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A condition for asset redundancy in the mean-variance model of portfolio investment*†

by

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Abstract: The mean-variance approach to portfolio investment exploits the fact that the diversification of investments by combination of different assets in one portfolio allows for reducing the financial risks significantly. The mean-variance model is formulated as a bi-objective optimization problem with linear (expected return) and quadratic (variance) objective functions. Given a set of available assets, the investor searches for a portfolio yielding the most preferred combination of these objectives. Naturally, the search is limited to the set of non-dominated combinations, referred to as the Pareto front. Due to the globalization of financial markets, investors nowadays have access to large numbers of assets. We examine the possibility of reducing the problem size by identifying those assets, whose removal does not affect the resulting Pareto front, thereby not deteriorating the quality of the solution from the investor's perspective.

We found a sufficient condition for asset redundancy, which can be verified before solving the problem. This condition is based on the possibility of reallocating the share of one asset in a portfolio to another asset without deteriorating the objective function values. We also proposed a parametric relaxation of this condition, making it possible to remove more assets for a price of a negligible deterioration of the Pareto front. Computational experiments conducted on five real-world problems have demonstrated that the problem size can be reduced significantly using the proposed approach

Keywords: modern portfolio theory, Markowitz model, mean-variance portfolio optimization, asset redundancy, problem size

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1. Introduction and problem formulation

The mean-variance model of portfolio investment has been formulated and first studied by Harry Markowitz in the 1950s (see Markowitz, 1952). He was later awarded a Nobel Prize in economics for that achievement, and the mean-variance model became the basis of the modern portfolio theory (Francis and Kim, 2013; Kolm, Tütüncü and Fabozzi, 2014). This model describes the situation, in which an investor constructs a portfolio of financial assets, aiming at maximizing the expected return of investment while minimizing risk. The latter is defined as the standard deviation or variance of portfolio return.

The mean-variance model is formulated as a problem of bi-objective optimization with linear (expected return) and quadratic (variance) objective functions. Given a set of assets available for the investor, the feasible solutions are portfolios represented by vectors of asset shares, which satisfy basic linear constraints and, optionally, additional constraints reflecting problem-specific requirements. The investor searches for a portfolio yielding the most preferred combination the two objectives. Naturally, the search is limited to the set of non-dominated combinations, referred to as the Pareto front. Various methods are proposed in the literature for assisting investors in this task (Anagnostopoulos and Mamanis, 2011; Kolm, Tütüncü and Fabozzi, 2014; Merton, 1972; Steuer, Hirschberger and Deb, 2016).

Deriving Pareto optimal portfolios may be computationally complex in the case of a large number of assets. If mixed-integer (Steuer, Hirschberger and Deb, 2016) or nonlinear (Lejeune, 2013) constraints are present, the problem complexity is much higher. As a result, the computation time may be unacceptably long. This issue is often addressed by developing approximate algorithms such as in Anagnostopoulos and Mamanis (2011), Lejeune (2013), Steuer, Hirschberger and Deb (2016). We approach this issue from another perspective, namely we try to simplify the mean-variance problem by reducing the number of assets.

It is obvious that the assets, which do not appear in any Pareto optimal portfolio are redundant and therefore can be removed from the model. We narrow down the definition of redundancy to the case where it is possible to reallocate the share of one asset in the portfolio by replacing it with another asset without deteriorating the objective function values. This allows us to formulate a sufficient condition for redundancy in terms of Pareto dominance relation between vectors of problem coefficients, corresponding to individual assets. For each asset, this vector is composed of its expected return, variance and covariances with the rest of assets. Furthermore, relaxing this condition by widening the Pareto domination cone allows for identifying more assets, which are likely to be redundant. The parameter of the relaxation controls the number of assets being removed.

The paper is organized as follows. First, we formulate the mean-variance problem of portfolio investment. The sufficient condition for asset redundancy is presented in Subsection 2.1 and its parametric relaxation in Subsection 2.2. In Section 3, we demonstrate on real-world problems that the proposed conditions allow for significant reductions in the numbers of assets with no or a negligible deterioration of the Pareto front. Section 4 closes the paper with the discussion of the practical value and the possible extensions of our results.

