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Regularized penalty method for non-coercive parabolic optimal control problems

by

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Abstract: The application of a proximal point approach to ill-posed convex control problems governed by linear parabolic equations is studied. A stable penalty method is constructed by means of multi-step proximal regularization (only w.r.t. the control functions) in the penalized problems. For distributed control problems with state constraints convergence of the approximately determined solutions of the regularized problems to an optimal process is proved.

Keywords: Proximal point methods, ill-posed parabolic control problems, distributed control, penalty methods.

1. Introduction

In this paper the proximal point approach coupled with the penalty technique is developed for solving ill-posed convex parabolic control problems with state constraints. The investigation is concentrated on problems governed by linear parabolic equations, the objective functional and the sets of admissible controls and states are assumed to be convex.

Usually, convergence of numerical methods for such problems is studied under the additional assumption that the objective functional is strictly (or strongly) convex w.r.t. the control, or that the optimal control possesses the bang-bang property. We refer here to Alt, Mackenroth (1989), Glashoff, Sachs (1977), Hackbusch, Will (1984), Knowles (1982), Lasiecka (1980, 1984), Mackenroth (1982-1983, 1987), Malanowski (1981), Troeltzsch (1987).

The first results, connected with the use of the penalty technique for control problems have been obtained by Lions (1968) and Balakrishnan (1968A, B). For further applications see Lions (1985). Penalization of the state equation permits to handle the control and state variables as independent ones. In Bergounioux (1992, 1994), for convex elliptic and parabolic control problems with state constraints, penalty methods have been used in order to prove the existence of Lagrange multipliers under weak qualification hypotheses. In all these investigations strong convexity of the objective functional was one of the essential conditions.

The paper here deals with convex parabolic control problems without the additional assumptions mentioned above. So, the problem may be non-uniquely solvable, moreover, we do not exclude that the set of optimal controls can be unbounded. Using the scheme of multi-step regularization developed in Kaplan, Tkilu1trmhke (HI94) for abstract convex variational problems, a partial proximal regularization (w.r.t. the control only) of the family of penalized problems is performed. This provides well-posedness of the auxiliary problems as well as weak convergence of their approximately determined solutions to an optimal process and convergence of the corresponding values of the objective functional to the optimal value of the original problem.

In comparison with the Tikhonov regularization, the use of the proximal regularization ensures an essentially better stability of the auxiliary problems and hence a faster convergence of SQP-methods.

For convex elliptic control problems an analogous approach has been realized in Hettich, Kaplan, Tichatschke (1994, 1997). In the last two decades proximal point technique has been successfully developed for solving variational inequalities with monotone operators, including convex optimization problems and saddle-point problems. Eckstein and Bertsekas (1992) have shown a relationship between the proximal point method and the Douglas-Rachford splitting method, pointing out new application fields, especially in mathematical physics. Nevertheless, besides the papers mentioned here, we do not know of the publications in which the proximal point technique was applied to control problems. Therefore, the main point of this paper is the theoretical investigation of the proximal-penalty approach for solving ill-posed optimal control problems by means of decoupling the state and the control.

2. Formulation of the control problem

Let n C JR^n (n :::; 3) be a bounded domain with a boundary *an* of the class C^2, n be locally situated on one side of *an*, and

$$Q = n \times]0,T[, \quad =an \times]0,T[.$$

In the sequel we use the following notation for functional spaces:
L2(0, *T; Z*) - space of functions with range in a Hilbert space *Z*, square integrable on (0, *T*),

$$\|v\|_{L_2(0,T;Z)} = ([\quad llv(t)ll\}dt) ,;, ;$$

ll· lo,Q - norm of an element in L2(0, *T;* L2(n));
C([O, T]; *Z*) - space of continuous functions on [O, T] with range in *Z*,

$$lvl_{C([D,T];Z)} = max_{o::;t::;r} [[v(t) ll_{Z};$$

$H^s(\Omega)$, $H_0^1(\Omega)$ - standard Sobolev spaces, $L_2(\Omega) = H^0(\Omega)$, $\|\cdot\|_{s,\Omega}$ - norm in $H^s(\Omega)$; $\|\cdot\|_{0,s,Q}$ - norm in $L_2(0,T;H^s(\Omega))$ for $s \ge 1$;
$(\cdot,\cdot)_\Omega$ - inner product in $L_2(\Omega)$;
$X \hookrightarrow H$ - continuous embedding of the space X into H.

We consider the parabolic equation

$$\frac{\partial y}{\partial t}(x,t) + Ay(x,t) = u(x,t) \quad \text{a. e. in } Q, \tag{1}$$

$$y(x,0) = 0 \text{ in } \Omega, \tag{2}$$

$$y(x,t) = 0 \text{ on } \Sigma, \tag{3}$$

where the elliptic operator A is given by

$$Ay = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y}{\partial x_j}\right) + a_0(x,t)y, \tag{4}$$

with $a_{ij} \in C^2(\bar{Q})$, $a_0 \in C^2(\bar{Q})$ such that for all $(x,t) \in \bar{Q}$, $\xi \in \mathbb{R}^n$ and some $d_0 > 0$

$$a_0(x,t) \ge 0 \quad \text{and} \quad \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ge d_0 \sum_{i=1}^{n} \xi_i^2. \tag{5}$$

