{O}}(\delta) \geq \delta$ and $g_{\mathcal{O}}$ is strictly increasing, it is sufficient to show that for a function f which is continuous from the right in 0 with $f(0) = 0$ we have $g_{\mathcal{O}} \leq f$ within some neighborhood of 0. The function f may tend to 0 arbitrarily slowly. As one can see from Example 3.2 as a special case, for domains with Lipschitz boundary with constants $l > 0, \nu > 0$ this holds for the function $f(\delta) = \delta\sqrt{l^2+1}$, which is linear in δ.*

One sees immediately that $g_{\mathcal{O}}$ is continuous from the right in 0. Therefore $g_\Omega(\delta) \longrightarrow 0$ uniformly for all $\Omega \in \mathcal{O}$ if $\delta \longrightarrow 0$. By the last point of Remark 3.1 this means that $\partial B_\delta(\Omega) \xrightarrow{h} \Gamma$ uniformly for all $\Omega \in \mathcal{O}$ if $\delta \longrightarrow 0$. Hence one can replace in Lemma 3.2 "*$g_{\mathcal{O}}$ is continuous from right in* 0" by "*$\partial B_\delta(\Omega) \xrightarrow[\delta\to 0]{h} \Gamma$ uniformly for all $\Omega \in \mathcal{O}$ if $\delta \longrightarrow 0$*".

Especially Theorem 3.1 can be rewritten as follows:

THEOREM 3.2 *Let $\mathcal{O}$ be a compact subset of $\mathcal{H}(\mathbb{R}^N)$. Then the following statements are equivalent:*

$$\partial B_\delta(\Omega) \xrightarrow[\delta\to 0]{h} \Gamma \text{ uniformly for all } \Omega \in \mathcal{O}, \tag{6}$$

$$\Omega_n \xrightarrow{h} \Omega \Longrightarrow \Gamma_n \xrightarrow{h} \Gamma \text{ for all sequences } (\Omega_n) \subset \mathcal{O}. \tag{7}$$

One gets immediately the following corollary:

COROLLARY 3.2 *Let Υ be a compact subset of $\mathbb{R}^N$ and let $\mathcal{O} \subset \mathcal{H}(\Upsilon)$. Then the following statements are equivalent:*

$$\partial B_\delta(\Omega) \xrightarrow[\delta\to 0]{h} \Gamma \text{ uniformly for all } \Omega \in \mathcal{O},$$

$$\Omega_n \xrightarrow{h} \Omega \Longrightarrow \Gamma_n \xrightarrow{h} \Gamma \text{ for all sequences } (\Omega_n) \subset \mathcal{O}.$$

Of course it can also be seen in Example 3.1 that for the first set $\mathcal{O}_1$ the boundaries $\partial B_\delta(\Omega_n)$ have distance greater than 1 to Γ_n if $\frac{1}{n} < \delta$, whereas for every Ω_n of the second set $\mathcal{O}_2$ it holds that $d_h(\partial B_\delta(\Omega_n), \Gamma_n) \leq \delta$. Also in Example 3.2 it can be seen that $d_h(\partial B_\delta(\Omega), \Gamma) \leq \delta\sqrt{l^2+1}$ for all $\Omega \in \mathcal{H}(l,\nu)$.

So, we have as the main result that if for a set of closed subsets of $\mathbb{R}^N$ the boundary of $B_\delta(\Omega)$ converges to the boundary of Ω uniformly for all Ω in this set then for any sequence Ω_n from $\Omega_n \xrightarrow{h} \Omega$ it follows that $\Gamma_n \longrightarrow \Gamma$. If the set of domains is compact also the reverse direction holds.

One should notice that for a sequence (Ω_n) for which the restriction (6) holds, the topology of the Ω_n may change in any stage of $\Omega_n \subset B_\varepsilon(\Omega)$ $\forall n \geq n_0$ for $\varepsilon > 0$ arbitrarily small, which is not possible in the case of Lipschitz boundaries. Consider finally an example, where Corollary 3.2 can be applied, but which is not satisfying any Lipschitz condition:

EXAMPLE 3.3 *In contrast to Example 3.2 consider now the following set $\mathcal{O}$ of domains: A domain in $\mathcal{O}$ is the union of a fixed number of k closed balls with*