Let $n > 1$ be the number of assets, which are indexed by natural numbers $1, \ldots, n$. A portfolio is a vector $\mathbf{x} \in \mathbf{R}^n$ representing the shares of the corresponding assets $(x_1, x_2, \ldots, x_n)$, which satisfy the conditions $x_i \geq 0$ for all $i \in \{1, 2, \ldots, n\}$ and $\sum_{i=1}^{n} x_i = 1$. By Ω we denote the set of all the vectors satisfying the above conditions. We assume that the following information is given: the expected return of each asset i, denoted $\mu_i \in \mathbf{R}$, and for all pairs of assets i and j, the covariance between them, denoted $\sigma_{ij} \in \mathbf{R}$.

The mean-variance model of portfolio investment, proposed in Markowitz (1952) and thereafter possibly extended with additional constraints, is defined as follows:

$$\begin{aligned} \max f_1(\mathbf{x}) &:= -\mathbf{x}^{\mathrm{T}} S \mathbf{x}, \\ \max f_2(\mathbf{x}) &:= \boldsymbol{\mu}^{\mathrm{T}} \mathbf{x}, \\ \text{s.t. } & \mathbf{x} \in X, \end{aligned}$$

where $X \subseteq \Omega$ is the set of feasible solutions, or feasible portfolios of assets; $S = (\sigma_{ij})_{n \times n}$ is the covariance matrix; $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ is the vector of expected returns. Note that we replaced the objective of minimizing the variance by the one of maximizing the negative variance for uniformity.

The modern portfolio theory aims at assisting the investor in solving the following decision making problem: find a portfolio, which is most preferred among all feasible portfolios with respect to the values of f_1 and f_2. It is always assumed that such a portfolio belongs to the set of Pareto optimal portfolios defined as follows:

$$P = \{\mathbf{x} \in X : \ \mathbf{f}(\mathbf{x}') \geq \mathbf{f}(\mathbf{x}) \Rightarrow \mathbf{f}(\mathbf{x}') = \mathbf{f}(\mathbf{x}) \text{ for all } \mathbf{x}' \in X\},$$

where $\mathbf{f} = (f_1, f_2)$, and the inequality between vectors is understood componentwise.

Taking into account that the portfolios are compared in the decision making only based on the values of the here specified objective functions, any two portfolios $\mathbf{x}$ and $\mathbf{x}'$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}')$ are regarded as equivalent. Therefore, solving the portfolio investment problem can be understood as finding the most preferred bi-objective vector in the set $\Pi = \{\mathbf{f}(\mathbf{x}) : \ \mathbf{x} \in P\}$, called the Pareto front, and identifying a corresponding portfolio in P.

2. Conditions for asset redundancy

The complexity of Pareto optimal portfolio derivation depends on the problem size, which is the number of assets n. Large n may hamper effective tackling of real-life portfolio investment problems, especially when mixed-integer or nonlinear constraints are imposed on the feasible solution set. Therefore, eliminating assets, whose absence does not significantly change the set Π, is of practical interest.

2.1. The sufficient condition

Clearly, if eliminating an asset from the model does not change the Pareto front, this asset is redundant. In order to make the definition of a redundant asset easy to operationalize, we get rid of referring to Pareto optimality in it.

Definition 1 *An asset r is called redundant, if for each portfolio $\mathbf{x} \in X$ containing asset r there exists another portfolio $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_n) \in X$ such that $x'_r = 0$ and $\mathbf{f}(\mathbf{x}') \geq \mathbf{f}(\mathbf{x})$.*

Given two different assets r and q in a portfolio $\mathbf{x} \in X$, we define an alternative portfolio, in which the share of asset r is reallocated to asset q:

$$\mathbf{y}^{r,q}(\mathbf{x}) = (y_1, \ldots, y_n)$$

where $y_r = 0$, $y_q = x_r + x_q$ and $y_i = x_i$ for $i \notin \{r, q\}$.

The following lemma results directly from Definition 1.

