

Algorithmic aspects of Newman polynomials and their divisors*

by

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Abstract: We study the problem of determining which integer polynomials divide Newman polynomials. First, we investigate the 8 438 known polynomials with Mahler measure less than 1.3. Apart from the polynomials having a positive real root, all but three of them are shown to divide a Newman polynomial of degree at most 1000. Second, using an algorithmic certification method of Hare and Mossinghoff, we exhibit several polynomials that divide no Newman polynomial. Among them is a degree-10 polynomial with Mahler measure approximately equal to 1.419404632, which improves the best known upper bound for any universal constant σ , if such a constant exists, governing the existence of Newman multiples in terms of Mahler measure. Finally, writing $l(x)$ for the polynomial obtained from Lehmer's polynomial by the substitution $x \mapsto -x$, we explicitly construct Newman polynomials divisible by $l(x)^2$ with degrees up to 150, and show that no Newman polynomial of degree at most 160 is divisible by $l(x)^3$.

Keywords: Newman polynomials, Mahler measure

1. Introduction

Recall that the Mahler measure of a polynomial p with complex coefficients is defined as

$$M(p) := |b_0| \prod_{k=1}^d \max(1, |\alpha_k|),$$

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where

$$p(x) = b_0x^d + b_1x^{d-1} + \cdots + b_d = b_0 \prod_{k=1}^d (x - \alpha_k), \text{ with } b_0 \neq 0.$$

It is well known (see Bombieri and Vaaler, 1987) that every polynomial $p(x) \in \mathbb{Z}[x]$, having Mahler measure less than 2 must divide a polynomial with all coefficients in $\{-1, 0, 1\}$. Given this result, it is tempting to ask the following question: does this result have any chance of persisting if we impose stricter conditions on the coefficients, namely that they are no longer in $\{-1, 0, 1\}$, but only in $\{0, 1\}$?

Recall that a *Newman polynomial* is a polynomial, whose coefficients are either 0 or 1, with constant term 1. In the remainder of this article, we denote by $\mathcal{N}$ the set of all Newman polynomials. In fact, we can immediately refine the question above through a few simple observations.

First, a polynomial with a positive real root cannot divide a Newman polynomial. More generally, one should exclude nonnegative real roots, since a zero root is also impossible for a divisor of a Newman polynomial, whose constant term is equal to 1. In the computations below, however, the polynomials under consideration have nonzero constant term; therefore this obstruction will usually be stated in the equivalent form of having a positive real root.

A second factor stems from the result of A. M. Odlyzko and B. Poonen (1993), which states that any nonzero root α of a Newman polynomial satisfies $1/g < |\alpha| < g$, where $g \approx 1.6180$ is the golden ratio. This result might lead us to think that an integer-coefficient polynomial without a nonnegative real root, provided that its Mahler measure is smaller than g , could divide a Newman polynomial. In this regard, a paper by K. G. Hare and M. J. Mossinghoff (2014) proves that the golden ratio is a natural bound for a wide range of classes of polynomials. Recall that a real algebraic integer strictly greater than 1 is a *Salem number* if all its conjugates (that is, the other roots of its minimal polynomial) have modulus less than or equal to 1, and at least one conjugate has modulus exactly 1. Recall also that a *Pisot number* is a real algebraic integer strictly greater than 1 whose conjugates all have modulus strictly less than 1. Given these definitions, we say that a real number α is a *negative Pisot number* (respectively, a *negative Salem number*) if $-\alpha$ is a Pisot number (respectively, a Salem number). In Hare and Mossinghoff (2014), the authors prove that every negative Pisot number within the interval $(-g, -1)$ that has no positive conjugates, as well as every negative Salem number in $(-g, -1)$, obtained through Salem's construction applied to small negative Pisot numbers, satisfies a Newman polynomial. Additionally, they identify all negative Salem numbers within $(-g, -1)$ with a degree of at most 20 and confirm that each of them is a root of a Newman polynomial.

Another interesting result is that of A. Dubickas (2003), which states that any product of cyclotomic polynomials that does not contain powers of $x - 1$ necessarily divides a Newman polynomial. Related questions for sparse polynomials with restricted coefficients have also been studied by D. Dutykh and J.-L. Verger-Gaugry. In Dutykh and Verger-Gaugry (2018), they investigate the almost Newman lacunary polynomials, with particular emphasis on reducibility, cyclotomic and noncyclotomic factors, reciprocal and nonreciprocal parts, and the geometry of their zero sets. Further connections between restricted coefficient polynomials, algebraic bases, rewriting trails, ultimately periodic representations, almost Newman polynomials, and Lehmer-type questions are developed in Dutykh and Verger-Gaugry (2021).

However, it turns out that the golden ratio is a bound that is too large, according to the same article by K. G. Hare and M. J. Mossinghoff (2014), where the authors exhibit five polynomials, whose Mahler measure is smaller than g , yet none of them divides a Newman polynomial. The smallest of the five Mahler measures is approximately equal to 1.55601, which corresponds to the polynomial $x^6 - x^5 - x^3 + x^2 + 1$.