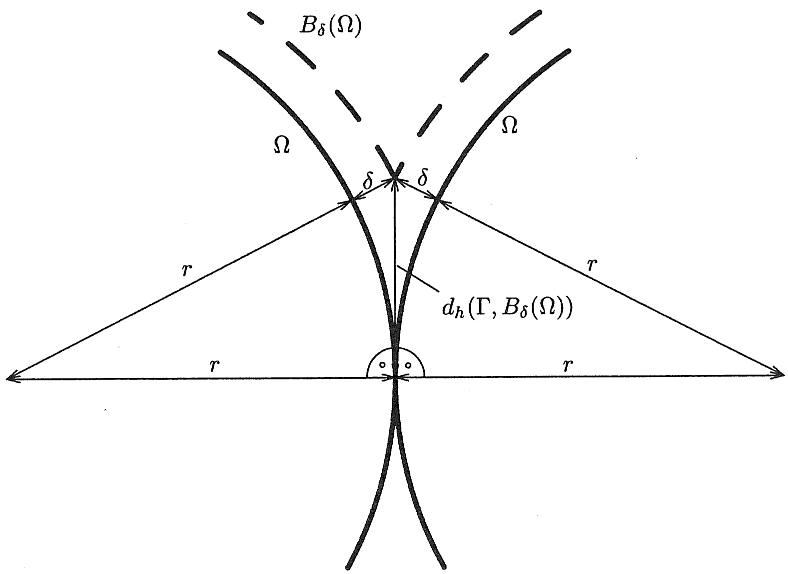

Figure 6. $d_h(\Gamma, B_\delta(\Omega))$ for balls with radius r

radius $r \in \mathbb{R}^+$, located within a compact subset Υ of $\mathbb{R}^2$. The radius may vary but is the same for each of the balls belonging to one domain. The balls of one domain are located arbitrarily in Υ and possibly touching others, but two of them have not more than one point in common, means the point of contact.

It is obvious that the sets in $\mathcal{O}$ are not satisfying any Lipschitz condition, because if two balls are touching each other, the boundary at the point of contact can neither be described as a single function nor can a Lipschitz condition be found.

Because all $\Omega \in \mathcal{O}$ are subsets of a compact set Υ, there exists a maximum possible radius r_{max} for the balls of Ω. The worst case for the distance $d_h(\Gamma, \partial B_\delta(\Omega))$ occurs at a point of contact. Assuming that δ is sufficiently small it can be calculated using the usual equations for triangles as $\sqrt{\delta^2 + 2r\delta}$, see Fig. 6. Hence, $d_h(\Gamma, \partial B_\delta(\Omega)) \leq \sqrt{\delta^2 + 2r_{max}\delta}$ means that $\partial B_\delta(\Omega)$ converges uniformly to Γ for $\delta \longrightarrow 0$. By Corollary 3.2 one has that $\Gamma_n \xrightarrow{h} \Gamma$ if $\Omega_n \xrightarrow{h} \Omega$.

4. The Lebesgue measure convergence of boundaries

Another important question in shape optimization is the following: Does the boundary preserve Lebesgue measure 0 when $\Omega_n \xrightarrow{h} \Omega$. In this section a similar criterion as before for boundary convergence is proven.

Denote from now on by μ the N-dimensional Lebesgue measure. As one can see, in the following example one may have $\mu(\Gamma_n) \not\longrightarrow \mu(\text{h-lim}(\Gamma_n))$:

EXAMPLE 4.1 *Let $\Omega_n := \left\{\frac{i}{2^n} \,\middle|\, i = 0, \ldots, 2^n\right\}$. It is clear that $\Gamma_n = \Omega_n$ and by $d_h(\Gamma_n, [0,1]) = \frac{1}{2^{n+1}}$ it holds that $\Gamma_n \xrightarrow{h} [0,1]$. But $\mu(\Gamma_n) \longrightarrow 0$ where $\mu(\text{h-lim}(\Gamma_n)) = \mu([0,1]) = 1$. Of course in this case also $\Gamma_n \overset{h}{\not\longrightarrow} \partial[0,1]$.*

EXAMPLE 4.2 *Now consider the sequence (Ω_n), where $\Omega_0 := [0,1]$ and Ω_n is that set obtained by removing an open interval of the length $1/(6(3^{n-1}))$ from the middle of each of the 2^{n-1} connected disjoint subsets of Ω_{n-1}. It is well known that $\Omega := \bigcap_{n\in\mathbb{N}} \Omega_n$ is a Cantor set with Lebesgue measure $1/2$ because the removed parts have the measure*

$$\sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{2}$$

as one can calculate by the formulas for geometric series. Furthermore, it is closed because it is an infinite intersection of closed sets. Therefore, $\Omega \in \mathcal{H}(\mathbb{R}^N)$ holds. Because Ω_n is divided into 2^n disjoint subsets with length of each less than $1/2^n$ it is clear that Ω consists of boundary points only (for a fixed $x \in \Omega$ and $\varepsilon > 0$ simply choose n such that $1/2^n < \varepsilon$, then $B_\varepsilon(x) \subset \Omega_n$ cannot be true and therefore $x \notin \mathring{\Omega}$).