Lemma 1 *Let the set of feasible portfolios satisfy the following condition:*

$$\begin{aligned} \mathbf{y}^{i,j}(\mathbf{x}) \in X \ &\text{for all } \mathbf{x} \in X \\ \text{and } i, j \in \{1, 2, \ldots, n\}, \ &i \neq j. \end{aligned} \tag{2.1}$$

An asset r is redundant, if for each portfolio $\mathbf{x} \in X$ there exists an asset $q \neq r$ such that $\mathbf{f}(\mathbf{y}^{r,q}(\mathbf{x})) \geq \mathbf{f}(\mathbf{x})$.

This lemma narrows down the definition of asset redundancy by considering only those alternative portfolios, which are obtained by reallocating shares between two assets. It is easy to see that for the classical mean-variance problem, where $X = \Omega$, the condition (2.1) holds. In the concluding section we give other examples of problems satisfying this condition.

Now we are in a position to formulate the sufficient condition for asset redundancy in a constructive way, i.e. enabling the possibility of its verification without solving the portfolio investment problem.

THEOREM 1 *Let the set of feasible portfolios satisfy condition (2.1). An asset r is redundant, if there exists another asset $q \neq r$ such that*

$$\begin{gathered} \mu_q \geq \mu_r, \ \ \sigma_{rr} \geq \sigma_{qq} \ \text{and} \\ \sigma_{rj} \geq \sigma_{qj} \ \text{for all} \ j \in \{1, \ldots, n\} \setminus \{r\}. \end{gathered} \tag{2.2}$$

PROOF Assume that the condition of the theorem holds for assets r and q, and consider an arbitrary portfolio $\mathbf{x} \in X$. According to Lemma 1, it is enough to prove that $\mathbf{f}(\mathbf{y}^{r,q}(\mathbf{x})) - \mathbf{f}(\mathbf{x}) \geq 0$.

The inequality $\mathbf{f}_1(\mathbf{y}^{r,q}(\mathbf{x})) - \mathbf{f}_1(\mathbf{x}) \geq 0$ follows directly from the evident equality $\boldsymbol{\mu}^{\mathrm{T}}\mathbf{y}^{r,q}(\mathbf{x}) - \boldsymbol{\mu}^T\mathbf{x} = x_r(\mu_q - \mu_r)$ and the inequality $\mu_q \geq \mu_r$. It remains to prove that

$$\mathbf{x}^{\mathrm{T}} S \mathbf{x} - \left(\mathbf{y}^{r,q}(\mathbf{x})\right)^{\mathrm{T}} S \mathbf{y}^{r,q}(\mathbf{x}) \geq 0. \tag{2.3}$$

Denote $\eta = \{1, \ldots, n\} \setminus \{r, q\}$ and $\Gamma = \sum\limits_{i \in \eta} \sum\limits_{j \in \eta} \sigma_{ij} x_i x_j$. Keeping in mind the symmetricity of matrix S, rewrite the components of (2.3) as follows:

$$\mathbf{x}^{\mathrm{T}} S \mathbf{x} = \Gamma + \sigma_{rr} x_r^2 + \sigma_{qq} x_q^2 + 2\sigma_{rq} x_r x_q + 2 \sum_{j \in \eta} \sigma_{rj} x_r x_j + 2 \sum_{j \in \eta} \sigma_{qj} x_q x_j,$$

$$\begin{aligned} &\left(\mathbf{y}^{r,q}(\mathbf{x})\right)^{\mathrm{T}} S \mathbf{y}^{r,q}(\mathbf{x}) = \Gamma + \sigma_{qq}(x_r + x_q)^2 + \\ &2 \sum_{j \in \eta} \sigma_{qj}(x_r + x_q) x_j = \Gamma + \sigma_{qq} x_r^2 + \sigma_{qq} x_q^2 + 2\sigma_{qq} x_r x_q + \\ &2 \sum_{j \in \eta} \sigma_{qj} x_r x_j + 2 \sum_{j \in \eta} \sigma_{qj} x_q x_j. \end{aligned}$$

