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On an application of an interval backward finite difference method for solving the one-dimensional heat conduction problem*

by

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Abstract: The paper concerns the interval method for solving the one-dimensional heat conduction problem. It is based on the conventional backward finite difference scheme with the appropriate local truncation error terms that are also taken into account. For the theoretical formulation of the interval approach we can show that the exact solution is included in the interval one. In practice, there are problems, for which we cannot determine the endpoints of the error term intervals exactly. Nevertheless, if we use the appropriate approximation, related to the endpoints considered, then the numerical experiments confirm that the interval solution includes the exact one.

Keywords: interval methods, interval arithmetic, finite difference methods, heat conduction problem

1. Introduction

As we know, when performing numerical calculations, considerable attention should be devoted to the accuracy of the results obtained. The main sources of errors are related to the initial data inaccuracy, the floating-point arithmetic, and the approximate numerical methods used. Fortunately, there are a few different approaches that enable taking into account a given number together with a related inaccuracy, and hence provide some information about the possible deviation of the result obtained from a desired exact value when the computations

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are finished. The techniques to be considered in this context are as follows: the interval arithmetic (Moore et al., 2009; Sunaga, 1958; Kulisch, 2013), the fuzzy arithmetic (Burczynski and Skrzypczyk, 1997; Kosinski, 2004) and the incremental arithmetic (Borawski, 2012). The paper concerns the first approach, i.e. the interval arithmetic and the interval methods (see also Hammer et al., 1993; Jaulin et al., 2001; Marciniak, 2009; Gajda et al., 2008) with a special attention directed to the interval methods used for solving some heat conduction problem.

The interval backward finite difference method proposed is based on the appropriate conventional scheme with its local truncation error term included. It is developed for the heat conduction problem (see Section 2.1) with the initial condition and the Dirichlet boundary conditions. For its theoretical formulation it can be shown that the interval solution obtained includes the exact one (see Section 2.2). Then, in Section 2.3, we propose a method of approximation of the endpoints of the error term intervals. Such approach is required in the case of problems, for which the exact determination of the endpoints considered is not possible. Note that if we use this approximation, then we cannot guarantee that the exact solution belongs to the interval one. In spite of this, the numerical experiments presented in Section 3 show that such inclusion occurs. Note that the interval methods based on some finite differences have been previously formulated for the heat conduction problem with the Dirichlet and mixed boundary conditions (Marciniak, 2012; Jankowska and Sypniewska-Kaminska, 2013) and also for the problems described by the diffusion, Poisson and wave equations (Hoffmann and Marciniak, 2013; Jankowska, 2012; Nakao, 2001; Nakao et al., 2013; Szyszka, 2012, 2015).

The authors directed the attention towards the interval methods based on finite differences and their implementation in the interval floating-point arithmetic due to several essential reasons. As we know, the conventional numerical methods used for solving the initial-boundary value problems, do not allow us to take into account the uncertain values of parameters present in a problem formulation. Such a necessity is particularly important when we address some bioheat transfer problems (e.g. in soft tissues, with particular reference to skin, see Majchrzak et al., 2011; Mochnacki and Piasecka-Belkhayat, 2013). The interval arithmetic applied allows us to represent some parameters, such as thermophysical properties of skin and blood, in the form of intervals. It is an important feature of the interval methods, given that we usually know a range of values that can be taken by parameters in the effect of influence of some environmental factors, i.e. age, state of health, lifestyle etc. Furthermore, the interval solutions obtained include also the error of the conventional method and the errors caused by the floating-point arithmetic used by computers, i.e. the rounding errors and the representation errors. An application of the interval finite difference methods to such a class of problems can be found in, e.g., Mochnacki and Piasecka-Belkhayat (2013), Jankowska and Sypniewska-Kaminska (2012), Jankowska (2014). On the other hand, the main reason for adopting the finite difference approach is the fact that we know the formula of the local truncation error term. As we mentioned, the exact values of the endpoints of the error term intervals are usually difficult or even impossible to determine. Nevertheless, we can apply a method of their approximation and, as many experiments confirm, the exact solution belongs to the interval solution obtained.

2. Interval backward finite difference method for solving the heat conduction problem

2.1. Heat conduction problem

Consider the one-dimensional heat conduction problem given by the governing equation

$$\frac{\partial u}{\partial t}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < L,\ t > 0, \tag{1}$$

subject to the initial condition and the Dirichlet boundary conditions

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{2}$$

$$u(0,t) = \varphi_1(t), \quad u(L,t) = \varphi_2(t), \quad t > 0. \tag{3}$$

The heat conduction problem (1)-(3) concerns the distribution of heat along an isotropic rod of length L (an isotropic infinite plate of thickness L) over time. A function $u = u(x,t)$ describes the temperature at a given location x and time t. We assume that the temperature within each cross-sectional element of the rod is uniform. Moreover, the rod is perfectly insulated on its lateral surface. The constant $\alpha = \sqrt{\kappa}$ is a material-specific quantity. It depends on the thermal diffusivity $\kappa = \lambda/(c\rho)$, where λ is the thermal conductivity, c is the specific heat, and ρ is the mass density of the body. It is assumed that λ, c, and ρ are independent of the position x in the rod.

2.2. Conventional and interval backward finite difference methods

Now we establish a grid on the domain. First, we set the maximum time $T_{\max}$. Then, we choose two integers, n, m, and we find the mesh constants h, k such that $h = L/n$ and $k = T_{\max}/m$. Hence, the grid points are (x_i, t_j), where $x_i = ih$ for $i = 0, 1, \ldots, n$ and $t_j = jk$ for $j = 0, 1, \ldots, m$.

Subsequently, we use the backward finite difference formula for $\partial u/\partial t\,(x_i, t_j)$ and the central finite difference formula for $\partial^2 u/\partial x^2\,(x_i, t_j)$, together with the appropriate local truncation errors, i.e.