For each $u \in U = L_2(0,T;L_2(\Omega))$ Problem (1)-(3) is uniquely solvable, and its solution y_u belongs to

$$W = \{y \in L_2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) : \frac{\partial y}{\partial t} \in L_2(0,T;L_2(\Omega)), y(x,0) = 0 \text{ in } \Omega\} \tag{6}$$

(see, for instance Lions, Magenes, 1968, vol. 1.). The space W endowed with the norm

$$\|y\|_W = \left(\|y\|_{0,2,\Omega}^2 + \left\|\frac{\partial y}{\partial t}\right\|_{0,Q}^2\right)^{1/2} \tag{7}$$

is a Hilbert space. Moreover (ibid., Theorem 1.3.1),

$$W \hookrightarrow C([0,T];H_0^1(\Omega)),$$

and the operator $\mathcal{T} : \mathcal{T}u = y_u$ is continuous as a mapping from $L_2(0,T;L_2(\Omega))$ into $C([0,T];H_0^1(\Omega))$ (see Lions, Magenes, 1968, vol. 2.).

In order to formulate the control problem we introduce the space

$$Y = \{y \in L_2(0,T;H_0^1(\Omega)) : \frac{\partial y}{\partial t} + Ay \in L_2(0,T;L_2(\Omega)),\ y(x,0) = 0 \text{ in } \Omega\} \tag{8}$$

which coincides algebraically with W: Indeed, regarding the smoothness of a_{ij}, a_0, the inclusion $W \subset Y$ is obvious, and the inclusion $Y \subset W$ is a consequence of the fact that $y_u \in W$ for each $u \in U$.

Using the inequality

$$\|y\|_W \leq c\|y\|_Y \quad \forall y \in Y,$$

which follows from the L_p-estimates for the solutions of parabolic equations (see Ladyshenskaja, Solonnikov, Ural'zewa, 1968, Theorem 4.9.1), one can easily show that the space Y with the norm

$$\|y\|_Y = \left\| \frac{\partial y}{\partial t} + Ay \right\|_{0,Q} \tag{9}$$

is a Hilbert space, too; moreover,

$$Y \hookrightarrow C([0,T]; H_0^1(\Omega)).$$

The approach suggested will be presented with the following model problem

Problem (P):

$$\text{minimize } J(u) = \|y_u(T) - y_d\|^2_{0,\Omega} \text{ subject to } u \in U_{ad},\ y_u \in G,$$

where U_{ad} and G are convex and closed sets in the spaces $L_2(0,T; L_2(\Omega))$ and Y, respectively; $y_d \in L_2(\Omega)$ is a given function and it is supposed that $\{u \in U_{ad} : y_u \in G\} \neq \emptyset$.

Due to the continuity of the mapping

$$\mathcal{T}: L_2(0,T;L_2(\Omega)) \to C([0,T];H_0^1(\Omega))$$

the functional J is continuous on $L_2(0,T; L_2(\Omega))$. Therefore, if U_{ad} is bounded, Problem (P) is solvable, but in general, non-uniquely solvable. If U_{ad} is unbounded, it may happen that the set of optimal controls be empty or unbounded.

We introduce the space

$$\mathcal{Z} = Y \times L_2(0,T; L_2(\Omega)),$$

endowed with the natural norm: For $z = (y, u)$ with $y \in Y$, $u \in L_2(0,T; L_2(\Omega))$,

$$\|z\|_{\mathcal{Z}} = (\|y\|_Y^2 + \|u\|^2_{0,Q})^{1/2}. \tag{10}$$

3. Regularized penalty method (RP-method)

We consider the following

Method (Multi-step regularization)

Let $\{r_i\}$, $\{\epsilon_i\}$, $\{\chi_i\}$ and $\{\delta_i\}$ be positive sequences with

$$\lim_{i\to\infty} r_i = 0,\ \sup_i r_i < 1,\ \lim_{i\to\infty} \frac{\epsilon_i}{\chi_i} = 0,\ \sup_i \chi_i \leq 2,$$

and $u^0 \in U_{ad}$.

Step i: Given $u^{i-1} \in U_{ad}$.

a) Set $u^{i,0} := u^{i-1}$, $s := 1$.
b) Given $u^{i,s-1}$, let

$$(\bar{y}^{i,s}, \bar{u}^{i,s}) = \operatorname{argmin}\{\Psi_{i,s}(y,u) : (y,u) \in G \times U_{ad}\} \tag{11}$$

with

$$\Psi_{i,s}(y,u) = \|y(T)-y_d\|^2_{0,\Omega} + \frac{1}{\epsilon_i}\|\dot{y} + Ay - u\|^2_{Q} + \frac{1}{\cdot}\|u - u^{i,s-1}\|^2_{Q}. \tag{12}$$

Compute an approximation $(y^{i,s}, u^{i,s}) \in G \times U_{ad}$ of $(\bar{y}^{i,s}, \bar{u}^{i,s})$ such that

$$\|(y^{i,s}, u^{i,s}) - (\bar{y}^{i,s}, \bar{u}^{i,s})\|_{\Xi} \le \frac{\cdot}{\chi_i} \tag{13}$$

c) If $\|u^{i,s} - u^{i,s-1}\|_{0,Q} > \delta_i$, set $s := s + 1$ and repeat b).
Otherwise, set $u^i := u^{i,s}$, $s(i) := s$, $i := i + 1$, and repeat Step i.