Furthermore, it can be seen that $\Omega_n \xrightarrow{h} \Omega$ *because* $d_h(\Omega_n, \Omega) \leq 1/3^n \longrightarrow 0$. *Also* $\Gamma_n \xrightarrow{h} \Gamma$ *because* $d_h(\Gamma_n, \Gamma) = d_h(\Gamma_n, \Omega) \leq 1/3^n \longrightarrow 0$, *where* $\mu(\Gamma_n) = 0$ *for all* $n \in \mathbb{N}$ *but* $\mu(\Gamma) = 1/2$. *This means that* $\Omega_n \xrightarrow{h} \Omega$, $\Gamma_n \xrightarrow{h} \Gamma$ *and* $\mu(\Gamma_n) = 0$ *for all* $n \in \mathbb{N}$ *does not guarantee that* Γ *has Lebesgue measure zero. Note that because every boundary is closed it is Lebesgue measurable.*

Consider now the following well-known theorem from measure theory, see for example De Barra (1974), p. 111:

THEOREM 4.1 *Let* (Ω_n) *be a sequence of measurable sets. If* $\Omega_n \supset \Omega_{n+1}$ *for all* $n \in \mathbb{N}$ *and* $\mu(\Omega_1) < \infty$ *then* $\mu(\bigcap_{n=1}^{\infty} \Omega_n) = \lim_{n\to\infty} \mu(\Omega_n)$ *holds.*

LEMMA 4.1 *Let* $(\Omega_n) \subset \mathcal{H}(\mathbb{R}^N)$ *with* $\Omega_n \supset \Omega_{n+1}$ *for all* $n \in \mathbb{N}$. *Then*

$$\bigcap_{n=1}^{\infty} \Omega_n = \text{h-lim}(\Omega_n).$$

Proof. As $\Omega_n \supset \Omega_{n+1}$ for all $n \in \mathbb{N}$ it holds that $(\Omega_n) \subset \mathcal{H}(\Omega_1)$ and therefore there exists a subsequence $(\Omega_k) \subset (\Omega_n)$ with $\Omega_k \xrightarrow{h} \Theta$. We will show now that $\bigcap_{n=1}^{\infty} \Omega_n = \Theta$:

- By Proposition 2.2 (3) and $\bigcap_{n=1}^{\infty} \Omega_n \subset \Omega_k$ for all $k \in \mathbb{N}$ it holds that $\bigcap_{n=1}^{\infty} \Omega_n \subset \Theta$.
- Assume now that there exists $x \in \Theta$ such that $x \notin \bigcap_{n=1}^{\infty} \Omega_n$. As $\bigcap_{n=1}^{\infty} \Omega_n = \{x \in \mathbb{R}^N \mid x \in \Omega_n \text{ for all } n \in \mathbb{N}\}$ there has to be $n_0 \in \mathbb{N}$ such that $x \notin \Omega_{n_0}$. Now $\Omega_n \supset \Omega_{n+1}$ for all $n \in \mathbb{N}$ hence $x \notin \Omega_n$ for all $n \geq n_0$. As (Ω_k) is a subsequence of (Ω_n) there has to be $k_0 \in \mathbb{N}$ such that $x \notin \Omega_k$ for all $k \geq k_0$. Because Ω_{k_0} is closed $\varepsilon := d(x, \Omega_{k_0}) > 0$ holds and as $\Omega_k \supset \Omega_{k+1}$ it follows that $d(x, \Omega_k) \geq \varepsilon$ for all $k \geq k_0$. But by Proposition 2.2 (2) Θ is the set of all points y for which a sequence (y_k) exists with $y_k \in \Omega_k$ and $y_k \longrightarrow y$. This is a contradiction to $d(x, \Omega_k) \geq \varepsilon$ for all $k \geq k_0$.
  Therefore $x \in \bigcap_{n=1}^{\infty} \Omega_n$ has to hold for every $x \in \Theta$.

As $\bigcap_{n=1}^{\infty} \Omega_n = \Theta$, for every cluster point Θ of (Ω_n) one has also the uniqueness of Θ and $\Omega_n \xrightarrow{h} \bigcap_{n=1}^{\infty} \Omega_n$. ■

LEMMA 4.2 *Let* $(\Omega_n) \subset \mathcal{H}(\mathbb{R}^N)$ *such that* $\Omega_n \xrightarrow{h} \Omega$. *Then* $\mu(\Omega_n \setminus \Omega) \longrightarrow 0$ *holds.*

Proof. As Ω and Ω_n are closed, they are Lebesgue measurable and so are $\Omega_n \cup \Omega$, $\Omega_n \setminus \Omega$ and $\Omega \setminus \Omega_n$. Note that Ω and $\Omega_n \setminus \Omega$ (respectively Ω_n and $\Omega \setminus \Omega_n$) are disjoint and therefore $\mu(\Omega_n \cup \Omega) = \mu(\Omega) + \mu(\Omega_n \setminus \Omega)$ (respectively $\mu(\Omega \cup \Omega_n) = \mu(\Omega_n) + \mu(\Omega \setminus \Omega_n)$).