From this we obtain:

$$\begin{aligned} &\mathbf{x}^{\mathrm{T}} S \mathbf{x} - \left(\mathbf{y}^{r,q}(\mathbf{x})\right)^{\mathrm{T}} S \mathbf{y}^{r,q}(\mathbf{x}) = (\sigma_{rr} - \sigma_{qq}) x_r^2 + \\ &2(\sigma_{rq} - \sigma_{qq}) x_r x_q + 2 \sum_{j \in \eta} (\sigma_{rj} - \sigma_{qj}) x_r x_j. \end{aligned}$$

Taking into account inequalities (2.2) and the non-negativity of vector $\mathbf{x}$, we conclude that all three summands of the latter expression are non-negative. This implies inequality (2.3). □

The condition, formulated in Theorem 1, has two practical drawbacks. First, if inequalities (2.2) for assets r and q turn into equalities, then both assets are redundant since Theorem 1 applies when r and q switch their roles. However, not necessarily both of them should be removed from the problem. If the redundancy of asset q follows only from the comparison with asset r and the latter is removed, then asset q is not redundant anymore. This means that additional check is needed before removing an asset when using the above condition. Secondly, the inequalities are formulated coefficient-wise, while it would be more convenient to formulate them vector-wise. That would enable applying algorithms of multidimensional sorting (see, e.g., Chen, Hwang and Tsai, 2012) for asset elimination. The following considerations allow for obtaining a formulation, which is devoid of the above drawbacks.

For each asset i, we introduce the $n+1$-vector of all problem parameters associated with i, referred to as the *representative vector* of asset i:

$$\mathbf{v}_i = (-\sigma_{i1}, -\sigma_{i2}, \ldots, -\sigma_{in}, \mu_i) .$$

Let for two different assets r and q, vector $\mathbf{v}_q$ dominate vector $\mathbf{v}_r$ in terms of Pareto, i.e. $\mathbf{v}_q \geq \mathbf{v}_r$ and $\mathbf{v}_q \neq \mathbf{v}_r$. Then, taking into account $\sigma_{rr} \geq \sigma_{qr} = \sigma_{rq} \geq \sigma_{qq}$ we see that inequalities (2.2) hold. Since the Pareto dominance relation is asymmetric, $\mathbf{v}_r$ cannot simultaneously dominate $\mathbf{v}_q$. Thus, we obtain the following statement:

Theorem 2 *Let the set of feasible portfolios satisfy condition (2.1). Removing all assets, whose representative vectors are dominated by representative vectors of other assets, does not change the Pareto front of the mean-variance problem.*

This theorem reduces the problem of eliminating redundant assets to multidimensional sorting of their representative vectors.

2.2. The relaxed condition for asset redundancy

The condition of Pareto dominance between representative vectors seems to be rather strong, as it assumes that $n+1$ inequalities between covariance coefficients simultaneously hold. Let us consider its relaxation by widening the Pareto domination cone. We use a simple technique, proposed in Kaliszewski and Michalowski (1997) and Kaliszewski (2000), and generalized in Podkopaev (2007). It consists in applying a linear transformation to the vectors being compared. By definition, if after the transformation one vector dominates another vector in terms of Pareto, then we say that the relaxed dominance relation takes place.

For a given vector $\mathbf{z} \in \mathbf{R}^k,\ k > 1$, the transformed vector is defined by $B\mathbf{z}$, where $B = (b_{ij})_{k \times k}$ is a matrix with diagonal elements equal to 1 and the rest of elements are small non-negative numbers. Each element $b_{ij},\ i \neq j$, can be interpreted in terms of the lower bound on acceptable trade-off ratio between the deterioration of the i-th component and the improvement of the j-th component (Podkopaev, 2007). In short, even when one vector is not dominated by another vector in terms of Pareto, the relaxed dominance relation can take place if the trade-off ratios between components of the vectors do not exceed the magnitudes of the corresponding elements of B.