Of course, the stopping rule (13) is not yet practicable. But, as it will be shown below, the functional $\Psi_{i,s}$ is strongly convex on Ξ. This usually permits to satisfy (13) by means of a stopping criterion of an algorithm, minimizing $\Psi_{i,s}$ on $G \times U_{ad}$.

As it has been mentioned in the introduction, penalization of the state equation enables to handle with y and u as independent variables. But, of course, this complicates the discretization procedure: For instance, applying finite element technique, one has to use elements of higher than first order. Concerning the application of high order finite element approximations to optimal control problems, see Lasiecka (1995) and Hendrickson (1995).

We do not suggest here a penalization of the state constraints. Such kind of penalization was applied, for instance, in Neittaanmaki, Tiba (1995).

4. Convergence of the RP-method

For shortness, in the sequel we will use the following abbreviations:

$$z = (y,u), \quad z^* = (y^*,u^*), \quad z^{i,s} = (y^{i,s},u^{i,s}) \text{ etc.} \tag{14}$$

Let

$$|z| = \left(\|\dot{y} + Ay - u\|^2_{0,Q} + \|u\|^2_{0,Q}\right)^{1/2}. \tag{15}$$

We start with some preliminary statements.

LEMMA 4.1 *On the space Ξ relation (15) defines a new norm $|\cdot|$ which is equivalent to the norm $\|\cdot\|_{\Xi}$:*

$$\frac{1}{\sqrt{3}}\|z\|_{\Xi} \le |z| \le \sqrt{3}\|z\|_{\Xi}. \tag{16}$$

Proof The right hand side inequality in (16) is obvious, and

$$|z|^2 = \|\dot{y} + Ay - u\|^2_{0,Q} + \|u\|^2_{0,Q}$$

$$= \left\|\sqrt{\frac{2}{3}}\left(\frac{dy}{dt}+Ay\right)-\sqrt{\frac{3}{2}}u\right\|^2_{0,Q}+\frac{1}{3}\left\|\frac{dy}{dt}+Ay\right\|^2_{0,Q}$$
$$+\frac{1}{3}\|u\|^2_{0,Q}+\frac{1}{6}\|u\|^2_{0,Q}\geq\frac{1}{3}\|z\|^2_{\Xi}$$

proves the left hand side inequality. □

LEMMA 4.2 *The functional $W_{i,s}$ is continuous and strongly convex on Ξ.*

Proof Due to Lemma 4.1, continuity of $\|y(T)-y_d\|_{0,\Omega}$ on Ξ ensures continuity of $W_{i,s}$. Now, let us prove strong convexity. To this end, we rewrite the functional as follows:

$$W_{i,s}(y,u) = \|y(T)-y_d\|^2_{0,\Omega}+\left(r_i-\frac{\chi_i}{2}\right)\left\|\frac{dy}{dt}+Ay-u\right\|^2_{0,Q}$$
$$+\frac{\chi_i}{2}\left\|\frac{dy}{dt}+Ay-u\right\|^2_{0,Q}+\frac{\chi_i}{2}\|u-u^{i,s-1}\|^2_{0,Q}$$
$$= \Big[\|y(T)-y_d\|^2_{0,\Omega}+\left(r_i-\frac{\chi_i}{2}\right)\left\|\frac{dy}{dt}+Ay-u\right\|^2_{0,Q} \tag{17}$$
$$+\frac{\chi_i}{2}\|u^{i,s-1}\|^2_{0,Q}-\chi_i\int_0^T(u(t),u^{i,s-1}(t))_0\,dt\Big]+\frac{\chi_i}{2}|(y,u)|^2.$$

Because of $r_i < 1$, $\chi_i \leq 2$ the term in the square brackets is a quadratic functional with a non-negative quadratic term in (y,u), hence, it is a convex functional. Therefore, taking into account Lemma 4.1, $W_{i,s}$ is strongly convex on the space Ξ with the norm $\|\cdot\|_\Xi$ or $|\cdot|$. □

The following result is an enlargement of the non-expansivity property of the proximal mapping in the case of partial prox-regularization.

Let Z be a Hilbert space with a norm $\|\cdot\|_Z$; Z_1 be a closed subspace of Z and $P: Z \to Z_1$ be the orthogonal projection operator. We consider the problem

$$\text{minimize } \Phi(z) = a(z,z) - R(z) \text{ subject to } z \in K, \tag{18}$$

where $a(\cdot,\cdot)$ is a continuous, symmetric and positive semi-definite bilinear form on $Z \times Z$, R is a linear, continuous functional on Z and $K \subset Z$ is a convex, closed set. Further, suppose that $b(\cdot,\cdot)$ is a second continuous, symmetric bilinear form on $Z \times Z$ such that

$$0 \leq b(z,z) \leq a(z,z) \text{ for } z \in Z, \tag{19}$$

and, with some $\beta > 0$,

$$b(z,z)+\|Pz\|^2_Z \geq \beta\|z\|^2_Z \text{ for all } z \in Z \tag{20}$$

By

$$|z|^2 = b(z, z) + \|Pz\|^2 \tag{21}$$

another norm is defined on Z, which is equivalent to $\|\cdot\|_Z$ according to the obvious relation

$$(M+1)\|z\|^2 \geq |z|^2 \geq \beta\|z\|^2$$

with $M \geq \sup_{z \neq 0} \frac{b(z,z)}{\|z\|_Z^2}$.