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_j) - u(x_i, t_{j-1})}{k} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j), \tag{4}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j). \tag{5}$$

Hence, if we substitute (4)-(5) to the equation (1), expressed at the grid points (x_i, t_j), we obtain

$$\begin{aligned} &(1+2\lambda)\,u(x_i,t_j) - \lambda u(x_{i-1},t_j) - \lambda u(x_{i+1},t_j) = u(x_i,t_{j-1}) \\ &\quad -\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_j) - \alpha^2 \frac{kh^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j), \\ &i = 1,2,\ldots,n-1,\ j = 1,2,\ldots,m, \end{aligned} \tag{6}$$

where $\lambda = \alpha^2 \left(k/h^2\right)$, $\eta_j \in (t_{j-1}, t_j)$, $\xi_i \in (x_{i-1}, x_{i+1})$. Finally, for the initial and boundary conditions (2)-(3), expressed at the grid points (x_i, t_j), we have

$$u(x_i, 0) = f(x_i), \quad i = 0,1,\ldots,n, \tag{7}$$

$$u(0,t_j) = \varphi_1(t_j), \ \ u(L,t_j) = \varphi_2(t_j), \quad j = 1,2,\ldots,m. \tag{8}$$

For the formulation of the interval couterpart of the conventional backward finite difference method here considered, we transform the exact formula (6) with (7)-(8) into the appropriate separate forms, according to the position in the grid. We have

$$\begin{aligned} &(1+2\lambda)\,u(x_1,t_j) - \lambda u(x_2,t_j) = \lambda u(x_0,t_j) + u(x_1,t_{j-1}) + \widehat{R}_{1,j}, \\ &\qquad i = 1,\ j = 1,2,\ldots,m, \end{aligned} \tag{9}$$

$$\begin{aligned} &(1+2\lambda)\,u(x_i,t_j) - \lambda u(x_{i-1},t_j) - \lambda u(x_{i+1},t_j) = u(x_i,t_{j-1}) + \widehat{R}_{i,j}, \\ &\qquad i = 2,3,\ldots,n-2,\ j = 1,2,\ldots,m, \end{aligned} \tag{10}$$

$$\begin{aligned} &(1+2\lambda)\,u(x_{n-1},t_j) - \lambda u(x_{n-2},t_j) = \lambda u(x_n,t_j) + u(x_{n-1},t_{j-1}) + \widehat{R}_{n-1,j}, \\ &\qquad i = n-1,\ j = 1,2,\ldots,m, \end{aligned} \tag{11}$$

where

$$\widehat{R}_{i,j} = -\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_j) - \alpha^2 \frac{kh^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j). \tag{12}$$

Note that the formulas (9)-(11) with (12) can be transformed to the following matrix representation

$$Cu^{(j)} = u^{(j-1)} + \widehat{E}_C^{(j)} + \widehat{E}_L^{(j)}, \quad j = 1,2,\ldots,m, \tag{13}$$

where

$$\begin{aligned} u^{(j)} &= \left[u(x_1,t_j), u(x_2,t_j), \ldots, u(x_{n-1},t_j)\right]^{\mathrm{T}}, \\ \widehat{E}_C^{(j)} &= [\lambda u(x_0,t_j), 0, \ldots, 0, \lambda u(x_n,t_j)]^{\mathrm{T}}, \\ \widehat{E}_L^{(j)} &= \left[\widehat{R}_{1,j}, \widehat{R}_{2,j}, \ldots, \widehat{R}_{n-1,j}\right]^{\mathrm{T}}, \end{aligned} \tag{14}$$

$$C = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & \vdots & 0 & 0 & 0 \\ -\lambda & 1+2\lambda & -\lambda & \vdots & 0 & 0 & 0 \\ 0 & -\lambda & 1+2\lambda & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 1+2\lambda & -\lambda & 0 \\ 0 & 0 & 0 & \vdots & -\lambda & 1+2\lambda & -\lambda \\ 0 & 0 & 0 & \vdots & 0 & -\lambda & 1+2\lambda \end{bmatrix}. \tag{15}$$

Note that $\dim C = (n-1) \times (n-1)$ and $\dim u^{(j)} = \dim \widehat{E}_C^{(j)} = \dim \widehat{E}_L(j) = (n-1) \times 1$. The matrix C is tridiagonal and symmetric. It is also positive definite and strictly diagonally dominant, due to the fact that $\lambda > 0$. The vectors of coefficients $\widehat{E}_C{}^{(j)}$, $j = 1, 2, \ldots, m$, in the formulas (14) depend on the stepsizes h, k, the parameter α, and the values of the functions φ_1, φ_2. They are different for each $j = 1, 2, \ldots, m$. On the other hand, the vectors $\widehat{E}_L{}^{(j)}$, $j = 1, 2, \ldots, m$, depend on the stepsizes h, k, and the values of the appropriate derivatives of u at the midpoints considered. What is most important, the components of $\widehat{E}_L{}^{(j)}$ represent the local truncation error terms of the conventional finite difference method at each mesh point.