Concerning the representative vectors of assets, we set elements of the transformation matrix $B_{(n+1)\times(n+1)}$ as $b_{ij} = \beta$ for $i \leq n,\ j \leq n,\ i \neq j$, and $b_{ij} = 0$ for $i = n + 1$ or $j = n + 1$, and $i \neq j$. Setting the latter elements to zero reflects the fact that we do not accept trade-offs between the covariance coefficients and the expected return due to different nature and scales of those values. The value $\beta \geq 0$ serves as the parameter of the relaxation. When $\beta = 0$, the new dominance relation coincides with the Pareto dominance relation. The higher is β, the wider is the domination cone, and the more assets would in consequence be potentially removed.

It is worth noting that Theorem 2 is not valid when the Pareto dominance relation is replaced with its relaxation. In other words, we cannot prove that after removing redundant assets according to the relaxed condition of redundancy, the Pareto front will remain the same. On the other hand, the condition for asset redundancy in this theorem is not a necessary condition, i.e. there is hope that some more assets can be removed without affecting the Pareto front. The computational experiments further on demonstrate the realization of this possibility.

3. Numerical experiments

In order to examine the practical use of the proposed approach, we did calculations for several real-life examples of the mean-variance problem. We used problem data (expected return and covariance coefficients) from the popular Beasley OR-Library (2018). It contains five collections of assets from the following stock market indexes (the numbers of assets are given in parentheses): Hang Seng (31), DAX (85), FTSE (89), Standard & Poor's (98), Nikkei (225).

First, for each of the mentioned sets of data, we enumerated redundant assets in accordance with Theorem 2 by identifying dominated representative vectors. Due to this operation it turned out that only the data set from Nikkei has redundant assets (45 out of 225).

Next, we examined the effect of relaxation on the number of redundant assets and the Pareto front. For each of the five problems and each value of β from the set $\{0.001, 0.002, 0.005, 0.01, 0.02, 0.05, 0.1, 0.2\}$, we calculated the number of redundant assets according to the relaxed dominance relation, and verified if the Pareto front changed significantly after removing them.

The difference between Pareto fronts of the original problem and the problem with the reduced number of assets is evaluated as follows. For both problems, we set 21 evenly distributed values of expected return by dividing the interval of expected return over the Pareto front into 20 equal parts. Then we derive 21 corresponding Pareto optimal solutions by solving the quadratic programming problem of minimizing variance, where the values, defined above, serve as the lower bound of expected return one at a time. We compare the obtained Pareto front representations of the two problems point-wise, calculating the absolute differences between corresponding objective function values of the corresponding solutions. For each of the objective functions, we take the maximum among those differences divided by the Pareto range of the corresponding objective function values for scaling. If both of the resulting maximum differences are not greater than 10^{-4} (i.e. 0.01% of the corresponding range), we consider that the two Pareto fronts are not significantly different.

The results are presented in Table 1. The columns correspond to the five considered data sets. The rows correspond to the considered values of β, while the first row contains original problem sizes. For each value of β and each data set, the cell contains two values: the number of assets remaining in the problem after removing the redundant assets under this β, and the percentage of assets removed, in parentheses. For each of the data sets, we highlight the cell corresponding to the maximal value of β, which did not result in significant change of the Pareto front with bold font and horizontal line.

Table 1. Numbers of remaining assets after applying the relaxed condition of redundancy for different values of β

β	Hang Seng	DAX	FTSE	S & P	Nikkei
Initial	31	85	89	98	225
0	31 (0%)	85 (0%)	89 (0%)	98 (0%)	180 (20%)
0.001	31 (0%)	85 (0%)	89 (0%)	98 (0%)	143 (36%)
0.002	31 (0%)	85 (0%)	89 (0%)	98 (0%)	106 (53%)
0.005	30 (3%)	85 (0%)	89 (0%)	98 (0%)	**46 (80%)**
0.01	28 (10%)	82 (4%)	88 (1%)	98 (0%)	20 (91%)
0.02	22 (29%)	**75 (12%)**	**80 (10%)**	**96 (2%)**	14 (94%)
0.05	**13 (58%)**	39 (54%)	49 (45%)	71 (28%)	7 (97%)
0.1	9 (71%)	12 (86%)	19 (79%)	43 (56%)	7 (97%)
0.2	5 (84%)	5 (94%)	9 (90%)	27 (72%)	6 (97%)

As one can notice from this table, large percentages of redundant assets can be removed without affecting the Pareto front in some of the considered examples. This confirms the potential usefulness of our approach. In practice, if a mean-variance problem of the portfolio investment is expected to be computationally hard, then filtering out redundant assets based on the proposed technique before solving the problem makes practical sense. This gives a chance of reducing the computational complexity quite significantly.