$\Omega_n \cup \Omega \xrightarrow{h} \Omega$ follows directly from Proposition 2.2 (3). For a fixed n let $\overline{n}$ denote that number for which by Proposition 2.2 (1) $\Omega_i \cup \Omega \subset B_{d_h(\Omega_n \cup \Omega, \Omega)}(\Omega)$ holds for all $i > \overline{n}$. Define now

$$\varepsilon_n := max\{d_h(\Omega_k \cup \Omega, \Omega) \mid \overline{n} \geq k \geq n\}.$$

As $\Omega_k \cup \Omega$ are all bounded, ε_n is well defined for all $n \in \mathbb{N}$ and one has that $\Omega_n \cup \Omega \subset \overline{B_{\varepsilon_n}(\Omega)}$ by Proposition 2.1 (3) and therefore

$$\mu(\overline{B_{\varepsilon_n}(\Omega)}) \geq \mu(\Omega_n \cup \Omega) = \mu(\Omega_n \setminus \Omega) + \mu(\Omega) \geq \mu(\Omega). \tag{8}$$

Furthermore, $\varepsilon_n \longrightarrow 0$ holds by definition (because $\Omega_n \cup \Omega \xrightarrow{h} \Omega$) and therefore $\overline{B_{\varepsilon_n}(\Omega)}$ is a decreasing sequence of sets, meaning that $\overline{B_{\varepsilon_n}(\Omega)} \supset \overline{B_{\varepsilon_{n+1}}(\Omega)}$ for all $n \in \mathbb{N}$.

By Lemma 4.1, $\overline{B_{\varepsilon_n}(\Omega)} \xrightarrow{h} \bigcap_{n=1}^{\infty} \overline{B_{\varepsilon_n}(\Omega)}$ and because $d_h(\Omega, \overline{B_{\varepsilon_n}(\Omega)}) = \varepsilon_n$ one has $\bigcap_{n=1}^{\infty} \overline{B_{\varepsilon_n}(\Omega)} = \Omega$.

By Theorem 4.1, $\mu(\overline{B_{\varepsilon_n}(\Omega)}) \longrightarrow \mu(\bigcap_{n=1}^{\infty} \overline{B_{\varepsilon_n}(\Omega)}) = \mu(\Omega)$ and considering (8) one has $\mu(\Omega_n \setminus \Omega) \longrightarrow 0$. ∎

REMARK 4.1 *From Lemma 4.2 one has* $\mu(\Omega_n \setminus \Omega) \longrightarrow 0$ *if* $\Omega_n \xrightarrow{h} \Omega$, *but* $\mu(\Omega \setminus \Omega_n)$ *may not converge to* 0*!*

Consider now the following theorem, which can be found in Beer (1974), p. 64:

THEOREM 4.2 *Let* $(\Omega_n) \subset \mathcal{H}(\mathbb{R}^N)$, $\Omega \in \mathcal{H}(\mathbb{R}^N)$ *be such that* $\Omega_n \xrightarrow{h} \Omega$. *Then it holds that*

$$\mu(\Omega \Delta \Omega_n) \longrightarrow 0 \iff \mu(B_\delta(\Omega_n)) \xrightarrow[\delta \to 0]{} \mu(\Omega_n) \textit{ uniformly for all } n \in \mathbb{N}$$

where $\Omega \Delta \Omega_n := \Omega \setminus \Omega_n \cup \Omega_n \setminus \Omega$ *denotes the symmetric difference of* Ω *and* Ω_n.

By using Lemma 4.2 and the fact that $\Omega \setminus \Omega_n$ and $\Omega_n \setminus \Omega$ are disjoint and measurable one has immediately:

COROLLARY 4.1 *Let* $(\Omega_n) \subset \mathcal{H}(\mathbb{R}^N)$, $\Omega \in \mathcal{H}(\mathbb{R}^N)$ *be such that* $\Omega_n \xrightarrow{h} \Omega$. *Then it holds that*

$$\mu(\Omega \setminus \Omega_n) \longrightarrow 0 \iff \mu(B_\delta(\Omega_n)) \xrightarrow[\delta \to 0]{} \mu(\Omega_n) \textit{ uniformly for all } n \in \mathbb{N}.$$

By using Corollary 4.1 and Lemma 4.2 one can also prove the following:

COROLLARY 4.2 *Let* $(\Omega_n) \subset \mathcal{H}(\mathbb{R}^N)$, $\Omega \in \mathcal{H}(\mathbb{R}^N)$ *be such that* $\Omega_n \xrightarrow{h} \Omega$. *Then it holds that*

$$\mu(\Omega_n) \longrightarrow \mu(\Omega) \iff \mu(B_\delta(\Omega_n)) \xrightarrow[\delta \to 0]{} \mu(\Omega_n) \textit{ uniformly for all } n \in \mathbb{N}.$$