Remark 1 *Consider the exact formulas (9)-(11) with (12) and the corresponding matrix representation (13) with (14)-(15). Let $u_{i,j}$ approximate $u(x_i, t_j)$. If we also neglect the error terms $\widehat{R}_{i,j}$, given in the equations (9)-(11) and in the components of the vectors $\widehat{E}_L{}^{(j)}$, then we get the conventional backward finite difference method (see also Anderson et al., 1984; Press et al., 2007) with the local truncation error $O(h^2 + k)$.*

Subsequently, we propose an interval backward finite difference method. It is formulated on the basis of the equations (9)-(11) with (12), or the appropriate matrix representation (13) with (14)-(15). Before that, we introduce some assumptions about the values of the derivatives of u at some midpoints considered. Hence, for the interval approach, we suppose that there exist the intervals $S_{i,j}$, $Q_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$, such that the following relations hold:

$$\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j) \in S_{i,j}, \quad \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j) \in Q_{i,j}. \tag{16}$$

Hence, by applying (16) to (12), we have that $\widehat{R}_{i,j} \in R_{i,j}$, where

$$R_{i,j} = -\frac{k^2}{2} S_{i,j} - \alpha^2 \frac{kh^2}{12} Q_{i,j}. \tag{17}$$

Then, we can formulate the interval backward finite difference method, related to the equations (9)-(11) with (12), as follows

$$(1+2\lambda)\,U_{1,j}-\lambda U_{2,j}=\lambda U_{0,j}+U_{1,j-1}+R_{1,j},\qquad i=1,\ j=1,2,\ldots,m, \tag{18}$$

$$(1+2\lambda)\,U_{i,j}-\lambda U_{i-1,j}-\lambda U_{i+1,j}=U_{i,j-1}+R_{i,j},\qquad i=2,3,\ldots,n-2,\ j=1,2,\ldots,m, \tag{19}$$

$$(1+2\lambda)\,U_{n-1,j}-\lambda U_{n-2,j}=\lambda U_{n,j}+U_{n-1,j-1}+R_{n-1,j},\qquad i=n-1,\ j=1,2,\ldots,m, \tag{20}$$

where

$$U_{i,0}=F\left(X_i\right),\quad i=0,1,\ldots,n, \tag{21}$$

$$U_{0,j}=\Phi_1\left(T_j\right),\quad U_{n,j}=\Phi_2\left(T_j\right),\quad j=1,2,\ldots,m. \tag{22}$$

Note that X_i, $i=0,1,\ldots,n$, T_j, $j=0,1,\ldots,m$, are the intervals such that $x_i\in X_i$ and $t_j\in T_j$. Furthermore, $F=F\left(X\right)$, $\Phi_1=\Phi_1\left(T\right)$, $\Phi_2=\Phi_2\left(T\right)$ denote the interval extensions of the functions $f=f\left(x\right)$, $\varphi_1=\varphi_1\left(t\right)$ and $\varphi_2=\varphi_2\left(t\right)$, respectively.

Similarly, the matrix representation of (18)-(20) with (17) is given as follows:

$$CU^{(j)}=U^{(j-1)}+E_C^{(j)}+E_L^{(j)},\quad j=1,2,\ldots,m, \tag{23}$$

where

$$U^{(j)}=\left[U_{1,j},U_{2,j},\ldots,U_{n-1,j}\right]^{\mathrm{T}}, \tag{24}$$

$$E_C^{(j)}=\left[\lambda U_{0,j},0,\ldots,0,\lambda U_{n,j}\right]^{\mathrm{T}},\quad E_L^{(j)}=\left[R_{1,j},R_{2,j},\ldots,R_{n-1,j}\right]^{\mathrm{T}}.$$

Theorem 1 *Let us assume that the local truncation error of the backward finite difference scheme can be bounded by the appropriate intervals at each step. Moreover, let $F=F\left(X\right)$, $\Phi_1=\Phi_1\left(T\right)$, $\Phi_2=\Phi_2\left(T\right)$ denote interval extensions of the functions $f=f\left(x\right)$, $\varphi_1=\varphi_1\left(t\right)$, $\varphi_2=\varphi_2\left(t\right)$, given in the initial and boundary conditions of the heat conduction problem (1)-(3). If $u\left(x_i,0\right)\in U_{i,0}$, $i=0,1,\ldots,n$, $u(0,t_j)\in\Phi_1\left(T_j\right)$, $u(L,t_j)\in\Phi_2\left(T_j\right)$, $j=1,2,\ldots,m$, and the interval linear system of equations (23) with (24) can be solved with an interval realization of some direct method, then for the interval solutions considered we have $u\left(x_i,t_j\right)\in U_{i,j}$, $i=1,2,\ldots,n-1$, $j=1,2,\ldots,m$.*

Remark 2 *Taking into consideration the formulas (9)-(11) and (18)-(20), with their appropriate matrix representations (13) and (23), we conclude that the proof of the above theorem is a natural consequence of the thesis of Theorem 2.*

Consider a finite system of linear algebraic equations of the form $Ax=b$, where A is an n-by-n matrix, b is an n-dimensional vector, and the coefficients of A and b are real or interval values. The existence of the solution to $Ax=b$ is provided by Theorem 2 (see Moore et al., 2009).

THEOREM 2 *If we can carry out all the steps of a direct method for solving $Ax = b$ in the interval arithmetic (if no attempted division by an interval containing zero occurs, nor any overflow or underflow), then the system has a unique solution for every real matrix in A and every real vector in b, and the solution is contained in the resulting interval vector X.*

2.3. Approximation of the endpoints of the error terms intervals

Before we apply the interval method considered, we have to compute the components of the vectors $E_L^{(j)}$. Consequently, the interval values of $S_{i,j}$, $Q_{i,j}$ are required (see (24) with (17)) for each mesh point (x_i, t_j), $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$. Note that determination of the exact values of the endpoints of the error term intervals $S_{i,j}$, $Q_{i,j}$ is possible only for some selected examples of the heat conduction problem (1)-(3). Generally, for any other case, such issue is still an open problem that deserves further investigation. Subsequently, we propose the method of approximation of the endpoints considered. It is based on the finite difference schemes.

Assumptions and preliminary steps

We establish the maximum time $T_{\max}$. For given values of n and m, we have the stepsizes $h = L/n$ and $k = T_{\max}/m$. Hence, the grid points are (x_i, t_j), where $x_i = ih$, $t_j = jk$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$. The indexes (i, j) are further used to indicate the interval solutions $U_{i,j}$, obtained by the interval backward finite difference method (18)-(20) (or (23)-(24)).