On the other hand, even among our small set of examples, we observe high variation of effects the relaxed dominance relation has on the resulting problems. Same values of β allow for reducing the number of assets to different extent for different problems. Significant reduction without changing the Pareto front cannot be always guaranteed. Additional research is needed in order to understand the factors, which influence the number of redundant assets.

It is reasonable to assume that this number depends on the distributional characteristics of asset returns. The paper by Hirschberger, Qi and Steuer (2007) presents a technique of random generation of covariance matrices for mean-variance portfolio investment models with desirable distribution characteristics. This technique can be used for a simulation study addressing the above issue.

4. Conclusion

Theorem 1 provides a sufficient condition for asset redundancy in a mean-variance problem of portfolio investment, which is operationalized in Theorem 2. This condition is only valid when the feasible solution set satisfies (2.2). It is easy to see that the basic mean-variance problem (where $X = \Omega$) satisfies this condition, since reallocating the shares between assets does not change the total sum of shares. Let us mention two problem modifications, well-known in the literature (see Steuer, Hirschberger and Deb, 2016), where condition (2.2) holds true.

The first modification consists in adding the cardinality constraint, which binds from above the number of assets in each portfolio:

$$|i:\ x_i > 0| \leq M, \text{ where } 1 \leq M < n.$$

The second modification prohibits investing too little money in individual assets:

$$x_i < L \ \Rightarrow\ x_i = 0 \text{ for all } i \in \{1, 2, \ldots, n\},$$
$$\text{where } 0 < L < 1.$$

It is easy to see that both modification do not violate condition (2.2): when the share of one asset is completely reallocated to another asset, the number of assets with positive shares does not increase, and an asset with a positive share smaller than L does not appear. Both of the mentioned modifications lead to mixed-integer problem formulations, which makes them computationally complex. On the other hand, the following problem modification, popular in the literature violates condition (2.2): it consists in binding all asset shares from above, i.e. introducing the constraints $x_i \leq U$ for $i \in \{1, 2, \ldots, n\}$, where $U < 1$. The inability to use the sufficient condition for asset redundancy in such problems can be considered a significant limitation of our approach. Overcoming this limitation is a research challenge worth taking.

Preliminary experiments demonstrate that in real-life problems, a significant number of assets can be removed without affecting the Pareto front. If the computational complexity of a mean-variance problem is much higher than the complexity of a multidimensional sorting procedure (which is usually low-degree polynomial, see, e.g., Chen, Hwang and Tsai, 2012), running this procedure before solving the problem gives a chance of reducing the total solution time.

References

Anagnostopoulos, K.P. and Mamanis, G. (2011) Multiobjective evolutionary algorithms for complex portfolio optimization problems. *Computational Management Science* 8, 259–279.

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D.(1999) Coherent measures of risk. *Mathematical Finance*, 9(3):203–228. URL https://onlinelibrary.wiley.com/doi/abs/10.1111 /1467-9965.00068.

D. Bertsimas, D. and Cory-Wright, R. (2018) A scalable algorithm for sparse portfolio selection.

Better, M. and Glover, F. (2006) Selecting project portfolios by optimizing simulations. *The Engineering Economist*, **51**(2):81–97. URL https://www.tandfonline.com/doi/abs/10.1080/00137910600695593.

Bienstock, D. (1996) Computational study of a family of mixed-integer quadratic programming problems. *Mathematical Programming, Series B*, **74**(2):121–140.

Briec, W., Kerstens, K. and de Woestyne, I. V. (2013) Portfolio selection with skewness: A comparison of methods and a generalized one fund result. *European Journal of Operational Research*, **230**(2):412–421. 10.1016/j.ejor.2013.04.021. URL http://www.sciencedirect.com/science /article/pii/S0377221713003196.