LEMMA 4.3 *For each $a^0 \in Z$ and*

$$a^1 = \arg\min\{\Phi(z) + \tfrac{\chi}{2}\|Pz - Pa^0\|^2 : z \in K\} \tag{22}$$

($\chi \in (0, 2]$ is kept fixed) the following inequalities are true for all $z \in K$:

$$|a^1 - z|^2 - |a^0 - z|^2 \leq -\|Pa^1 - Pa^0\|^2 + \frac{2}{\chi}[\Phi(z) - \Phi(a^1)]$$

and

$$|a^1 - z|_Z \leq |a^0 - z|_Z + \eta(z), \tag{23}$$

with

$$\eta(z) = \begin{cases} \left[\frac{2}{\chi}(\Phi(z) - \Phi(a^1))\right]^{1/2} & \text{if } \Phi(z) > \Phi(a^1) \\ 0 & \text{otherwise} \end{cases}.$$

If, moreover, $\|Pa^1 - Pa^0\|_Z \geq \delta \geq \eta(z)$, then

$$|a^1 - z|_Z \leq |a^0 - z|_Z + \frac{\eta^2(z) - \delta^2}{2|a^0 - z|_Z}. \tag{24}$$

This lemma was proved in Hettich, Kaplan, Tichatschke (1997) for $\chi = 2$. The modification of the proof for arbitrary $\chi \in (0, 2]$ is quite evident.

Now we come back to the control problem.

4.1. The case of a bounded set U_{ad}

Assume there exists a point $\breve{u} \in U_{ad}$ such that

$$y_{\breve{u}} = T\breve{u} \in \operatorname{int} G \ (\text{in } Y). \tag{25}$$

LEMMA 4.4 *Let (y^*, u^*) be an optimal process of Problem (P), $\nu \in (0, \frac{1}{4})$ be an arbitrary number. Suppose that $(\bar{y}^{i,s}, \bar{u}^{i,s})$, $(y^{i,s}, u^{i,s})$ are defined by (11) and (13), where in the function (12) the previous iterate $u^{i,s-1}$ is replaced by an* *arbitrary point $u^{i,s-1} := u \in U_{ad}$. Then there exist two constants $d(\nu)$ and* d_1, *independent of* i, *s* 1, $u^{i,s-1}$, $\{\epsilon_i\}$, $\{r_i\}$ *and* $\{\chi_i\}$, *such that*

$$J_i(y^*, u^*) - J_i(\bar{y}^{i,s}, \bar{u}^{i,s}) < d(\nu) r_i^{1-2\nu} \qquad (26)$$

and

$$|(y^{i,s}, u^{i,s}) - (y^*, u^*)| < d_1, \qquad (27)$$

with

$$J_i(Y, u) = \|y(T) - Yd\|_{,n} + \cdot_i \| \cdot \| + Ay - u\|_{,Q} . \qquad (28)$$

Underline that $|\cdot|$ is defined by (15), and the controlling sequences $\{r_i\}$, $\{\epsilon_i\}$, and $\{\chi_i\}$ are chosen according to the RP-method.

Proof The existence of the points $(\bar{y}^{i,s}, \bar{u}^{i,s})$ and $(y^{i,s}, u^{i,s})$ is guaranteed by Lemma 4.2. Now, we introduce the following notation:

$$\hat{y}^{i,s} = Tu^{i,s}, \quad \hat{z}^{i,s} = (\hat{y}^{i,s}, u^{i,s}), \quad z(u) = (Tu, u),$$

$$\tau_{min} = \begin{cases} \inf_{w \in \partial G} \|Tu - w\|_Y & \text{if } \partial G \neq \emptyset \\ +\infty & \text{if } \partial G = \emptyset \end{cases},$$

$$\tau_{max} = \max_{u \in U_{ad}} \|z(u) - z(u)\|_S,$$

$$w^{i,s} = \arg\min_{v \in G \times U_{ad}} \left\| z^{i,s} - v \right\|_S .$$

Note that $\tau_{max} > \tau_{min}$ if $\{Tu : u \in U_{ad}\} \cap \partial G \neq \emptyset$. In case of $\hat{z}^{i,s} \notin G$ we define the points

$$h^{i,s} \in \{z(u) + \lambda (\hat{z}^{i,s} - z(u)) : \lambda \geq 0\} \cap \{\partial G \times U_{ad}\}$$

and (if $h^{i,s} \neq w^{i,s}$)

$$k^{i,s} \in \{z(u) + \lambda (\hat{z}^{i,s} - w^{i,s}) : \lambda \geq 0\} \cap \{h^{i,s} + \mu (h^{i,s} - w^{i,s}) : \mu \geq 0\}.$$