Due to the initial and boundary conditions, the interval values $U_{i,j}$ for $i = 0$, $i = n$, $j = 1, \ldots, m$, and $i = 0, 1, 2, \ldots, n-1$, $j = 0$, can be computed from the interval extensions $\Phi_1 = \Phi_1(T)$, $\Phi_2 = \Phi_2(T)$, $F = F(X)$ of the functions $\varphi_1 = \varphi_1(t)$, $\varphi_2 = \varphi_2(t)$, $f = f(x)$, respectively. The unknown values of the solution, i.e. the components of the vectors $U^{(j)}$, where $j = 1, 2, \ldots, m$, can be computed successively, when we apply, e.g., an interval realization of some direct method for solving the interval linear system of equations (23)-(24). Before that, we have to compute the endpoints of the error term intervals $S_{i,j}$, $Q_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$. An appropriate algorithm of the approximation of the endpoints of these intervals can be given in the way given below:

ALGORITHM

Step 1. Take $iter = -1$. Set $\widetilde{n} = n$, $\widetilde{m} = m$ and $\widetilde{h} = h$, $\widetilde{k} = k$. Based on that, the grid points are (x_s, t_q), where $x_s = s\widetilde{h}$, $t_q = q\widetilde{k}$, $s = 0, 1, \ldots, \widetilde{n}$, $q = 0, 1, \ldots, \widetilde{m}$.

Step 2. Take $iter = iter + 1$.

If $iter = 0$, then take $mul = 1$.
Else, if $iter \geq 1$, then take $mul = 2 \cdot mul$.
EndIf.
Then, take $\widetilde{n} = mul \cdot n$, $\widetilde{m} = mul \cdot m$ and $\widetilde{h} = h/mul$, $\widetilde{k} = k/mul$.

Step 3. Solve the initial-boundary value problem (1)-(3) with the interval realization of the conventional backward finite difference scheme. Such interval realization can be understood as the interval backward finite difference method (18)-(20) (or (23)-(24)) with the components of $E_L^{(q)}$ that represent the local truncation error terms all equal to zero, i.e., we have $R_{s,q} = 0$. The interval solutions, obtained in this way, are further denoted by $U_{s,q}^{IA}$, where IA is the abbreviation for the *interval arithmetic* that allows us to consider in the interval solution the inexact initial data, uncertain values of parameters and the errors caused by the floating-point arithmetic (i.e. representation and rounding errors) in the proper way. Note that the indexes (s, q) are further used to indicate the mesh points, at which the interval solutions $U_{s,q}^{IA}$ are being computed. Nevertheless, because of the way, in which the denser grid is established, we can always find a pair of indexes (s, q) such that $(x_i, t_j) = (x_s, t_q)$.

Step 4. Check, if $U_{s,q}^{IA}$ can be further used for the approximation of the endpoints of the error term intervals.
If $iter = 0$, then go back to Step 2.
Else, if $iter \geq 1$, then

(a) Compute the maximum distance $q_{\max}$ between $U_{i,j}^{IA\,(iter-1)}$ and $U_{i,j}^{IA\,(iter)}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$. Note that a distance between such two intervals is defined as

$$q_{i,j} = \max\left\{ |\underline{U}_{i,j}^{IA\,(iter-1)} - \underline{U}_{i,j}^{IA\,(iter)}|, |\overline{U}_{i,j}^{IA\,(iter-1)} - \overline{U}_{i,j}^{IA\,(iter)}| \right\}. \quad (25)$$

(b) Choose a tolerance value TOL. If $q_{\max} \leq TOL$, then stop the iteration process and go further to Step 5. Else, go back to Step 2.

EndIf.

Step 5. Compute the endpoints of $S_{i,j}$, $Q_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$. We assumed (see Section 2.2) that these intervals are such that for $\eta_j \in (t_{j-1}, t_j)$, $\xi_i \in (x_{i-1}, x_{i+1})$, we have

$$\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j) \in S_{i,j} = \left[\underline{S}_{i,j}, \overline{S}_{i,j}\right], \quad \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j) \in Q_{i,j} = \left[\underline{Q}_{i,j}, \overline{Q}_{i,j}\right].$$

For an approximation of the partial derivatives $\partial^2 u/\partial t^2$ and $\partial^4 u/\partial x^4$ at a given point, we can use some finite difference schemes. Subsequently, we choose the sixth order ones for the partial derivatives with respect to time t and the fourth order ones for the partial derivatives with respect to space x (see also Fornberg, 1998). In general, for the point (x_s, t_q), we have

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x_s, t_q) &= \diamond_{F,\widetilde{k}}\, u(x_s, t_q) + O\left(\widetilde{k}^6\right), \\
\frac{\partial^2 u}{\partial t^2}(x_s, t_q) &= \diamond_{C,\widetilde{k}}\, u(x_s, t_q) + O\left(\widetilde{k}^6\right), \\
\frac{\partial^2 u}{\partial t^2}(x_s, t_q) &= \diamond_{B,\widetilde{k}}\, u(x_s, t_q) + O\left(\widetilde{k}^6\right),
\end{aligned} \quad (26)$$

$$
\begin{aligned}
\frac{\partial^4 u}{\partial x^4}(x_s, t_q) &= \diamond_{F,\widetilde{h}}\, u(x_s, t_q) + O\left(\widetilde{h}^4\right), \\
\frac{\partial^4 u}{\partial x^4}(x_s, t_q) &= \diamond_{C,\widetilde{h}}\, u(x_s, t_q) + O\left(\widetilde{h}^4\right), \\
\frac{\partial^4 u}{\partial x^4}(x_s, t_q) &= \diamond_{B,\widetilde{h}}\, u(x_s, t_q) + O\left(\widetilde{h}^4\right),
\end{aligned}
\tag{27}
$$