In the light of all this, the natural question that arises is the following: *Does there exist a constant σ , such that if the integer-coefficient polynomial p has no nonnegative real root and has a Mahler measure smaller than σ , then p necessarily divides a Newman polynomial?*

Naturally, the result of K. G. Hare and M. J. Mossinghoff (2014) implies that σ , if it exists, is at most approximately 1.55601. This upper bound was improved by P. Drungilas, J. Jankauskas, and J. Šiurys (2018), who studied the existence of Littlewood and Newman multiples of Borwein polynomials by means of a decision algorithm for monic integer polynomials with no roots on the unit circle. As a part of this work, they produced new examples of polynomials with no Newman multiple and showed that σ , if it exists, must in fact be less than 1.436632261.

We first chose to investigate polynomials with small Mahler measure, since such polynomials have been studied extensively for decades. In particular, tables of such polynomials are available, some of which are certified as complete. More precisely, we will examine the list of all known polynomials having a Mahler measure less than 1.3 and a degree less than or equal to 180, with the aim of determining whether the polynomials in this list divide any Newman polynomials. This list was available for many years on a website maintained by M. J. Mossinghoff. Since that website is no longer available, we use the copy, which is reproduced in the file `Known180.txt`, included in our dataset, Idris and Sac-Épée (2026). For clarity and readability, we refer to this list throughout the article as *Known180*. This non-exhaustive list of 8 438 polynomials is complete up to degree 44 (Mossinghoff, Rhin and Wu, 2008). In the following section, we

will present an algorithm, designed to explicitly provide a Newman polynomial (if it exists) that a polynomial from this list divides, provided that the sought Newman polynomial does not have a degree strictly greater than 1 000. Apart from polynomials that have a positive real root, we will see that only three polynomials remain unresolved, in the sense that no Newman multiple of degree at most 1 000 was found for them.

2. The case of polynomials with small Mahler measure

As announced, in this first approach we will focus on polynomials in $\mathbb{Z}[x]$, whose Mahler measure is strictly less than 1.3. All polynomials in *Known180* are monic and irreducible. Moreover, they are reciprocal by a classical result of C. J. Smyth proven in Smyth (1971). We have developed and implemented an algorithm designed to take as input each polynomial from *Known180* and search (up to degree 1 000), for a Newman polynomial that has this input polynomial as one of its factors. In the next subsection, we will precisely explain how this algorithm works.

2.1. Description of the algorithm

An input polynomial, chosen from *Known180*, is a reciprocal polynomial of even degree (say $2d$). It can be written down as

$$p(x) = x^{2d} + a_1x^{2d-1} + a_2x^{2d-2} + \dots + a_{d-1}x^{d+1} + a_dx^d + a_{d-1}x^{d-1} + \dots + a_2x^2 + a_1x + 1,$$

where the a_i 's are explicitly given. If p is not already a Newman polynomial, and if it does not have any strictly positive real root (which would prevent it from dividing a Newman polynomial), we search for a polynomial

$$q(x) = x^m + b_{m-1}x^{m-1} + \dots + b_1x + 1 \in \mathbb{Z}[x]$$

such that $p(x)q(x)$ is a Newman polynomial. The constant term of q is forced to be 1, since the input polynomial p has constant term 1 and the product $p(x)q(x)$ must also have constant term 1. Expanding the product $p(x)q(x)$ and identifying the coefficients c_k ($k = 0, 1, \dots, 2d + m$) of each power of x in this product (given that the a_i 's are explicitly known, it follows that each c_k is a linear expression in $b_1, \dots, b_{m-1}$), this comes down to solving the system of linear inequalities with integer unknowns $b_1, \dots, b_{m-1}$:

$$0 \leq c_k \leq 1, \quad \text{for } k = 0, 1, \dots, 2d + m. \quad (2.1)$$

We increase the value of m , starting from 1, until we find a solution (that is, a polynomial q) satisfying the system of constraints (2.1), in which case $p(x)$

is indeed a factor of the Newman polynomial $p(x)q(x)$. We do not allow m to keep increasing if $2d + m$ becomes greater than 1 000.