Chang, T.-J., Meade, N., Beasley, J. and Sharaiha, Y. (2000) Heuristics for cardinality constrained portfolio optimisation. *Computers & Operations Research*, **27**(13):1271–1302. http://www.sciencedirect.com /science/article/pii/S030505489900074X.

Chen, W.M., Hwang, H.K., Tsai, T.H. (2012) Maxima-fnding algorithms for multidimensional samples: A two-phase approach. *Computational Geometry* 45, 33–53.

Ehrgott, M., Klamroth, K. and Schwehm, C. (2004) An MCDM approach to portfolio optimization. *European Journal of Operational Research*, **155**(3):752–770. doi: 10.1016/S0377-2217(02)00881-0. http://www.sciencedirect.com/science/article/pii/ S0377221702008810.

Elton, E., Gruber, M., Brown, S. and Goetzmann, W. (2014) *Modern Portfolio Theory and Investment Analysis.* John Wiley & Sons, United States of America, 9th edition.

Elton, E. J. and Gruber, J. (2000) The rationality of asset allocation recommendations. *The Journal of Financial and Quantitative Analysis*, **35**(1):27–41. http://www.jstor.org/stable/2676237.

Emerich, M. and Yevseyeva, I. (2018) Portfolio optimization in non-financial application domains. The presentation given at the conference "Large-scale portfolio investments (Beyond the Markowitz portfolio theory)", Warsaw, Poland, Apr. 2018. https://www.impan.pl/en/activities/banach-center/ conferences/18-portfolio/program.

Francis, J. and Kim, D. (2013) *Modern Portfolio Theory: Foundations, Analysis, and New Developments.* Wiley Finance, Wiley.

Frangioni, A. and Gentile, C. (2006) Perspective cuts for a class of convex 1 mixed integer programs. *Math. Program.*, **106**(2):225–236. doi: 10.1007/s10107-005-0594-3.

Hirschberger, M., Qi, Y. and Steuer, R. E. (2007) Randomly generating portfolio-selection covariance matrices with specified distributional characteristics. *European Journal of Operational Research*, **177**(3):1610–1625, mar 2007. 10.1016/j.ejor.2005.10.014.

Jacob, N. and Pettit, R. (1984) *Investments.* Richard D. Irwin Inc., Homewood, USA.

Jayasekara, P. L., Adelgren, N. and Wiecek, M. M. (2019) On convex multiobjective programs with application to portfolio optimization. *Journal of Multi-Criteria Decision Analysis*, n/a(n/a):1–14, 2019. https://onlinelibrary.wiley.com/doi/abs /10.1002/mcda.1690.

Kalayci, C. B., Ertenlice, O. and Akbay, M. A. (2019) A comprehensive review of deterministic models and applications for mean-variance portfolio optimization. *Expert Systems with Applications*, 125:345–368. http://www.sciencedirect.com /science/article/pii/S0957417419301162.

Kaliszewski, I. (2000) Using trade-off information in decision-making algorithms. *Computers & Operations Research* 27, 161–182.

Kaliszewski, I. and Michalowski, W. (1997) Efcient solutions and bounds on tradeoffs. *Journal of Optimization Theory and Applications* 94, 381–394.

Kolm, P.N., Tütüncü, R. and Fabozzi, F.J. (2014) 60 years of portfolio optimization: Practical challenges and current trends. *European Journal of Operational Research* 234, 356–371.

Konno, H. and Suzuki, K. (1992) A fast algorithm for solving large scale mean-variance models by compact factorization of covariance matrices. *Journal of the Operations Research Society of Japan*, **35**(1):93–104, 1992. 10.15807/jorsj.35.93.

Lejeune, M.A. (2013) Portfolio optimization with combinatorial and downside return constraints. In: L.F. Zuluaga, T. Terlaky, eds., *Modeling and*

Optimization: Theory and Applications. Springer New York, New York, 31–50.

Lo, A. W., Petrov, C. and Wierzbicki, M. (2003) It's 11pm - do you know where your liquidity is? The mean-variance liquidity frontier. *Journal Of Investment Management*, **1**(1). https://ssrn.com/abstract=495103.