Proof. From

$$0 = \mu(\Omega_n \cup \Omega) - \mu(\Omega \cup \Omega_n) = \mu(\Omega \setminus \Omega_n) + \mu(\Omega_n) - \mu(\Omega) - \mu(\Omega_n \setminus \Omega)$$

one has $\mu(\Omega_n) = \mu(\Omega) + \mu(\Omega_n \setminus \Omega) - \mu(\Omega \setminus \Omega_n)$. By $\Omega_n \xrightarrow{h} \Omega$ and Lemma 4.2, $\mu(\Omega_n \setminus \Omega) \longrightarrow 0$ holds. Therefore, $\mu(\Omega_n) \longrightarrow \mu(\Omega)$ if and only if $\mu(\Omega \setminus \Omega_n) \longrightarrow 0$. The proof follows directly from Corollary 4.1. ∎

LEMMA 4.3 *Let $\mathcal{O} \subset \mathcal{H}(\mathbb{R}^N)$ be such that $\mu(B_\delta(\Gamma)) \xrightarrow[\delta \to 0]{} \mu(\Gamma)$ uniformly for all $\Omega \in \mathcal{O}$. Then it holds that*

$$\Gamma_n \xrightarrow{h} \Theta \Longrightarrow \mu(\Gamma_n) \longrightarrow \mu(\Theta).$$

Proof. Follows directly from Corollary 4.2. ∎

THEOREM 4.3 *Let $\mathcal{O} \subset \mathcal{H}(\mathbb{R}^N)$ be compact. Then the following statements are equivalent:*

- $\mu(B_\delta(\Gamma)) \xrightarrow[\delta \to 0]{} \mu(\Gamma)$ *uniformly for all* $\Omega \in \mathcal{O}$.
- $\Gamma_n \xrightarrow{h} \Theta \Longrightarrow \mu(\Gamma_n) \longrightarrow \mu(\Theta)$.

Proof. "$\Longrightarrow$": Follows directly from Lemma 4.3.

"$\Longleftarrow$": Assume that $\mu(B_\delta(\Gamma)) \xrightarrow[\delta \to 0]{} \mu(\Gamma)$ uniformly for $\Omega \in \mathcal{O}$ does not hold. This means that there is a $\varepsilon > 0$ such that for all $\delta > 0$ there exists a $\Omega \in \mathcal{O}$ with $|\mu(B_\delta(\Gamma)) - \mu(\Gamma)| > \varepsilon$.

Let now $\varepsilon > 0$ be such that the above statement holds. So by selecting a sequence $\delta_k \longrightarrow 0$ one can choose Ω_k such that $|\mu(B_{\delta_k}(\Gamma_k)) - \mu(\Gamma_k)| > \varepsilon$. By compactness of $\mathcal{O}$ and Lemma 3.1 there exists a subsequence (again denoted by (Ω_k)) such that both $\Omega_k \xrightarrow{h} \Omega \in \mathcal{O}$ and $\Gamma_k \xrightarrow{h} \Theta$, but $|\mu(B_{\delta_k}(\Gamma_k)) - \mu(\Gamma_k)| > \varepsilon$. So by Corollary 4.2 $\mu(\Gamma_n) \longrightarrow \mu(\Theta)$ cannot be true. ∎

REMARK 4.2 *Note that for Lemma 4.3 and Theorem 4.3 only $\Gamma_n \xrightarrow{h} \Theta$ is required, $\Gamma = \Theta$ and $\Omega_n \xrightarrow{h} \Omega$ is not necessary and may not hold. For an example, where $\Gamma = \Theta$ does not hold consider $\mathcal{O}_1$ of Example 3.1. For an example where $\Omega_n \xrightarrow{h} \Omega$ does not hold consider the example of $\Omega_n := [-1, 1]$ for n even, $\Omega_n := \{-1, 1\}$ for n odd. In both cases $\Gamma_n = \{-1, 1\}$, therefore $\mu(\Gamma_n) \longrightarrow \mu(\{-1, 1\}) = 0$, but (Ω_n) has no Hausdorff limit. Of course one has by Lemma 3.1 under the assumptions of Lemma 4.3 or Theorem 4.3 that if $\Omega_n \xrightarrow{h} \Omega$ then $\mu(\Gamma_k) \longrightarrow \mu(\Theta)$ holds for each cluster point Θ of (Γ_n), where (Γ_k) denotes the corresponding subsequence of (Γ_n) with $\Gamma_k \xrightarrow{h} \Theta$.*

So, one has as an important special case the following criterion for Lebesgue measure preservation 0 of the boundary. This case is very important for shape optimization.