As an example, let us suppose we want to determine whether the polynomial $p(x) = x^{16} + x^{15} - x^{11} - x^8 - x^5 + x + 1$ from *Known180* is a factor of a Newman polynomial. For m from 1 to 21, the system (2.1) has no solution. For $m = 22$, we search for integers $b_1, \dots, b_{21}$, that is, a polynomial $q(x) = x^{22} + b_{21}x^{21} + \dots + b_1x + 1$ (not necessarily reciprocal), such that $p(x)q(x)$ is a Newman polynomial. In this particular case, the system, referred to in (2.1) becomes:

find $b_1, b_2, \dots, b_{21}$ in $\mathbb{Z}$ such that

$$\left\{ \begin{array}{l} 0 \leq b_{21} + 1 \leq 1 \\ 0 \leq b_{21} + b_{20} \leq 1 \\ 0 \leq b_{20} + b_{19} \leq 1 \\ 0 \leq b_{19} + b_{18} \leq 1 \\ \vdots \\ 0 \leq b_5 + b_4 - 1 \leq 1 \\ 0 \leq b_4 + b_3 \leq 1 \\ 0 \leq b_3 + b_2 \leq 1 \\ 0 \leq b_2 + b_1 \leq 1 \\ 0 \leq b_1 + 1 \leq 1. \end{array} \right. \quad (2.2)$$

Solving the system with integer unknowns (2.2) yields the polynomial

$$q(x) = x^{22} + x^{20} + x^{18} + x^{15} + x^{13} + 2x^{11} + x^9 + x^7 + x^4 + x^2 + 1$$

and it is easy to verify that

$$p(x)q(x) = x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{27} + x^{11} + x^4 + x^3 + x^2 + x + 1,$$

which is indeed a Newman polynomial. We solve the system of inequalities (2.2) using the mixed-integer linear programming package Gurobi, which is called from a program written in Julia. We will provide further technical details later in the article. We did this for the 8 438 polynomials in *Known180*.

2.2. Calculation results

We have recorded the results of the computations in the accompanying dataset, see Idris and Sac-Épée (2026). Unlike the result files described below, the file *Known180.txt*, which contains the list *Known180*, records only the nonredundant coefficients of each polynomial, since all polynomials in *Known180* are reciprocal. The other files contain all the coefficients of each polynomial.

The file `newman.txt` contains all the polynomials from *Known180* that are already Newman polynomials. Each line contains the polynomial along with its degree, Mahler measure, and the full list of coefficients.

The file `positive.txt` contains all the polynomials from *Known180* that, having a positive real root, cannot divide a Newman polynomial. Each line contains the polynomial with its degree, Mahler measure, one of its positive real roots, and the list of its coefficients.

The last three files, namely `outputP.txt`, `outputQ.txt`, and `outputD.txt`, are interconnected. On each line k of the file `outputP.txt`, there is a polynomial p from *Known180*, with its degree, Mahler measure, and list of coefficients. On the same line k of `outputQ.txt`, there is a polynomial q such that the product pq is a Newman polynomial. Line k of `outputQ.txt` contains the degree of q and its list of coefficients. Finally, each line k of the file `outputD.txt` contains the product of the polynomials p and q , found on lines k of `outputP.txt` and `outputQ.txt`. The file `outputD.txt`, therefore, contains only Newman polynomials.

To verify that the Newman multiples are correct, the reader may use the short script `verifNewman.gp` (available at Idris and Sac-Épée, 2026), written in GP/PARI, which should be placed in the directory, where the files `outputP.txt`, `outputQ.txt`, and `outputD.txt` have previously been downloaded. One simply needs to indicate, in the first line of the script, which line k of the files `outputP.txt`, `outputQ.txt`, and `outputD.txt` is to be checked. The script displays the polynomial p , appearing on line k of `outputP.txt` (the degree and the Mahler measure are skipped), the polynomial q appearing on line k of `outputQ.txt` (the degree is skipped), computes and displays the product pq , and then displays the Newman polynomial d appearing on line k of `outputD.txt` (the degree is skipped). The script confirms that one indeed has $pq = d$.

It is worth noting that among the 8 438 polynomials examined, if one excludes those that were discarded from the outset because they have a positive real root, only three polynomials remain unresolved, and it is unknown whether they divide a Newman polynomial or not.

The only three polynomials in *Known180*, for which we were unable to exhibit multiples in the set $\mathcal{N}$ of Newman polynomials, are recorded in the file `nonsol.txt` in the dataset Idris and Sac-Épée (2026). Our algorithm certifies only that these polynomials do not divide any Newman polynomial of degree at most 1 000. It remains open whether they may occur as factors of Newman polynomials of higher degree.

In the following section, we examine a method which, under certain conditions, can certify that a given polynomial divides no Newman polynomial.

3. Polynomials that divide no Newman polynomial

In this section, we employ an algorithmic method, introduced in Hare and Mossinghoff (2014) by K. G. Hare and M. J. Mossinghoff to construct examples of polynomials with integer coefficients that do not divide any Newman polynomial. We begin by briefly outlining the underlying principles of this algorithm. For the sake of clarity, we strictly adhere to the notations, particularly the names of the sets, as introduced in Hare and Mossinghoff (2014).