where the operators $\diamond_{F,\widetilde{k}}$, $\diamond_{C,\widetilde{k}}$, $\diamond_{B,\widetilde{k}}$ and $\diamond_{F,\widetilde{h}}$, $\diamond_{C,\widetilde{h}}$, $\diamond_{B,\widetilde{h}}$ are defined as

$$
\begin{aligned}
\diamond_{F,\widetilde{k}}\, u(x_s, t_q) &= \frac{1}{180\widetilde{k}^2}\left[938\, u(x_s, t_q) - 4014\, u(x_s, t_{q+1}) + 7911\, u(x_s, t_{q+2})\right. \\
&\quad -9490\, u(x_s, t_{q+3}) + 7380\, u(x_s, t_{q+4}) \\
&\quad \left. -3618\, u(x_s, t_{q+5}) + 1019\, u(x_s, t_{q+6}) - 126\, u(x_s, t_{q+7})\right], \\
\diamond_{C,\widetilde{k}}\, u(x_s, t_q) &= \frac{1}{180\widetilde{k}^2}\left[2\, u(x_s, t_{q-3}) - 27\, u(x_s, t_{q-2})\right. \\
&\quad +270\, u(x_s, t_{q-1}) - 490\, u(x_s, t_q) + 270\, u(x_s, t_{q+1}) \\
&\quad \left. -27\, u(x_s, t_{q+2}) + 2\, u(x_s, t_{q+3})\right], \\
\diamond_{B,\widetilde{k}}\, u(x_s, t_q) &= \frac{1}{180\widetilde{k}^2}\left[938\, u(x_s, t_q) - 4014\, u(x_s, t_{q-1}) + 7911\, u(x_s, t_{q-2})\right. \\
&\quad -9490\, u(x_s, t_{q-3}) + 7380\, u(x_s, t_{q-4}) \\
&\quad \left. -3618\, u(x_s, t_{q-5}) + 1019\, u(x_s, t_{q-6}) - 126\, u(x_s, t_{q-7})\right],
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\diamond_{F,\widetilde{h}}\, u(x_s, t_q) &= \frac{1}{6\widetilde{h}^4}\left[56\, u(x_s, t_q) - 333\, u(x_{s+1}, t_q) + 852\, u(x_{s+2}, t_q)\right. \\
&\quad -1219\, u(x_{s+3}, t_q) + 1056\, u(x_{s+4}, t_q) \\
&\quad \left. -555\, u(x_{s+5}, t_q) + 164\, u(x_{s+6}, t_q) - 21\, u(x_{s+7}, t_q)\right], \\
\diamond_{C,\widetilde{h}}\, u(x_s, t_q) &= \frac{1}{6\widetilde{h}^4}\left[-u(x_{s-3}, t_q) + 12\, u(x_{s-2}, t_q) - 39\, u(x_{s-1}, t_q)\right. \\
&\quad +56\, u(x_s, t_q) - 39\, u(x_{s+1}, t_q) + 12\, u(x_{s+2}, t_q) \\
&\quad \left. -u(x_{s+3}, t_q)\right], \\
\diamond_{B,\widetilde{h}}\, u(x_s, t_q) &= \frac{1}{6\widetilde{h}^4}\left[56\, u(x_s, t_q) - 333\, u(x_{s-1}, t_q) + 852\, u(x_{s-2}, t_q)\right. \\
&\quad -1219\, u(x_{s-3}, t_q) + 1056\, u(x_{s-4}, t_q) \\
&\quad \left. -555\, u(x_{s-5}, t_q) + 164\, u(x_{s-6}, t_q) - 21\, u(x_{s-7}, t_q)\right].
\end{aligned}
\tag{29}
$$

As we know, $\eta_j \in (t_{j-1}, t_j)$ and $\xi_i \in (x_{i-1}, x_{i+1})$. Hence, we propose to utilize the idea of finite differences for the approximation of the endpoints of the error term intervals, taking the intervals $U_{s,q}^{IA}$ instead of $u(x_s, t_q)$ in (28)-(29). Since for a given pair of indexes (i, j), we can always find the indexes (s, q) such that $(x_i, t_j) = (x_s, t_q)$, then based on the operators defined above, we can easily compute the following intervals

$$\begin{gathered} S^*_{i,j-1} = \diamond_{\circ,\widetilde{k}}\, U^{IA}_{i,j-1}, \quad S^*_{i,j} = \diamond_{\circ,\widetilde{k}}\, U^{IA}_{i,j}, \\ Q^*_{i-1,j} = \diamond_{\circ,\widetilde{h}}\, U^{IA}_{i-1,j}, \quad Q^*_{i,j} = \diamond_{\circ,\widetilde{h}}\, U^{IA}_{i,j}, \quad Q^*_{i+1,j} = \diamond_{\circ,\widetilde{h}}\, U^{IA}_{i+1,j}, \end{gathered} \tag{30}$$

where $\circ \in \{F, C, B\}$ and is used to specify a forward, central and backward finite difference, respectively. We used to apply the central finite differences in the case of most mesh points. Nevertheless, the forward and backward finite differences are necessary for the points that are located near the boundary.