COROLLARY 4.3 *Let $\mathcal{O} \subset \mathcal{H}(\mathbb{R}^N)$ be compact. Then the following statements are equivalent:*

- $\mu(B_\delta(\Gamma)) \xrightarrow[\delta\to 0]{} 0$ *uniformly for all* $\Omega \in \mathcal{O}$.
- *For each sequence* $(\Omega_n) \subset \mathcal{O}$ *it holds that* $\Omega_n \xrightarrow{h} \Omega \Longrightarrow \mu(\Theta) = 0$, *where* Θ *is any cluster point of* (Γ_n).

Proof. It remains to show that from $\mu(B_\delta(\Gamma)) \xrightarrow[\delta\to 0]{} 0$ it follows that $\mu(\Gamma) = 0$. The rest follows directly from Theorem 4.3 and Lemma 3.1:

As for any $\delta_k \longrightarrow 0$ it holds that $\bigcap_{k=1}^{\infty} \overline{B_{\delta_k}(\Gamma)} = \Gamma$ ("$\supset$" holds by definition, "$\subset$" because Γ is closed, otherwise there would exist $x \in \bigcap_{k=1}^{\infty} \overline{B_{\delta_k}(\Gamma)}$ with $d(x,\Gamma) = \varepsilon > 0$) and as $\overline{B_{\delta_k}(\Gamma)} \subset B_{2\delta_k}(\Gamma)$ and $0 \le \mu\left(\overline{B_{\delta_k}(\Gamma)}\right) \le \mu(B_{2\delta_k}(\Gamma)) \longrightarrow 0$ also, we obtain by Theorem 4.1, $0 = \lim_{k\to\infty} \mu(\overline{B_{\delta_k}(\Gamma)}) = \mu(\bigcap_{k=1}^{\infty} \overline{B_{\delta_k}(\Gamma)}) = \mu(\Gamma)$. ∎

EXAMPLE 4.3 *Similarly as in Example 3.2 it should be demonstrated that the subset $\mathcal{H}(l,\nu)$ of closed, bounded, domains $\Omega \subset \mathbb{R}^N$ which have a Lipschitz boundary with constants l,ν, has the property*

$$\Omega_n \xrightarrow{h} \Omega \Longrightarrow \mu(\Gamma) = 0$$

for every converging sequence $(\Omega_n) \subset \mathcal{H}(l,\nu)$. Using Lemma 4.3 this can be done without proving that $\mathcal{H}(l,\nu)$ is complete (meaning that the limit of every converging sequence of $\mathcal{H}(l,\nu)$ is again in $\mathcal{H}(l,\nu)$), which may cause a lot of technical effort to prove or counterprove.

Let now $(\Omega_n) \subset \mathcal{H}(l,\nu)$ be a converging sequence $\Omega_n \xrightarrow{h} \Omega$. From Example 3.2 we know already that in that case $\Gamma_n \xrightarrow{h} \Gamma$. Because $\mathcal{H}(l,\nu) \subset \mathcal{H}(\mathbb{R}^N)$ and $\mathcal{H}(\mathbb{R}^N)$ is complete also $\Omega \in \mathcal{H}(\mathbb{R}^N)$, meaning that Ω is compact.

Hence, for $\varepsilon > 0$ arbitrary, but fixed, there exists (by Proposition 2.2 (1)) an n_0 such that for all $n \ge n_0$, $\Omega_n \subset B_\varepsilon(\Omega)$. So, we have for all $n \in \mathbb{N}$ that $\Omega_n \subset \Upsilon := \bigcup_{i=1}^{n_0} \Omega_i \cup \overline{B_\varepsilon(\Omega)}$, where Υ is compact. This means that for every $\vartheta > 0$ there exists a finite covering of Υ with balls of diameter ϑ. Because $\Gamma \subset \Omega \subset \Upsilon$, this finite covering will also cover Γ.

Consider now an arbitrary, but fixed, ball of this covering such that a point $x \in \Gamma$ is within this ball. Then, by the assumption that Ω has a Lipschitz boundary with constants l,ν there exists a coordinate system and an $N-1$ dimensional cube $K_\nu^{1,\dots,N-1}(x)$ such that Γ can be represented as a Lipschitz function $\varphi_x(y_{1,\dots,N-1})$ with Lipschitz constant l. Furthermore, there must not exist any other part of Γ within

$$\{y \in \mathbb{R}^N \mid y_{1,\dots,N-1} \in K_\nu^{1,\dots,N-1}(x) \text{ and}$$
$$y_N \in]\varphi_x(y_{1,\dots,N-1}) - l\nu\sqrt{N-1}, \varphi_x(y_{1,\dots,N-1}) + l\nu\sqrt{N-1}[\}. \quad (9)$$

Taking into consideration that the slope is bounded by l ($-l$ respectively), the boundary must in vertical direction be between the lines L_1 and L_2 of Fig. 7.