Obviously, the points $h^{i,s}$ and $k^{i,s}$ are uniquely determined, and, due to the similitude of the triangles with vertices $k^{i,s}, h^{i,s}, z(u)$ and $w^{i,s}, h^{i,s}, \hat{z}^{i,s}$, we obtain

$$\left\| \hat{z}^{i,s} - h^{i,s} \right\|_S = \frac{\|h^{i,s} - z(u)\|_S}{\|k^{i,s} - z(u)\|_S} \left\| \hat{z}^{i,s} - w^{i,s} \right\|_S$$
$$< \frac{\tau_{max}}{\tau_{min}} \left\| \hat{z}^{i,s} - w^{i,s} \right\|_S . \qquad (29)$$

In the standard manner the Gateaux-differentiability of the functional

$$\frac{1}{r_i}\left\|\frac{dy}{dt}+Ay-u\right\|^2_{0,Q}+\frac{\chi_i}{2}\left\|u-u^{i,s-1}\right\|^2_{0,Q}$$

in the space B with the norm $|\cdot|$ can be established. Regarding the definition of $(\bar{y}^{i,s},\bar{u}^{i,s})$, we obtain by means of Proposition II.2.2 in Ekeland, Temam (1976) that, for all $(y,u)\in G\times U_{ad}$,

$$\begin{aligned}
&\|y(T)-y_d\|^2_{0,\Omega}-\|\bar{y}^{i,s}(T)-y_d\|^2_{0,\Omega}\\
&+\frac{2}{r_i}\int_0^T\Big(\frac{d\bar{y}^{i,s}(t)}{dt}+A\bar{y}^{i,s}(t)-\bar{u}^{i,s}(t),\\
&\quad\frac{dy(t)}{dt}+Ay(t)-u(t)-\frac{d\bar{y}^{i,s}(t)}{dt}-A\bar{y}^{i,s}(t)+\bar{u}^{i,s}(t)\Big)_0\,dt\\
&+\chi_i\int_0^T\left(\bar{u}^{i,s}(t)-u^{i,s-1}(t),\,u(t)-\bar{u}^{i,s}(t)\right)_0dt\ \ge 0.
\end{aligned}\tag{30}$$

Setting $y=y^*$, $u=u^*$ in (30), in view of $\frac{dy^*}{dt}+Ay^*-u^*=0$ and the obvious inequality

$$\begin{aligned}
&\|u-u^{i,s-1}\|^2_{0,Q}-\|\bar{u}^{i,s}-u^{i,s-1}\|^2_{0,Q}\\
&\ge 2\int_0^T\left(\bar{u}^{i,s}(t)-u^{i,s-1}(t),\,u(t)-\bar{u}^{i,s}(t)\right)_0dt,
\end{aligned}\tag{31}$$

one can conclude that

$$\frac{1}{r_i}\left\|\frac{d\bar{y}^{i,s}}{dt}+A\bar{y}^{i,s}-\bar{u}^{i,s}\right\|^2_{0,Q}\le\|y^*(T)-y_d\|^2_{0,\Omega}+\frac{\chi_i}{2}\|u^*-u^{i,s-1}\|^2_{0,Q}.\tag{32}$$

Thus,

$$\|\bar{y}^{i,s}\|_Y\le\left(\frac{r_i}{2}\right)^{1/2}\left(\|y^*(T)-y_d\|^2_{0,\Omega}+\frac{\chi_i}{2}\|u^*-u^{i,s-1}\|^2_{0,Q}\right)^{1/2}+\|\bar{u}^{i,s}\|_{0,Q}.\tag{33}$$

Now, regarding the boundedness of U_{ad}, $r_i<1$, $\chi_i\le 2$, $\lim_{i\to+\infty}\frac{\delta_i}{r_i}=0$ and condition (13), inequality (33) yields

$$\|\bar{y}^{i,s}\|_Y<c_1,\quad\|y^{i,s}\|_Y<c_1\tag{34}$$

(throughout the whole paper the constants c_k do not depend on (i,s)). The estimate (27) follows immediately from (34) and the equivalence of the norms $\|\cdot\|_B$ and $|\cdot|$.

Due to (34) and

$$\|y\|_W\le c\|y\|_Y\quad\text{for all } y\in Y,$$

one gets

$$
\|\bar{y}^{s}\|_W < C2, \quad \|\bar{Y}^{i,s}\|_W < C2 \text{ with } C2 = CC1 \tag{35}
$$

Inequality (32) ensures also that

$$
\left\| \frac{d}{dt}\bar{y}^{s} + A J \bar{f}^{s} - \bar{u}^{i,s} \right\|_{\alpha Q} < C3 r_i/2. \tag{36}
$$

Taking into account that $\frac{d\hat{y}^{i,s}}{dt} + A\hat{y}^{i,s} - u^{i,s} = 0$, this leads to

$$
\|\bar{y}^{s} - \hat{y}^{i,s}\|_Y < c3 r_i/2 \tag{37}
$$

and, due to $\left\|\bar{y}^{i,s} - \hat{y}^{i,s}\right\|_Y = \left\|\bar{z}^{i,s} - \hat{\bar{z}}^{i,s}\right\|_\Xi$, we obtain

$$
\left\| \bar{z}^{i,s} - \hat{z}^{i,s} \right\|_S < C3 r_i^{1/2}. \tag{38}
$$

Because $z^{i,s} \in G \times U_{ad}$ the estimate

$$
\left\| w^{i,s} - \hat{z}^{i,s} \right\|_S < C3 r_i^{1/2} \tag{39}
$$

follows from (38) and from the definition of $w^{i,s}$.
Denote by

$$
Z_f = \{ z = (Tu, u) : z \in G \times U_{ad} \}
$$

the set of feasible processes and

$$
\bar{\bar{z}}^{i,s} \equiv \left(\bar{\bar{y}}^{i,s}, \bar{\bar{u}}^{i,s}\right) = \arg\ \min\left\{\|\bar{z}^{i,s} - z\|_\Xi : \ z \in Z_f\right\}.
$$