Finally, we compute the interval hulls of the intervals $S^*_{i,j-1}$, $S^*_{i,j}$ and $Q^*_{i-1,j}$, $Q^*_{i,j}$, $Q^*_{i+1,j}$, respectively. Then, we take the endpoints of the results obtained as the approximations for the endpoints of $S_{i,j}$ and $Q_{i,j}$. We have

$$\begin{gathered} \underline{S}_{i,j} \approx \min\left\{\underline{S}^*_{i,j-1}, \underline{S}^*_{i,j}\right\}, \quad \overline{S}_{i,j} \approx \max\left\{\overline{S}^*_{i,j-1}, \overline{S}^*_{i,j}\right\}, \\ \underline{Q}_{i,j} \approx \min\left\{\underline{Q}^*_{i-1,j}, \underline{Q}^*_{i,j}, \underline{Q}^*_{i+1,j}\right\}, \quad \overline{Q}_{i,j} \approx \max\left\{\overline{Q}^*_{i-1,j}, \overline{Q}^*_{i,j}, \overline{Q}^*_{i+1,j}\right\}. \end{gathered} \tag{31}$$

Step 6. Use the error term intervals $S_{i,j}$, $Q_{i,j}$ to compute the components of the interval vectors $E_L^{(j)}$, $j = 1, 2, \ldots, m$.

3. Numerical results

Consider the heat conduction problem given by the governing equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1,\ t > 0, \tag{32}$$

and the initial and boundary conditions

$$u(x,0) = 1 - 0.8x + \sin \pi x, \quad 0 \le x \le 1, \tag{33}$$

$$u(0,t) = 1, \quad u(1,t) = 0.2, \quad t > 0. \tag{34}$$

The analytical solution of the problem (32)-(34) and the partial derivatives of u, present in the error terms are known and given as follows

$$u(x,t) = 1 - 0.8x + e^{-t}\sin(\pi x), \tag{35}$$

$$\partial^2 u/\partial t^2\,(x,t) = e^{-t}\sin(\pi x), \quad \partial^4 u/\partial x^4\,(x,t) = \pi^4 e^{-t}\sin(\pi x). \tag{36}$$

We choose $T_{\max} = 1$, and then we use the exact solution (35) to obtain the temperature distribution, described by the problem (32)-(34), as presented in Figs. 1 (a)-(b). The partial derivatives (36) of the function u (present in the error term (12)) are given in Figs. 2 (a)-(b).

We carry out a number of numerical experiments to investigate the interval method considered. The computations were performed with the C++ libraries (dedicated for the Intel C++ compiler) for the floating-point conversions and interval arithmetic using the double extended precision format (see also Jankowska, 2010). For a given grid of points (x_i, t_j) defined by the constants n and m, we choose a denser grid of points (x_s, t_q), generated with the constants $\widetilde{n}$ and $\widetilde{m}$ that are further used for an approximation of the endpoints of the error term intervals (see the Algorithm). For most experiments we set $n = m$, $\widetilde{n} = \widetilde{m}$, where $\widetilde{n}$, $\widetilde{m}$ are chosen such that their values are greater or equal to 520 and less or equal to 1000. In this way we have $q_{\max} \leq TOL = 3\text{E-}4$.

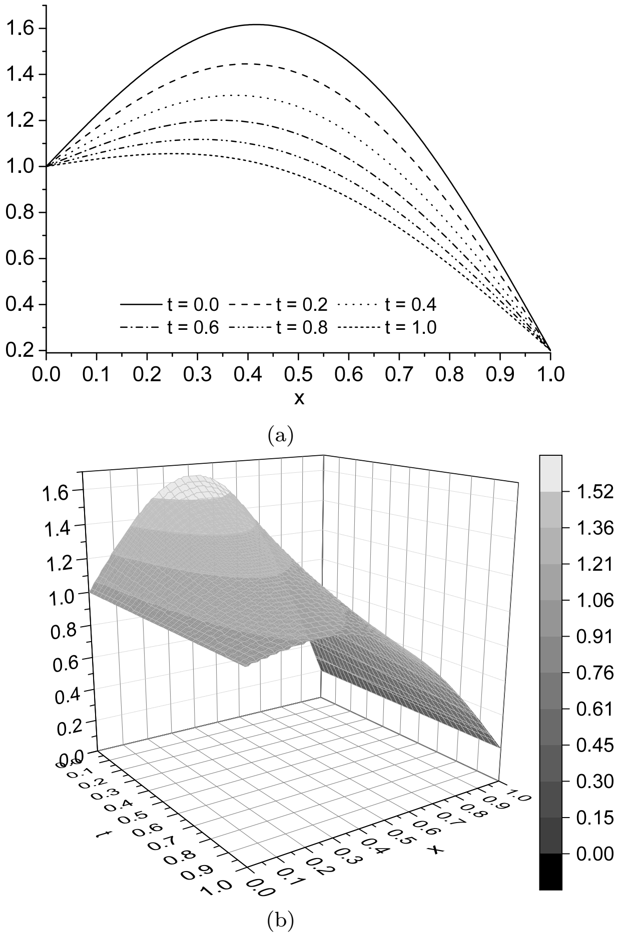
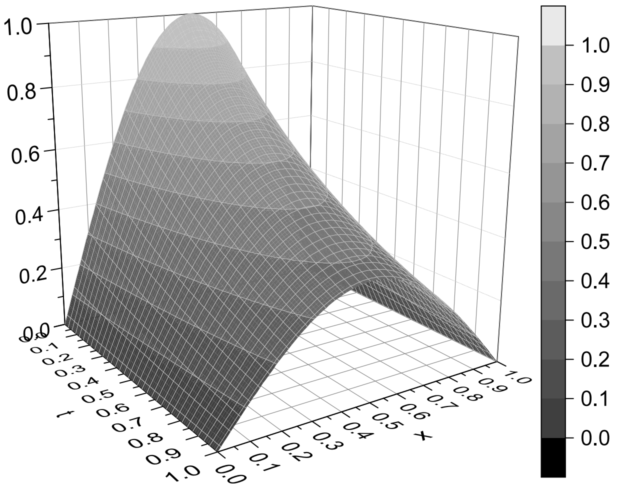
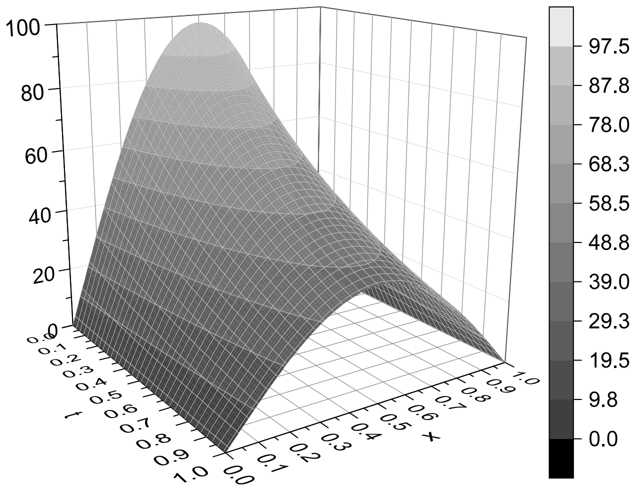