Lwin, K., Qu, R. and Kendall, G. (2014) A learning-guided multi-objective evolutionary algorithm for constrained portfolio optimization. *Applied Soft Computing*, 24:757–772. 10.1016/j. asoc.2014.08.026.

Mansini, R., Ogryczak, W. and Speranza, M. G. (2014) Twenty years of linear programming based portfolio optimization. *European Journal of Operational Research*, **234**(2):518–535, 2014. 10.1016/j.ejor.2013.08.035. http://www.sciencedirect.com/science/article/pii /S0377221713007194.

Mansini, R., Ogryczak, W. and Speranza, M. G. (2015) *Linear and Mixed Integer Programming for Portfolio Optimization. EURO Advanced Tutorials on Operational Research*. Chapter "Portfolio optimization with other real features", 63–72. Springer International Publishing.

Markowitz, H. (1952) Portfolio selection. *Journal of Finance*, **7**(1):77–91. 10.2307/2975974.

Markowitz, H.M. (1959) *Portfolio Selection: Efficient Diversification of Investments*. Yale University Press.

Mashayekhi, Z. and Omrani, H. (2016) An integrated multi-objective Markowitz-DEA cross-efficiency model with fuzzy returns for portfolio selection problem. *Applied Soft Computing*, 38:1–9. http://www.sciencedirect. com/science/article/pii/S1568494615005906.

Merton, R. (1972) An analytic derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis* 7, 1851–1872.

OR-Library (2018) http://people.brunel.ac.uk/ mastjjb/jeb/info.html, last accessed: August 2018.

Podkopaev, D. (2007) An approach to finding trade-off solutions by a linear transformation of objective functions. *Control and Cybernetics* 36, 347–356.

Podkopaev, D. (2008) Representing partial information on preferences with the help of linear transformation of objective space. In: T. Trzaskalik, ed., *Multiple Criteria Decision Making '07*. Karol Adamiecki University of Economics Press, 175–194.

Pogue, G. A. (1970) An extension of the Markowitz portfolio selection model to include variable transactions' costs, short sales, leverage policies and taxes. *The Journal of Finance*, **25**(5):1005–1027. http://www.jstor.org /stable/2325576.

Salehpoor, I. B. and Molla-Alizadeh-Zavardehi, S. (2019) A constrained portfolio selection model at considering risk-adjusted measure by using hybrid meta-heuristic algorithms. *Applied Soft Computing*, 75:233–253. 10.1016/j.asoc.2018.11.011. http://www.sciencedirect.com/science /article/pii/S1568494618306434.

Steuer, R. E., Hirschberger, M. and Deb, K. (2016) Extracting from the relaxed for large-scale semi-continuous variable nondominated frontiers. *Journal of Global Optimization*, **64**(1):33–48. doi: 10.1007/s10898-015-0305-4.

Tobin, J. (1958) Liquidity preference as behavior towards risk. *Review of Economic Studies*, **25**(2):65–86. https://EconPapers.repec.org/RePEc:oup:restud:v:25:y:1958:i:2:p:65-86.

Utz, S., Wimmer, M., Hirschberger, M. and Steuer, R. E. (2014) Tri-criterion inverse portfolio optimization with application to socially responsible mutual funds. *European Journal of Operational Research*, **234**(2): 491–498. doi: 10.1016/j.ejor.2013.07.024. http://www.sciencedirect.com/science/article/pii/S037722171300605X.

Utz, S., Wimmer, M. and Steuer, R. E. (2015) Tri-criterion modeling for constructing more-sustainable mutual funds. *European Journal of Operational Research*, **246**(1):331–338. https://epub.uni-regensburg.de/31634/.

Yoshimoto, A. (1996) The mean-variance approach to portfolio optimization subject to transaction costs. *Journal of the Operations Research Society of Japan*, **39**(1):99–117. doi: 10.15807/jorsj.39.99.

Zhang, Y., Li, X. and Guo, S. (2018) Portfolio selection problems with Markowitz's mean–variance framework: A review of literature. *Fuzzy Optimization and Decision Making*, **17**(2):125–158, Jun 2018. 10.1007/s10700-017-9266-z.