Because Z_f is convex and closed in Ξ, such an element $\bar{\bar{z}}^{i,s}$ exists.
If $\hat{y}^{i,s} \notin G$ and $w^{i,s} \neq h^{i,s}$, then on account of $h^{i,s} \in Z_f$, we obtain from (29), (38) and (39) that

$$
\begin{aligned}
\|\bar{z}^{i,s} - \bar{\bar{z}}^{i,s}\|_S &\leq \|\bar{z}^{i,s} - h^{i,s}\|_S \\
&\leq \|\bar{z}^{i,s} - \hat{z}^{i,s}\|_S + \|\hat{z}^{i,s} - h^{i,s}\|_S < \left(\frac{\cdot}{\cdot} + 1\right) c3 r_i/2
\end{aligned} \tag{40}
$$

If $\hat{y}^{i,s} \notin G$, but $w^{i,s} = h^{i,s}$, estimate (40) follows from (38), (39) and $\tau_{max} > \tau_{min}$.

In case $\hat{y}^{i,s} \in G$, the inequality

$$
\|\bar{z}^{i,s} - \bar{\bar{z}}^{i,s}\|_S < c3 r_i/2 \tag{41}
$$

is an immediate consequence of (38), and so (40) is also true.

By inserting $y = \bar{y}^{i,s}$, $u = u^{i,s}$ into relation (30), one gets

$$\|\bar{y}^{i,s}(T) - y_d\|_{0,\Omega}^2 + \frac{2}{r_i}\left\|\frac{d\bar{y}^{i,s}}{dt} + A\bar{y}^{i,s} - \bar{u}^{i,s}\right\|_{0,Q}^2$$
$$\leq \left\|\bar{\bar{y}}^{i,s}(T) - y_d\right\|_{0,\Omega}^2 + \chi_i \left\|\bar{u}^{i,s} - \bar{\bar{u}}^{i,s}\right\|_{0,Q} \left\|\bar{u}^{i,s} - u^{i,s-1}\right\|_{0,Q},$$

and hence,

$$\frac{r_i}{2}\left\|\frac{d\bar{y}^{i,s}}{dt} + A\bar{y}^{i,s} - \bar{u}^{i,s}\right\|_{0,Q}^2 \leq \left\|\bar{\bar{y}}^{i,s}(T) - \bar{y}^{i,s}(T)\right\|_{0,\Omega} \left\|\bar{y}^{i,s}(T) + \bar{\bar{y}}^{i,s}(T) - 2y_d\right\|_{0,\Omega}$$
$$+ \chi_i \left\|\bar{u}^{i,s} - \bar{\bar{u}}^{i,s}\right\|_{0,Q} \left\|\bar{u}^{i,s} - u^{i,s-1}\right\|_{0,Q}. \tag{42}$$

Because of $Y \hookrightarrow C([0,T]; H_0^1(\Omega))$, (35), (40), $r_i \leq 1$, $\chi_i \leq 2$ and the boundedness of U_{ad}, inequality (42) leads to

$$\left\|\frac{d\bar{y}^{i,s}}{dt} + A\bar{y}^{i,s} - \bar{u}^{i,s}\right\|_{0,Q} \leq c_4 r_i^{3/4}. \tag{43}$$

Using (43) instead of (36), the estimates (37)-(39) can be improved (w.r.t. the order) and we obtain

$$\left\|\bar{z}^{i,s} - \bar{\bar{z}}^{i,s}\right\|_{S} \leq c_4 r_i^{3/4}, \tag{44}$$

$$\left\|w^{i,s} - \bar{z}^{i,s}\right\|_{S} \leq c_4 r_i^{3/4}. \tag{45}$$

Thus, similar to (40), the inequality

$$\left\|\bar{z}^{i,s} - \bar{\bar{z}}^{i,s}\right\|_{\Xi} \leq \left(\frac{\tau_{max}}{\tau_{min}} + 1\right) c_4 r_i^{3/4}$$

can be established.

A multiple repetition of this operation (using in each step the current estimates) leads to the conclusion that, with arbitrarily fixed $\nu \in (0, \frac{1}{4})$ and some constant $c(\nu)$, the estimates

$$\left\|\frac{d\bar{y}^{i,s}}{dt} + A\bar{y}^{i,s} - \bar{u}^{i,s}\right\|_{0,Q} \leq c(\nu) r_i^{1-\nu},$$

$$\left\|\bar{z}^{i,s} - \bar{\bar{z}}^{i,s}\right\|_{\Xi} \leq \left(\frac{\tau_{max}}{\tau_{min}} + 1\right) c(\nu) r_i^{1-2\nu} \tag{46}$$