Figure 1: Temperature distribution described by the heat conduction problem for: (a) selected values of t; (b) $t \in [0, 1]$

First, we examine how a decrease of the stepsizes h and k (i.e. an increase of n and m) affects the widths of the interval solutions obtained. We use the interval method considered for the equally spaced grid points in relation to x and t, i.e. we have $n = m$ (see Fig. 3(a)). Then, the grid points are such that the stepsize k is much smaller than the stepsize h (see Fig. 3(b)). In



(a)



(b)

Figure 2: Partial derivatives of the function u that are present in the error terms: (a) $\partial^2 u/\partial t^2(x,t)$; (b) $\partial^4 u/\partial x^4(x,t)$

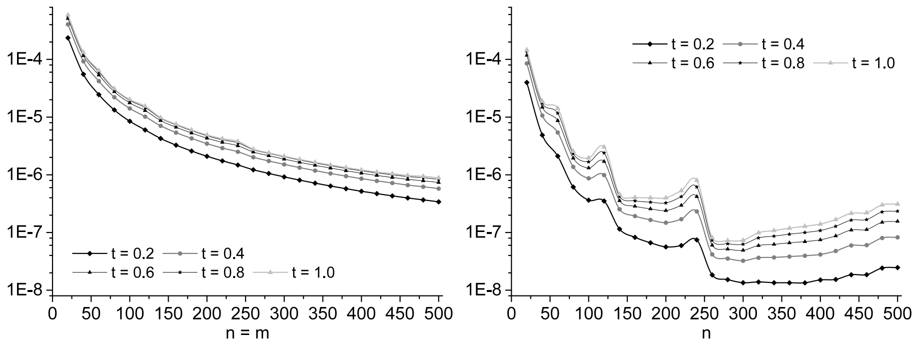
the second case we apply the stability criterion required for the conventional backward finite difference scheme, formulated for the heat conduction problem of the form (1)-(3) with the coefficient α^2 given as a function dependent on the position x. Such condition (see e.g. Press et al., 2007) states that the stepsize k should satisfy the relation $k \leq h^2/(2\alpha^2)$. It is not required in the case of the conventional and interval methods considered. Nevertheless, if we compare the results presented in Figs 3(a)-(b), then we can see that a decrease of the stepsize k with respect to the stepsize h causes a decrease of the interval solution widths. Such a behavior is desired and understandable (e.g. if $n = 20$, $h = 0.05$, then we take $m = 160$, $k = 0.00625$, so that the condition required was satisfied in the example problem considered). On the other hand, the decreasing tendency observed in Fig. 3(b) is not so regular as we can see in Fig. 3(a). This is caused by a number of factors. Note that in the case when $n \neq m$, the condition imposed on $\widetilde{n}$ is the same as when $n = m$, i.e. $520 \leq \widetilde{n} \leq 1000$. Simultaneously, $\widetilde{m}$ is many times greater than $\widetilde{n}$. Furthermore, for the selected subsequent grids (e.g. $n = 120$ and $n = 140$), the values of $\widetilde{m}$ differ significantly one from the other. If we take into account also the fact that too small values of $\widetilde{k}$ can, as a result, lead to an increase of widths of the error term intervals, then some irregularities in Fig. 3(b) become understandable.

Secondly, we compare the widths of the interval solutions $U(x = 0.5, t)$, $U^{IA}(x = 0.5, t)$, obtained with the interval method considered, and the appropriate component $E_L(x = 0.5, t)$ of the vector $E_L(t)$ that contains all the error term intervals (see Fig.4). We consider two cases that correspond to $t = 0.1$ and $t = 1$. The widths of $U^{IA}(x = 0.5, t)$ are very small in comparison to $U(x = 0.5, t)$. Nevertheless, since in the interval realization of the conventional method the local truncation error of the conventional scheme is neglected, then the exact solution is usually not included in $U^{IA}(x, t)$ (see Table 1). This is in contrast to the interval solution $U(x, t)$ such that the exact solution belongs to it (see Table 2).

Finally, we analyze the widths of the error term intervals $S(x = 0.5, t)$, $Q(x = 0.5, t)$, together with the interval $E_L(x = 0.5, t)$ that depends on both of them as defined in the equation (17) (see Fig.5). Note that $E_L(x = 0.5, t)$ is an interval such that it should include (see the remarks given in the Algorithm proposed in Section 2.3) the local truncation error terms of the conventional scheme (always neglected in the conventional approach).

