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To the memory of

Alexei Alexeevich MILYUTIN

Alexei Alexeevich Milyutin, an outstanding Russian mathematician, died a sudden death at the age of 75 on April 20, 2001. A bright researcher, A.A. Milyutin spent his whole life as an ardent servant of science. The lust for the unknown, the unrivaled will, knowledge and the God-given gift helped him be on the move all the time, heading farther and farther towards new areas in the field of mathematics which became a part of his life—extremum theory, and, particularly, optimal control. He never stopped his intellectual toil until the very last moment in life and died of a heart attack as he was giving a talk at a seminar on optimal control which he had unfailingly supervised for more than 30 years.

A.A. Milyutin was born in Moscow on July 27, 1925. In 1948, he graduated from the Mechanics and Mathematics Department of the Moscow State University and became a post-graduate student at the same department. The subject he chose for his candidate (Ph.D.) dissertation arose from the question discussed in the university corridors, whether the spaces of continuous functions on a segment and on a square are linearly isomorphic or not. A.A. Milyutin gave a positive answer to this question in 1951. He did not know that he had actually solved a problem of Stefan Banach. The dissertation was successfully defended in the same year, with I.M. Gelfand and L.A. Lyusternik as opponents. However, the result was left unpublished and, for another 15 years, unknown to many mathematicians who kept trying to answer the same question. The issue was raised again at the ICM in Moscow in 1966, during a talk by A. Pełczyński. Luckily, the manuscript of the dissertation was not lost, and the result was reported at a congress. Later, due to the efforts of mathematicians from Kharkov

(Ukraine) and Poland, it was published in its entirety (in *Theory of functions, functional analysis, and applications*, Kharkov, 1966, no. 2).

In 1954, A.A. Milyutin, together with other graduates of the Mech-Math MSU, joined a computational group that worked with Academician L.D. Landau at the Institute of Physical Problems, USSR Academy of Sciences, and then he worked for several years on numerical solutions of applied problems at the Institute of Chemical Physics. However, his basic mathematical interests still lay in theoretical areas.

The Pontryagin's Maximum Principle (MP) obtained in the late 1950s determined the life of many mathematicians, including A.A. Milyutin and his colleague and friend A.Ya. Dubovitskiĭ. In 1965, they published their seminal paper, "Extremum problems in the presence of constraints" (*USSR J. Comput. Math. Math. Phys.*, **5**, 3), which quickly gained wide popularity and became a true milestone both for the authors and for a number of other researchers, due to exceptional clarity, simplicity and efficiency of the ideas (the so-called Dubovitskiĭ–Milyutin scheme). Those ideas made it immediately possible to extend the MP to the new classes of problems including problems with state constraints.

The first account of the research on the MP was done by A.A. Milyutin in his proper doctorate (D.Sc.), brilliantly defended at the Institute of Applied Mathematics in 1966. Alongside the main results, the dissertation contained an example of an extremal reaching the boundary of a state constraint and having a countable number of junction points with the boundary. Later, a similar example was independently found by H. Robbins.

In a series of their joint projects in the late 1960s and 1970s, A.Ya. Dubovitskiĭ and A.A. Milyutin developed a theory of the MP for problems with regular and non-regular mixed constraints. Their remarkable achievement was a "local MP" for non-regular mixed constraints, published in the book *Necessary conditions of extremum in a general optimal control problem* (Nauka, Moscow, 1971, in Russian).

Further efforts of the authors were aimed at obtaining an "integral MP" for problems with non-regular mixed constraints. This was the subject of A.Ya. Dubovitskiĭ's doctorate. Later A.A. Milyutin found a new form of MP conditions for the general problem that reflects their non-uniqueness and hierarchy, and he also found new methods of obtaining them. These results are presented in A.A. Milyutin's book *The maximum principle for the general optimal control problem* (Fizmatlit, Moscow, to appear this year; in Russian).

Apart from research, Alexei Alexeevich Milyutin lectured and conducted seminars at the Mech-Math MSU from the late 1960s to the mid 1970s. Here, at his joint seminar with E.S. Levitin, A.A. Milyutin started an intensive research on the theory of higher order conditions. He raised the issue of obtaining second order necessary conditions in optimal control problems that would be as closely connected with sufficient conditions as in problems of finite-dimensional calculus and the classical calculus of variations. This research resulted in an abstract

theory of higher order conditions for problems with constraints, published in a joint paper of E.S. Levitin, A.A. Milyutin and N.P. Osmolovskiĭ (*Russian Math. Surveys*, **33**, 6, 1978). The abstract theory offered radically new ways to obtaining higher-order conditions in optimal control problems and made it possible to develop a complete theory of quadratic conditions both for non-singular (N.P. Osmolovskiĭ) and singular (A.V. Dmitruk) extremals.

In the late 1970s, A.A. Milyutin proved a remarkable "theorem on finite codimensions" (in *Metody teorii ekstremal'nykh zadach v ekonomike*, Nauka, Moscow, 1981, in Russian), that clarified the essence of a number of results obtained by other mathematicians (A.A. Agrachev, R.V. Gamkrelidze, A. Krener, et al.) on higher order necessary conditions for singular optimal regimes.

At that time, researchers working in the extremum theory, like A.D. Ioffe, V.M. Tikhomirov, V.F. Sukhinin and others discovered new forms of the key theorem in that theory, the Lyusternik theorem on a tangent subspace. A.A. Milyutin proposed his version of this theorem, interpreting it as a "theorem on covering". Other interpretations followed. A review of the results was published by A.V. Dmitruk, A.A. Milyutin and N.P. Osmolovskiĭ in *Russian Math. Surveys*, **35**, 6, 1980. However, the theorem on covering turned out to be the simplest and clearest in formulation and at the same time quite efficient, which makes it increasingly popular.