are valid uniformly w.r.t. (i, s). Now, from the obvious equality

$$J_i(z^*) - J_i(\bar z^{i,s}) =$$

$$\|y^*(T) - y_d\|^2_{0,\Omega} - \|\bar y^{i,s}(T) - y_d\|^2_{0,\Omega} + \|\bar y^{i,s}(T) - y_d\|^2_{0,\Omega}$$

$$- \|\tilde y^{i,s}(T) - y_d\|^2_{0,\Omega} - \frac{\chi_i}{2}\|\bar u^{i,s} - u^{i,s-1}\|^2_{0,Q}, \tag{47}$$

due to $\|y^*(T) - y_d\|_{0,\Omega} \leq \|\bar y^{i,s}(T) - y_d\|_{0,\Omega}$, (34), (46), and the embedding $Y \hookrightarrow C([0, T]; H_0^1(\Omega))$, we get

$$J_i(z^*) - J_i(\bar z^{i,s}) \leq d(\nu) r_i^{1-2\nu},$$

with $d(\nu)$ independent from (i, s), i.e. estimate (26) is true. □

REMARK 4.1 *It should be emphasized that the inequalities (32), (33) and (42) are established without the assumption of boundedness of U_{ad}. Estimate (29) can be preserved for unbounded U_{ad} if η_{max} is replaced by $\rho \geq \|z(u) - \bar z^{i,s}\|_S$.*

THEOREM 4.1 *Assume that U_{ad} is a bounded set and condition (25) is valid; that $\nu \in (0, \frac{1}{4})$ is a fixed number and that constants $d(\nu)$, d_1 are defined according to Lemma 4.4. Let the positive sequences $\{r_i\}$, $\{\epsilon_i\}$, $\{\chi_i\}$ and $\{\delta_i\}$ in the RP-method satisfy the conditions*

$$\sup_i r_i < 1, \quad \sup_i \chi_i \leq 2, \quad \sum_{i=1}^{\infty} \frac{r_i^{1/2-\nu}}{\chi_i} < \infty, \quad \sum_{i=1}^{\infty} \frac{\epsilon_i}{\chi_i} < \infty \tag{48}$$

and

$$\frac{1}{2d_1}\left(2d(\nu)\frac{r_i^{1-2\nu}}{\chi_i} - \left(\delta_i - \frac{\epsilon_i}{\chi_i}\right)^2\right) + \sqrt{3}\,\frac{\epsilon_i}{\chi_i} < 0, \quad \delta_i > \frac{\epsilon_i}{\chi_i}. \tag{49}$$

Then, for any starting point $u^0 \in U_{ad}$, the RP-method is well-defined, i.e. $s(i) < \infty$ for each i; $\{u^{i,s}\}, \{y^{i,s}\}$ converge weakly in $L_2(Q), Y$ to $\bar u, \bar y$ respectively, where $(\bar y, \bar u)$ is an optimal process for Problem (P); $\{\|y^{i,s}(T) - y_d\|^2_{0,\Omega}\}$ converges to $J(\bar u)$.

Proof Let us assume that $s(i) < \infty$ for $i = 1, \dots, k-1$. Then, starting in step $i = k$ with $u^{k-1} = u^{k-1,s(k-1)}$, due to the definition of $s(i)$, (13) and (49), we conclude

$$\|\bar u^{k,s} - u^{k,s-1}\|_{0,Q} \geq \|u^{k,s} - u^{k,s-1}\|_{0,Q} - \|u^{k,s} - \bar u^{k,s}\|_{0,Q}$$

$$> \delta_k - \frac{\epsilon_k}{\chi_k} > 0 \text{ for } 1 \leq s < s(k).$$

Together with inequality (26) and

$$2d(v\{t\frac{2v}{X_k} - (Dk -)_{X_k}^{2} < 0$$

(cf. (49)), this implies

$$\|u^{k,s} - u^{k,s-1}\|^2_{O,Q} > \frac{2}{X_k}[J_k(z^*) - J_k(z^{k,s})] \text{ for } 1::; s < s(k). \tag{50}$$

Let $z^{1,0} = (Tu^0, u^0)$. Applying (26) and Lemma 4.3 with

$$Z = 3, \quad Z1 = \{z = (_y, u) \text{ E } 3: \ _y = 0\}, \quad 4 > = Jk,$$

$$a(z, z) = (_y(T), i)(T))_n$$
$$+2\frac{-}{r_k}\int_0^T \left(\frac{d_y(t)}{dt} + A_y(t) - u(t)\ ', \frac{d_y(t)}{dt} + Ay(t) - u(t)\right)_n dt,$$

$$b(z, z) = \int^T\left(\frac{d\ t)}{-} + A_y(t)\_ u(t), \frac{d!\ t)}{-} + Ai)(t) - u(t)\right)_n dt,$$

$$.e(z) = 2(_y(T), Y_d)_n, \quad K = GXU_{ad}, \quad a^0 = z^{k,s-1}, \quad z = z^*, \quad x = x_k$$

and $D = D_k - \frac{Es}{Xk}$,
we obtain from (24) and (27) that

$$|z^{k,s} - z^*| < |z^{k,s-1} - z^*| + \frac{1}{2d1}\left(2d(v)\frac{1-2v}{X_k} - (D_k - \ )_{X_k}^2\right) \tag{51}$$

holds for $1 ::; s < s(k)$. Using (13), (16) and (49), inequality (51) yields

$$|z^{k,s} - z^*| - |z^{k,s-1} - z^*| < \frac{1}{2d1}\left(2d(v\{t\frac{2v}{X_k} - (D_k -)_{X_k}^2\right) + v'3_{X_k} < 0. \tag{52}$$

Inequality (52) proves that $s(k) <$ oo, because the middle term in (52) is independent of s.