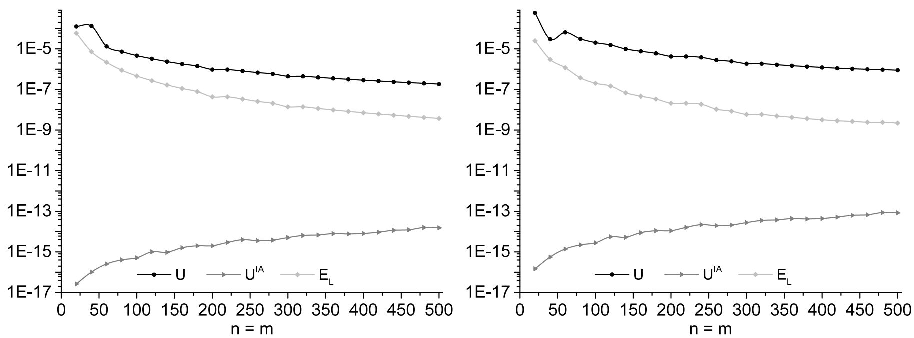
4. Conclusions

The interval method proposed is formulated on the basis of finite differences. It constitutes another approach in the area of verified computing related to the heat conduction problems. In its theoretical formulation it can be shown that the exact solution of the problem belongs to the interval solution. Nevertheless, in general, the endpoints of the error term intervals have to be approximated with some possibly high order finite difference schemes. One of our aims was to present an appropriate algorithm that can be used for this task. Then, the



(a) (b)

Figure 3: Widths of the interval solution $U(x = 0.5, t)$ obtained with the interval method considered and different values of n, where: (a) $m = n$; (b) m is such that $k \leq h^2/(2\alpha^2)$



(a) (b)

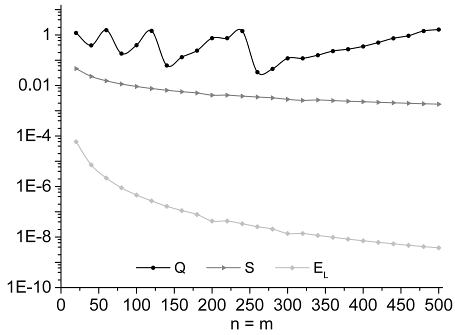
Figure 4: Widths of the interval solutions $U(x = 0.5, t)$, $U^{IA}(x = 0.5, t)$ and the appropriate component $E_L(x = 0.5, t)$ of the vector $E_L(t)$ that contains the error term intervals for: (a) $t = 0.1$; (b) $t = 1$

x	$u(x, t = 1)$	$U^{IA}(x, t = 1)$	*width*
0.1	1.033E+0	[1.034256351106114E+0,1.034256351106115E+0]	1.15E-15
0.2	1.056E+0	[1.057328494495153E+0,1.057328494495155E+0]	1.98E-15
0.3	1.057E+0	[1.059127010626348E+0,1.059127010626351E+0]	2.52E-15
0.4	1.029E+0	[1.031644890817003E+0,1.031644890817006E+0]	2.78E-15
0.5	9.678E-1	[9.697413190404690E-1,9.697413190404718E-1]	2.79E-15
0.6	8.698E-1	[8.716448908170034E-1,8.716448908170060E-1]	2.57E-15
0.7	7.376E-1	[7.391270106263488E-1,7.391270106263509E-1]	2.15E-15
0.8	5.762E-1	[5.773284944951539E-1,5.773284944951554E-1]	1.56E-15
0.9	3.936E-1	[3.942563511061142E-1,3.942563511061151E-1]	8.30E-16

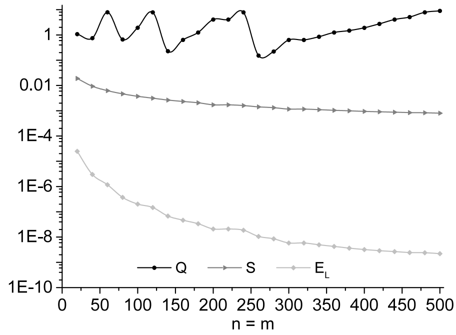
Table 1: Values of the exact solution $u(x, t = 1)$ and the interval solution $U^{IA}(x, t = 1)$ obtained with the interval realization of the conventional method, where $h = k = 0.01$.

x	$u(x, t = 1)$	$U(x, t = 1)$	*width*
0.1	1.0336809E+0	[1.0336773603200E+0,1.0336837965741E+0]	6.43E-06
0.2	1.0562341E+0	[1.0562272678769E+0,1.0562393576005E+0]	1.20E-05
0.3	1.0576207E+0	[1.0576113907519E+0,1.0576278583704E+0]	1.64E-05
0.4	1.0298741E+0	[1.0298632472572E+0,1.0298824601789E+0]	1.92E-05
0.5	9.6787944E-1	[9.6786802368765E-1,9.6788815492366E-1]	2.01E-05
0.6	8.6987413E-1	[8.6986326277521E-1,8.6988244382515E-1]	1.91E-05
0.7	7.3762071E-1	[7.3761141697840E-1,7.3762783154001E-1]	1.64E-05
0.8	5.7623411E-1	[5.7622729572120E-1,5.7623933001132E-1]	1.20E-05
0.9	3.9368099E-1	[3.9367737891274E-1,3.9368377898258E-1]	6.40E-06

Table 2: Values of the exact solution $u(x, t = 1)$ and the interval solution $U(x, t = 1)$ obtained with the interval method, where $h = k = 0.01$.



(a)



(b)

Figure 5: Widths of the error term interval $E_L(x = 0.5, t)$ and the intervals $S(x = 0.5, t)$, $Q(x = 0.5, t)$, which are all taken into account when the interval solution is produced with the interval method considered, where: (a) $t = 0.1$; (b) $t = 1$

analysis of the widths of the interval solution as well as the error term intervals was carried out. Since the analytical solution of the problem is known, we were able to check if the exact solution belongs to the interval solution. Furthermore, the similar tests were performed with respect to the interval values of the partial derivatives present in the error terms in each iteration step of the algorithm. The interval method considered just validates the conventional one. However, as the numerical experiments show, the exact solution is included in the interval solutions obtained.

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