In the mid 1980s, A.A. Milyutin got increasingly engaged in transforming the known conditions of extremum into a working tool to investigate optimal control problems rather than in obtaining new conditions of extremum. Hence his theorems on the absence of jumps and singular components in the measures—Lagrange multipliers for the state constraints in the conditions of MP (published in the book *Necessary condition in optimal control*, Nauka, Moscow, 1990; in Russian). Alexei Alexeevich himself conducted an active investigation of new phenomena in optimal control and other fields of mathematics by using the MP tools and promoted the application of these tools in every possible way.

Using the MP, he studied singularities of extremals reaching and leaving the boundary of state constraint (see his joint book with V.V. Dikusar, *Qualitative and numerical methods in the maximum principle*, Nauka, Moscow, 1989; in Russian). Later, A.A. Milyutin obtained quite general conditions under which the reaching occurs through a countable number of contacts with the boundary. These results made up his last book, almost completed.

At the same time, A.A. Milyutin used the MP to study the singularities of extremals in their passage from a non-singular to a singular regime (the junction phenomenon), the corresponding funnels of extremals' non-uniqueness, and conditions for the presence of the so-called chattering phenomenon, or, more generally, of a second kind discontinuity in control. A major contribution to these studies was made by S.V. Chukanov. The results are published in the *Russian J. Math. Phys.*, **2**, 1, 1994, and also in a joint monograph by A.A. Milyutin, A.E. Ilyutovich, N.P. Osmolovskiĭ, and S.V. Chukanov, *Optimal control in linear systems* (Nauka, Moscow, 1993; in Russian).

Using the quadratic conditions, obtained by A.V. Dmitruk and himself, A.A. Milyutin studied the concept of rigidity of trajectories of control systems, making rather an unexpected transition from the given system to a degenerate optimal control problem. This helped him obtain characteristics of a set of quadratically rigid trajectories in contact structures for the space of arbitrary dimension (*Russian J. Math. Phys.*, **4**, 2, 1998, and *Trans. Moscow Math. Soc.*, **60**, 1999). A.A. Milyutin constructed a surprising example of a quadratically rigid trajectory which has a non-unique collection of normalized Lagrange multipliers, and for each collection, the corresponding second variation is not even nonnegative on the subspace of critical variations (*Russian J. Math. Phys.*, **7**, 1, 2000).

Also, A.A. Milyutin applied the quadratic conditions to the study of abnormal geodesics in sub-Riemannian metrics. He proved that in the C^1-topology every abnormal, locally quadratically rigid geodesic is a limit of a sequence of normal extremals of two types: sequences where the distance between all pairs of conjugate points on these extremals tends to zero, and sequences where these distances are bounded from zero by a positive constant (to appear in *Trans. Moscow Math. Soc.*).

Questions arising in the quadratic theory of singular extremals encouraged the investigation of the problem of approximating an arbitrary vector field in a finite-dimensional space by gradient fields. A.A. Milyutin found a duality formula that connects a normalized circulation of the given vector field with the distance (in some metric) from this field to the set of gradient fields (*Russian J. Math. Phys.*, **3**, 1, 1995).

At another time A.A. Milyutin addressed the duality theory in connection with the known mass transfer problem of Monge–Kantorovich. He obtained, jointly with V.L. Levin, a complete description of the cost functions on a given compact set that provide equality between the optimal values in the primal and dual problems for all right hand sides. They developed an essentially new approach to the general duality problem for infinite-dimensional problems of linear programming in the mass setting (*Russian Math. Surveys*, **34**, 3, 1979).

Together with V.L. Bodneva, A.A. Milyutin obtained new and interesting results on the mathematical theory of vibrations. They proposed a generalization of the Krylov–Bogolyubov asymptotic method that allows to write recurrent formulas for the terms of asymptotic expansion. This approach cannot be reduced to the known methods of averaging (*Russian Math. Surveys*, **42**, 3, 1987, and *Russian J. Math. Phys.*, **5**, 2, 1997).

Together with N.P. Osmolovskiĭ, A.A. Milyutin studied the interrelation of ideas from the calculus of variations and optimal control. By analogy with the theory of weak minimum, which is the basis of the classical calculus of variations, they developed a theory of the so-called Pontryagin minimum which is characteristic to optimal control. This theory is presented in their book *Calculus of Variations and Optimal Control* (Amer. Math. Soc., 1998).

After the trip to Israel in 1998 for the conference dedicated to the 300th anniversary of the calculus of variations, A.A. Milyutin's attention was attracted by the results from the theory of MP for differential inclusions. He found finer forms for the necessary optimality condition than those known earlier and showed that the assumption of Lipschitz continuity of the inclusion is essential (to appear in *Mat. Sbornik*).

The last studies of A.A. Milyutin were devoted to integral quadratic functionals on an infinite time interval.

Alexei Alexeevich Milyutin was a doubtless leader in the field of optimal control, who influenced many people, including us, his students. He also was an outstanding personality who lived an exceptionally honest and uncompromising life. His sudden death was absolutely unexpected to all people who knew him—his students, colleagues, and relatives. A cherished memory of Alexei Alexeevich Milyutin will stay long in our hearts.

A.V. Dmitruk, N.P. Osmolovski