Let β be an algebraic integer with $|\beta| > 1$, and define the closed disk

$$I'(\beta) = \left\{ z \in \mathbb{C} : |z| \leq \frac{|\beta|}{|\beta| - 1} \right\}.$$

We iteratively build the set of values

$$\mathcal{N}'(\beta, d) = \{F(\beta) : F \in \mathcal{N}, \deg(F) \leq d\} \cap I'(\beta),$$

and define the *total reachable* set as

$$\mathcal{N}'(\beta) = \bigcup_{d \geq 0} \mathcal{N}'(\beta, d).$$

The set $\mathcal{N}'(\beta, d)$ can be computed recursively: starting with $\mathcal{N}'(\beta, 0) = \{1\}$, each subsequent set is generated by the recurrence

$$\begin{aligned} &\mathcal{N}'(\beta, d+1) \\ &= \mathcal{N}'(\beta, d) \cup (\{\beta\omega : \omega \in \mathcal{N}'(\beta, d)\} \cap I'(\beta)) \cup (\{\beta\omega + 1 : \omega \in \mathcal{N}'(\beta, d)\} \cap I'(\beta)). \end{aligned}$$

This construction exploits the fact that for any $F \in \mathcal{N}$, if $F(\beta) \notin I'(\beta)$, then neither $\beta F(\beta)$ nor $\beta F(\beta) + 1$ can lie in $I'(\beta)$.

Given that we are looking for a polynomial that divides no Newman polynomial, there is no point in continuing the construction of $\mathcal{N}'(\beta, d)$ if $0 \in \mathcal{N}'(\beta, d)$ for some d , since this implies that β satisfies a certain Newman polynomial F , and therefore the minimal polynomial of β divides this Newman polynomial F . The case of interest to us arises when, for some d , it turns out that $\mathcal{N}'(\beta, d+1) = \mathcal{N}'(\beta, d)$, and, simultaneously, $0 \notin \mathcal{N}'(\beta, d)$. In this case, it becomes clear that $\mathcal{N}'(\beta)$ is finite and that $0 \notin \mathcal{N}'(\beta)$, so that β is a root of no Newman polynomial, and therefore the minimal polynomial of β divides no Newman polynomial.

In Hare and Mossinghoff (2014), the authors provide a list of monic polynomials with integer coefficients, all of whose roots lie in the slit annulus

$$\{z \in \mathbb{C} : g^{-1} < |z| < g\} \setminus \mathbb{R}^+,$$

and prove, using the algorithmic method, described above, that these polynomials do not divide any Newman polynomial. Among these polynomials, the one with the smallest Mahler measure is the polynomial $x^6 - x^5 - x^3 + x^2 + 1$, whose Mahler measure is approximately 1.556014485. The significance of their result lies in establishing that any constant σ , such that every integer-coefficient polynomial of Mahler measure less than σ and with no nonnegative real root that divides a Newman polynomial, if such a constant exists, must satisfy

$$\sigma \leq M(x^6 - x^5 - x^3 + x^2 + 1) \approx 1.556014485.$$

This bound was improved by P. Drungilas, J. Jankauskas and J. Šiurys (2018), where the authors enlarged the list given in Hare and Mossinghoff (2014) by adding several polynomials of Mahler measure less than 1.556014485, one of which has Mahler measure approximately equal to 1.436632261. Consequently, any such constant σ , if it exists, must be no greater than this Mahler measure.

Our aim is now to further improve the bound established in Drungilas, Jankauskas and Šiurys (2018). Along the way, we obtain a list of certified examples of degrees 8, 9, 10, 11, and 12, with Mahler measure not exceeding 1.556014485; some of these recover previously known examples, while others appear to be new. In particular, one degree-10 polynomial has Mahler measure approximately equal to 1.419404632, thereby improving the bound of Drungilas, Jankauskas and Šiurys (2018).

We proceed in two stages. First, we identify candidate polynomials, which we hope do not divide any Newman polynomial, by rapidly testing, with the aid of the algorithm described in Section 2, all monic polynomials of degrees 6, 7, 8, 9, 10, 11, and 12, whose coefficients lie between -2 and 2 . In view of the high speed of this preliminary screening step, we did not attempt to reduce the amount of computation by exploiting symmetries or other similar reductions. This step took fewer than three hours on a laptop running Ubuntu 24.04.4 LTS, equipped with an Intel(R) Core(TM) Ultra 9 185H processor (16 cores: 6 performance, 8 efficient, and 2 low-power efficient cores; 22 threads) and 64 GB of installed RAM (about 62 GB usable under Linux). In brief, for each tested polynomial, we solve systems of inequalities with integer unknowns in order to test whether the candidate polynomial divides some Newman polynomial of degree at most 200. The algorithm used in this first stage has been fully implemented in the Julia programming language (JULIA 1.11.6). The optimization problem is formulated within the framework of the JuMP modeling language, while the search for a solution that respects the constraints imposed is carried out using the Gurobi library (Gurobi Optimizer 12.0.3) via a wrapper maintained by the JuMP community. Gurobi was run with its default parameters. Each candidate polynomial found at the preselection stage is then verified using the algorithm from Hare and Mossinghoff (2014), and is either retained or discarded according to the outcome of this verification. The preselection step

produced 38 irreducible polynomials, all of whose roots lie in the slit annulus $\{z \in \mathbb{C} : g^{-1} < |z| < g\} \setminus \mathbb{R}^+$, and none of which divides any Newman polynomial up to degree 200. When several polynomials had the same Mahler measure, only one representative was retained. The verification step certified 33 of them; the remaining 5 were left undecided by the computation, since the number of elements in the corresponding set does not stabilize within the maximum allowed depth (set to 200), while the maximum allowed cardinality of the set was fixed at 40,000,000 in order to avoid excessive memory consumption.