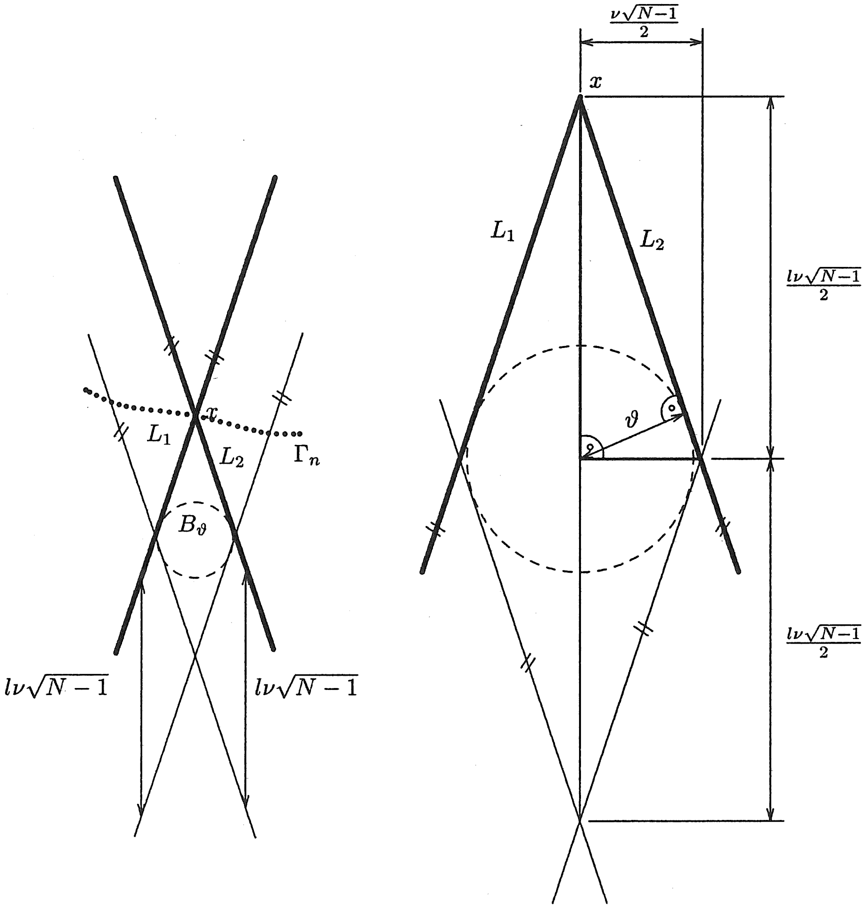

Figure 7. A ball B_ϑ possibly touching φ_x only

Hence a ball B_ϑ, possibly touching the boundary (meaning touching L_1 and L_2) but not touching any other part of Γ than represented by φ_x must be within the region given by (9). So in the worst case this ball must be located like in Fig. 7. By the usual equations for triangles the maximum radius for such a ball calculates as $\vartheta \leq \frac{l\nu\sqrt{N-1}}{2\sqrt{l+1}}$ (see Fig. 7). If ϑ was selected suitably small (meaning $\vartheta < \min\left(\frac{\nu}{2}, \frac{l\nu\sqrt{N-1}}{2\sqrt{l^2+1}}\right)$) there will be no other part of Γ within B_ϑ than that one represented by φ_x within $K_\nu^{1,\dots,N-1}(x)$. Denote now by k the number of balls with that radius ϑ used for the finite covering of Υ.

So, select now a ball B_ϑ from the above covering with $\Gamma \cap B_\vartheta \neq \emptyset$. Then select a point $x \in \Gamma \cap B_\vartheta$ and consider the corresponding function φ_x within

$K_\nu^{1,\dots,N-1}(x)$. By a similar calculation as in Example 3.2 one obtains that a ball of radius δ must be covered by (see Fig. 8):

$$\{y \in \mathbb{R}^N \mid y_{1,\dots,N-1} \in K_\nu^{1,\dots,N-1}(x) \text{ and } \\ y_N \in]\varphi_x(y_{1,\dots,N-1}) - \delta\sqrt{l^2+1}, \varphi_x(y_{1,\dots,N-1}) + \delta\sqrt{l^2+1}[\}. \tag{10}$$

The same holds for each other point of the graph of φ_x within $K_\nu^{1,\dots,N-1}(x)$. Therefore the Lebesgue measure of $B_\delta(graph(\varphi_x))$ is (within $K_\nu^{1,\dots,N-1}(x)$) lim-