After performing all computations and verifications, the resulting polynomials are listed in Table 1, which also specifies, for each polynomial, the chosen root β used in the computation, the stabilization depth that was obtained, and the cardinality of the corresponding set.

Table 1: Certified examples of polynomials with Mahler measure not exceeding 1.556014485 and not dividing a Newman polynomial

Polynomial	Mahler measure	Chosen conjugate β	d_{stab}	$ \mathcal{N}'(\beta) $
$x^{10} - x^8 - x^5 + x + 1$	1.419404632	$1.1307 - 0.1621i$	17	768
$x^9 - x^6 - x^5 - x^4 + x^2 + x + 1$	1.436632261	$1.1403 - 0.1518i$	14	433
$x^{12} - x^{11} - x^6 + x^5 + 1$	1.448290492	$1.1426 - 0.1884i$	15	388
$x^{10} - x^7 - x^6 - x^5 + x^2 + x + 1$	1.475517312	$1.1562 - 0.0996i$	13	232
$x^{10} - x^9 - x^5 + x^4 + 1$	1.475928627	$1.1569 - 0.2252i$	12	190
$x^{11} - x^9 - x^8 + x + 1$	1.477652735	$1.1928 - 0.0507i$	10	82
$x^{10} - x^8 - x^7 + x^5 - x^3 + x + 1$	1.481909009	$1.1576 - 0.1629i$	12	200
$x^9 - x^7 - x^6 + x + 1$	1.483444878	$1.1860 - 0.1451i$	10	85
$x^{12} - x^{11} - x^8 + x^7 + 1$	1.488053700	$1.1620 - 0.1901i$	11	165
$x^8 - x^7 - x^4 + x^3 + 1$	1.489581321	$1.1736 - 0.2801i$	10	95
$x^{12} - x^9 - x^7 - x^5 + x^4 + x^2 + 1$	1.503425675	$1.1164 - 0.1208i$	19	1932
$x^{12} + x^{11} - x^9 - x^8 - x^7 - x^6 - x^5 + x^3 + x^2 + x + 1$	1.504539808	$1.0939 - 0.0894i$	27	15322
$x^{12} - x^{10} - x^7 - x^6 + x^4 + x + 1$	1.505811890	$1.1615 - 0.0544i$	13	196
$x^{12} - x^9 - x^7 - x^5 + x^2 + x + 1$	1.515809085	$1.1305 - 0.0819i$	17	753
$x^8 - x^6 - x^5 + x + 1$	1.518690904	$1.1765 - 0.1929i$	10	102
$x^{10} - x^9 - x^7 + x^6 + x^4 - x^3 + 1$	1.519065984	$1.1666 - 0.2719i$	11	120
$x^{12} - x^{10} - x^7 + x + 1$	1.519151476	$1.1402 - 0.1065i$	15	451
$x^{12} - x^8 - x^7 - x^6 - x^5 + x^3 + x^2 + x + 1$	1.520043072	$1.1339 - 0.0516i$	16	637
$x^{12} - x^{10} - x^9 + x^7 + 1$	1.521929346	$1.1381 - 0.1765i$	15	491
$x^{12} - x^{11} - x^{10} + x^9 + x^8 - x^6 + x^4 + x^3 - x^2 - x + 1$	1.522096479	$1.1440 - 0.4619i$	17	320
$x^{12} - 2x^{11} + x^{10} + x^9 - x^8 + x^7 - x^6 + x^5 - x^4 + x^3 + x^2 - 2x + 1$	1.526433287	$1.0877 - 0.5859i$	28	1325
$x^{12} - x^{10} - x^7 + x^3 + 1$	1.531401303	$1.1328 - 0.1357i$	16	655
$x^{11} - x^9 - x^8 + x^6 + x^5 - x^4 - x^3 + x + 1$	1.534099819	$1.1333 - 0.1744i$	17	664
$x^{11} - x^8 - x^6 + x + 1$	1.534872750	$1.1048 - 0.1448i$	53	89226
$x^8 - x^6 - x^5 - x^4 + x^2 + x + 1$	1.536566472	$1.2220 - 0.1122i$	8	41
$x^9 - x^8 - x^6 + x^5 + 1$	1.536913983	$1.1920 - 0.2548i$	8	60

Polynomial	Mahler measure	Chosen conjugate β	d_{stab}	$ \mathcal{N}'(\beta) $
$x^{11} - x^{10} - x^9 + x^8 + x^7 - x^6 - x^5 + x^4 + x^3 - x^2 + 1$	1.537810676	$1.1679 - 0.2533i$	11	123
$x^{11} - x^7 - x^6 + x^3 + x^2 + x + 1$	1.539383026	$1.1336 - 0.2531i$	33	5293
$x^{11} - x^{10} + x^3 + 1$	1.542629108	$1.1786 - 0.3131i$	10	75
$x^{12} - x^{11} - x^9 + x^7 + x^5 - x^4 + 1$	1.547469802	$1.2124 - 0.1812i$	8	45
$x^9 - x^7 - x^6 + x^4 + 1$	1.550687063	$1.1423 - 0.2286i$	22	454
$x^{11} - x^{10} - x^5 + x^4 + 1$	1.553591690	$1.1450 - 0.2051i$	15	348
$x^6 - x^5 - x^3 + x^2 + 1$	1.556014485	$1.1910 - 0.3710i$	8	44

The certification procedure used to prove that a polynomial divides no Newman polynomial was implemented independently in two different environments: first in Julia, and then in GP/PARI (version 2.15.4). In the Julia implementation, polynomial manipulations were carried out using the package Polynomials.jl, while irreducibility tests were performed with Nemo.jl. In GP/PARI, the corresponding computations were reproduced using its native polynomial routines. The GP/PARI implementation is slower than the Julia version, but it provided an independent cross-check of the entries, which are reported in Table 1. We note that this table is not certified to be exhaustive, since 5 polynomials were left undecided by the certification procedure.

In the following section, we investigate Newman polynomials from a different perspective, focusing on their ability to have repeated zeros outside the unit disk. At the same time, this second application aims to highlight the flexibility of the MILP framework developed in this article by showing that it can also be adapted to the construction of Newman polynomials divisible by prescribed repeated noncyclotomic factors.

4. Newman polynomials with repeated zeros

4.1. State of the art

In the article by P. Borwein and T. Erdélyi (1996), the authors recall that, after observing through numerical computations that no Newman polynomial of degree ≤ 25 has a repeated root of modulus strictly greater than 1, A. Odlyzko raised the question of whether every Newman polynomial necessarily has its repeated roots either at 0 or on the unit circle. This question was resolved in the paper by M. J. Mossinghoff (2003).

Following Mossinghoff's notation, we set

$$l(x) = x^{10} - x^9 + x^7 - x^6 + x^5 - x^4 + x^3 - x + 1,$$

which is obtained from Lehmer's polynomial by the substitution $x \mapsto -x$. We

then define $g(x) = l(x)^2$. Mossinghoff carried out an exhaustive search up to high degrees in order to identify Newman polynomials, which are divisible by $l(x)^2$, the interest being that $l(x)$ has a real root of modulus strictly greater than 1. In the first part of his article, he examined all Newman polynomials up to degree 60, discovering two suitable examples, one of degree 59 and one of degree 60. He then extended his search by focusing on reciprocal Newman polynomials up to degree 100, leading to the discovery of 21 additional Newman polynomials, which are divisible by $l(x)^2$. Six further examples were obtained using alternative choices for $g(x)$, the largest degree among these being 105.

Mossinghoff's method, which is highly efficient, consists in encoding each Newman polynomial f as an integer $w = f(2)$. Prior to performing any polynomial division, a large number of candidates are discarded by imposing the necessary congruence conditions

$$f(2) \equiv 0 \pmod{g(2)} \quad \text{and} \quad f(-2) \equiv 0 \pmod{g(-2)}.$$

This substantial arithmetic filtering reduces the search space to a small subset of cases, for which exact polynomial divisibility then needs to be tested.

An alternative approach, if one wishes to avoid exhaustive searches—which remain time-consuming even when the number of computations is reduced—is to apply our integer optimization method by treating the polynomial $l(x)^2$ in the same manner as we did for all polynomials in the list *Known180*. In other words, it suffices to search for a polynomial $q(x)$ with integer coefficients such that $q(x)l(x)^2$ is a Newman polynomial. As in Section 2.1, this immediately reduces to solving a system of integer inequalities.

To carry out these computations efficiently, we opted for an accelerated alternative version of our algorithm, in which the system of inequalities is replaced by a small number of linear equations. In practice, this approach allowed us to handle high degrees directly (we rapidly reached degree 150) while also removing the restriction to reciprocal Newman polynomials. The presentation of this method is the subject of the following section.

4.2. Algorithm for cancelling the remainder of the Euclidean division

4.2.1. Outline of the algorithm

With the notation, introduced in Section 4.1, we consider the polynomial

$$l(x) = x^{10} - x^9 + x^7 - x^6 + x^5 - x^4 + x^3 - x + 1,$$

and we set $g(x) = l(x)^2$. For an integer $d \geq \deg(g)$, we seek a Newman polynomial of degree d , that is, a polynomial of the form

$$N(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + 1, \quad a_1, \dots, a_{d-1} \in \{0, 1\},$$

which is a multiple of $g(x)$.