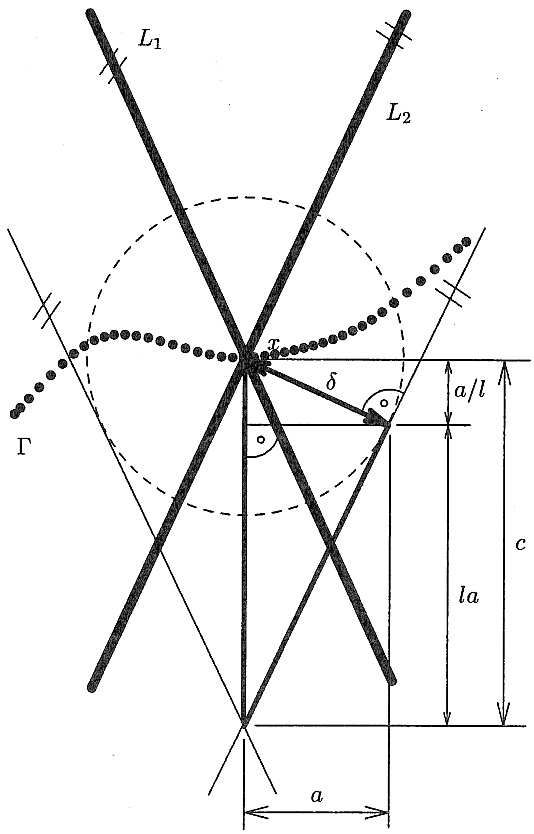

Figure 8. A ball B_δ covered by (10)

ited by the Lebesgue measure of (10), which is $(2\nu)^{N-1}(2\delta\sqrt{l^2+1})$. Remembering that the finite covering consists of k balls B_ϑ we have $\mu(B_\delta(\Gamma)) \leq k2^N\nu^{N-1}\delta\sqrt{l^2+1}$, which is linear in δ.

This proves that $\mu(B_\delta(\Gamma_n)) \xrightarrow[\delta\to 0]{} \mu(\Gamma_n) = 0$ *uniformly for all* Ω_n *and by Lemma 4.3 using Example 3.2 one has* $\Omega_n \xrightarrow{h} \Omega \Longrightarrow \mu(\Gamma) = 0$. *Because this holds for any sequence* $(\Omega_n) \subset \mathcal{H}(l,\nu)$ *with* $\Omega_n \xrightarrow{h} \Omega$, *this completes the proof.* ■

EXAMPLE 4.4 *Consider now again the set* $\mathcal{O}$ *of domains from Example 3.3, which does not satisfy any Lipschitz condition. If for each* $\Omega \in \mathcal{O}$ *it holds that* $\mu(B_\delta(\Gamma)) \leq k(r+\delta)^2\pi - k(r-\delta)^2\pi = 4kr\delta\pi$ *when* δ *is sufficiently small, meaning* $\delta \leq r$. *Hence, for each* $\Omega \in \mathcal{O}$ $\mu(B_\delta(\Gamma)) \leq 4kr_{max}\delta\pi \xrightarrow[\delta\to 0]{} 0 = \mu(\Gamma)$. *Using the result of Example 3.3 one has by Lemma 4.3 that* $\Omega_n \xrightarrow{h} \Omega \Longrightarrow \mu(\Gamma_n) \longrightarrow \mu(\Gamma) = 0$.

5. Concluding remarks

The final result for shape optimization is that for any set $\mathcal{O}$ of admissible domains it is sufficient to show that

- $\partial B_\delta(\Omega) \xrightarrow[\delta\to 0]{h} \Gamma$ uniformly for all $\Omega \in \mathcal{O}$,
- $\mu(B_\delta(\Gamma)) \xrightarrow[\delta\to 0]{} 0$ uniformly for all $\Omega \in \mathcal{O}$

in order to make sure that

$$\Omega_n \xrightarrow{h} \Omega \Longrightarrow \Gamma_n \xrightarrow{h} \Gamma \text{ and } \mu(\Gamma_n) \longrightarrow \mu(\Gamma) = 0.$$

REMARK 5.1 *Please note that for a sequence of domains* $\Omega_n \xrightarrow{h} \Omega$ *the properties* $\Gamma_n \xrightarrow{h} \Gamma$ *and* $\mu(\Gamma_n) \longrightarrow \mu(\Gamma)$ *are completely independent. For an example where* $\Omega_n \xrightarrow{h} \Omega$ *and* $\Gamma_n \xrightarrow{h} \Gamma$ *but* $\mu(\Gamma_n) \not\longrightarrow \mu(\Gamma)$ *consider Example 4.2. In contrast, for an example where* $\Omega_n \xrightarrow{h} \Omega$ *and* $\mu(\Gamma_n) \longrightarrow \mu(\Gamma)$ *but* $\Gamma_n \overset{h}{\not\longrightarrow} \Gamma$ *consider* $\mathcal{O}_1$ *from Example 3.1.*

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