The Euclidean division of $N(x)$ by $g(x)$ can be written as

$$N(x) = g(x)H(x) + R(x), \quad \text{with } \deg(R) < \deg(g).$$

Clearly,

$$g(x) \mid N(x) \quad \text{is equivalent to} \quad R(x) = 0.$$

For a fixed degree d , the coefficients of the remainder $R(x)$ are simple linear expressions in the binary variables $a_1, \dots, a_{d-1}$, with the leading and constant coefficients of $N(x)$ fixed equal to 1. Consequently, the divisibility condition $g(x) \mid N(x)$ reduces to the simultaneous vanishing of these linear expressions. As a result, the search for a multiple of $l(x)^2$ among Newman polynomials of degree d amounts to solving a system of linear equations consisting of $\deg(g)$ equations in the $d - 1$ binary unknowns $a_1, \dots, a_{d-1}$.

To solve this system, we employ an algorithm of the same nature as that used in Section 2.1, with the difference that Section 2.1 deals with a system of linear inequalities, whereas the present setting involves a system of linear equations. In both cases, the core of the method relies on a mixed-integer linear programming (MILP) solver based on the branch-and-cut technique. These computations were carried out in the same software and hardware environment as described in Section 3, again using Gurobi with its default parameters.

4.2.2. Computational results and comments

We have chosen to present one polynomial per degree (whenever a suitable polynomial exists for the degree under consideration), up to degree 150. As in Mossinghoff (2003), Newman polynomials are represented in Table 2 using hexadecimal encoding. Since a Newman polynomial has coefficients in $\{0, 1\}$, it may be identified with the binary word formed by its coefficients, read from the highest-degree term down to the constant term. Equivalently, this binary word is the base-2 expansion of the integer $f(2)$. The hexadecimal notation used in Table 2 is therefore nothing more than a compact representation of this binary encoding, with each hexadecimal digit encoding four consecutive binary digits. We recover several polynomials that already appear in Table 1 of Mossinghoff (2003), but new polynomials also arise well before exceeding degree 100, which is where Table 1 of Mossinghoff (2003) ends, since our polynomials are not

required to be reciprocal. Thus, for degree 93, we provide the non-reciprocal polynomial

$$\begin{aligned} & x^{93} + x^{92} + x^{86} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} \\ & + x^{69} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{48} \\ & + x^{47} + x^{45} + x^{42} + x^{40} + x^{39} + x^{37} + x^{34} + x^{32} + x^{28} + x^{27} + x^{25} \\ & + x^{24} + x^{22} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12} + x^{11} + x^8 + x^5 + x + 1 \end{aligned}$$

which is indeed a multiple of $l(x)^2$.

Table 2: Newman polynomials divisible by $l(x)^2$, encoded in hexadecimal representation

deg(f)	Hexadecimal encoding of f
59	C49E23C93C47923
60	1B33F1364D91F99B
79	C42E67FE42427FE67423
84	1BF66C43EC4446F846CDB
85	37EDB84FDB61B6FC876DFB
90	639D99136318C63644CDCE3
92	1A59E126B6D4A56DAC90F34B
93	305B8BE5BAB9A5A51B497923
94	694A44C6EF9B7A006117F99B
95	C49E22400624F1364D91F99B
96	182345F5AD226C896B5F45883
97	3722F1D73627FF91B3AE3D13B
99	DFB6E49A554BFFE5EFAE3D13B
100	1BC3DC3ED48DFBF6256F87787B
101	34A5C8AF55736DB3AABD44E94B
102	691F21E23252CA266365057ADB
103	DCBB435B62E0BD0746DAC2DD3B
104	1B27583B6960DA5F53B9982DD3B
105	3787883F743027DA9D6EACA5DCB
106	6217333EE5D5EABD5D3BE667423
107	C49E225B33E55E6B60AC954F99B
108	1A52CC31D2106C65C0A298414923
109	3667E26C9B23FF7FE23C93C47923
110	62495296F6CD2243B8C1DB5F5883
111	DB5EAOA6DEF59751FF6DD407FB7B
112	1885CC23413B3AB8CD03B39E95323
113	3667EF3AF08B1450C892ACDDC7923
114	621730B5A7FFBD71616397953FB7B
115	C10F28AC0AC78958180C6DC97BDFB
116	1BF1C8850D11BFDA96DEB370352123
117	31212827E3D4D487E93F27FFE67423
118	60EF5650D1DE83A1E9D48981463723

$\deg(f)$	Hexadecimal encoding of f
119	C4FAD7BA97C2CFD6B94237A41AD7FB
120	18995E6FE1AA04687E78537E4B41F9B
121	37EF7A677A9AB5E1D59EEC9B8C7DE7B
122	60B7736745938332D21C93A79D37923
123	C7CE2C0A5A59DD516336EB1318D7CE3
125	30468A627162BA07CCECF121B0A75323
126	6CCFF45B93FFFE6675739863EA735323
127	C73EB19A97352444B67C10D1D4C556A9
128	189F5AF154AAD290602D01EE4DC52F723
129	342AD5C57853DAA3E09EDFD732A54E94B
130	690766AEDA956835DF11B2276465572B3
131	C49ECA4E15B08C277234B3BA5C8F77F9B
132	1A5FE9F11B73F71F0AF05B5EBC6141F99B
133	366648BB4F195FCFF9B85E4C71A07787B
134	69D7AED27383B4060B55FD28E957C5FDAF
135	F5A72ACE882379E14F5AD706C1904C1923
136	1A746E02E27973B9E2480F7867E7BB235CB
137	310BFBD1989D599740B3B4917675C67FE3
138	694B24458A39ABB9D401777F80CF5E6694B
139	F77EABAFE546557B39FD2FE643A57D572B3
140	1BDBF94508CC7F61534EE99497E25C52F723
141	319396AAA2ACEFA97B570D49D3711992FF4B
142	6905A4AE7133EA8ECAD9EB9EF97FA7C47923
143	D23F21F4DD494583F0CE0456FC0701F97923
144	1885C9D6A84A3F250955F671BDACEAD727423
145	34BF60CF7EEF23F169BC3C0DEA8515D0FA5FB
146	6E5CC38A712EDCA345F90B3F1ED96FDD37683
147	C49E87DABEB95ED9DC8182996C4E23ED5F923
148	189684517CAAD4D9863DFA9CA16CDCFC03F1FB
149	36428A55D5405C555FA57D0AD1ED6B2011294B
150	624C6D4B4351EE0A34728E6B7B6056085064E3

The Newman polynomials obtained are available both in hexadecimal representation and as lists of coefficients in the files `resulthexa.txt` and `resultcoeffs.txt`, which can be found at Idris and Sac-Épée (2026). For example, two entries of Table 2, namely

18995E6FE1AA04687E78537E4B41F9B

and

624C6D4B4351EE0A34728E6B7B6056085064E3,

correspond to the following Newman polynomials:

$$\begin{aligned}
& x^{120} + x^{119} + x^{115} + x^{112} + x^{111} + x^{108} + x^{106} + x^{104} + x^{103} + x^{102} \\
& + x^{101} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{84} \\
& + x^{83} + x^{81} + x^{79} + x^{77} + x^{70} + x^{66} + x^{65} + x^{63} + x^{58} + x^{57} + x^{56} \\
& + x^{55} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{42} + x^{40} + x^{37} + x^{36} \\
& + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{23} + x^{21} + x^{20} + x^{18} \\
& + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^4 + x^3 + x + 1,
\end{aligned}$$

and

$$\begin{aligned}
& x^{150} + x^{149} + x^{145} + x^{142} + x^{139} + x^{138} + x^{134} + x^{133} + x^{131} + x^{130} \\
& + x^{128} + x^{126} + x^{123} + x^{121} + x^{120} + x^{118} + x^{113} + x^{112} + x^{110} + x^{108} \\
& + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{91} + x^{89} + x^{85} + x^{84} \\
& + x^{82} + x^{78} + x^{77} + x^{76} + x^{73} + x^{71} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{59} \\
& + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{38} + x^{36} \\
& + x^{34} + x^{33} + x^{27} + x^{22} + x^{20} + x^{14} + x^{13} + x^{10} + x^7 + x^6 + x^5 + x + 1.
\end{aligned}$$

For an independent cross-check, we also verified using GP/PARI that these polynomials are indeed Newman multiples of $l(x)^2$.

A natural next step after obtaining these multiples of $l(x)^2$ is to apply the same method in order to search for multiples of $l(x)^3$ within the set $\mathcal{N}$ of Newman polynomials. M. J. Mossinghoff (2003) verified that $l(x)^3$ does not divide any Newman polynomial of degree less than 120. Using our method, we verify that $l(x)^3$ does not divide any Newman polynomial of degree at most 160.

5. Conclusion

The results obtained in this work naturally lead to several open directions for further investigation. In particular, the existence of Newman multiples for the three unresolved polynomials of Mahler measure less than 1.3 is still unknown, and it remains unclear whether a universal constant, governing the existence of Newman multiples for integer-coefficient polynomials according to their Mahler measure exists. From a computational perspective, it is likely that some of the bounds and constructions obtained in this work could be further improved by testing alternative optimization solvers or software environments beyond the one employed here. More generally, the mixed-integer linear programming formulations developed in this article appear very promising and should provide a flexible framework for addressing a wider range of problems involving

polynomials with restricted coefficients, including questions related to Borwein, Littlewood and Newman polynomials.

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