Now, for $s = s(k)$, the use of Lemma 4.3 with the same data as above, leads to

$$|z^{k,s(k)} - z^*| < |z^{k,s(k)-1} - z^*| + \sqrt{2d(v)\frac{1-2v}{X_k}},$$

hence,

$$|z^{k,s(k)} - z^*| < |z^{k,s(k)-1} - z^*| + \sqrt{2d(v)\frac{1-2v}{X_k}} + v'3_{X_k}. \tag{53}$$

Taking into account that the finiteness of $s(1)$ can be proved quite analogously, we infer that

$$s(i) < \infty \text{ for each } i,$$

and the inequalities (52) and (53) are valid for each k.

In view of (52), (53) and (48), Lemma 2.2.2 from Polyak (1987) ensures the convergence of the sequence $\{\|z^{i,s} - z^*\|\}$, and with regard to (13), (16) and the last inequality in (48), the sequence $\{\|\bar{z}^{i,s} - z^*\|\}$ converges to the same limit.

Suppose that $\{z^{i_k,s_k}\}$, with $s_k > 0$ for each k, converges weakly to $\bar{z} = (\bar{y}, \bar{u}) \in Z$. Due to (46), (48), $\{\bar{z}^{i_k,s_k}\}$ converges weakly to $\bar{z}$, too. Observing the convexity and the closedness of $\bar{Z}_1$ and that $\{z^{i_k,s_k}\} \subset \bar{Z}_1$, we conclude that $\bar{z} \in \bar{Z}_1$.

But Lemma 4.3 yields also

$$\|\bar{z}^{i,s} - z^*\|^2 - \|z^{i,s-1} - z^*\|^2 \leq \frac{2}{\chi_i}\left[\bar{J}_i(z^*) - \bar{J}_i(\bar{z}^{i,s})\right],$$

and by definition of $\bar{J}_i$ (cf (28))

$$\bar{J}_i(z^*) = \|y^*(T) - y_d\|^2_{0,\Omega} = J(u^*), \quad \bar{J}_i(\bar{z}^{i,s}) = \|\bar{y}^{i,s}(T) - y_d\|^2_{0,\Omega},$$

hence,

$$\left|\bar{z}^{i,s} - z^*\right|^2 - \left|z^{i,s-1} - z^*\right|^2 \leq \frac{2}{\chi_i}\left[J(u^*) - \left\|\bar{y}^{i,s}(T) - y_d\right\|^2_{0,\Omega}\right].$$

Due to the convexity and the continuity (in Y), the functional $\|y(T) - y_d\|^2_{0,\Omega}$ is weakly lower semi-continuous. Taking limit in the last inequality w.r.t. the subsequence $\{z^{i_k,s_k}\}$, we obtain

$$J(u^*) \geq \|\bar{y}(T) - y_d\|^2_{0,\Omega},$$

hence, $\bar{z}$ is an optimal process. Finally, Lemma 1 in Opial (1967) ensures weak convergence of both $\{z^{i,s}\}$ and $\{\bar{z}^{i,s}\}$ to $\bar{z} \in Z$. □

4.2. The case of an unbounded set U_{ad}

Now, solvability of Problem (P) is supposed. As before, we assume that there exists a point $\tilde{u} \in U_{ad}$ such that

$$y_{\tilde{u}} = T\tilde{u} \in \operatorname{int} G \ \ (\text{in } Y). \tag{54}$$

Let $z^* = (y^*, u^*)$ be an optimal prozess, $T_{min} = \inf_{w \in \partial G} \|T\tilde{u} - w\|_Y$, and let $\rho_{i,s-1} > 0$ and

$$c_0 \geq \max\{T_{min}, \|y^*(T) - y_d\|_{0,\Omega}, \|z(\tilde{u}) - z^*\|_S\}$$

be given. To simplify the further consideration, we suppose additionally that $0 < \underline{\chi} \leq \chi_i$ holds in the PR-method.

LEMMA 4.5 *Let* $\nu \in (0, \frac{3}{4})$ *and*

$$u^{i,s-1} \in \{u \in \mathcal{U}_{ad} : \|u - u^*\|_{0,Q} \le \rho_{i,s-1}\} \tag{55}$$

be arbitrarily chosen, and $(\bar{y}^{i,s}, \bar{u}^{i,s})$ *be defined by (11) with this element* $u^{i,s-1}$. *Then, there exist constants* d *and* $d(\nu)$, *independent of* $i, s \ge 1$, $\{r_i\}$, $\{\epsilon_i\}$, $\{\chi_i\}$ *and* $u^{i,s-1}$, *such that*

$$J_i(y^*, u^*) - J_i\left(\bar{y}^{i,s}, \bar{u}^{i,s}\right) \le d(c_0 + \rho_{i,s-1})^3 \left((c_0 + \rho_{i,s-1}) r_i^{1/2} + 1\right) r_i^{1/2}. \tag{56}$$

Moreover, in case $G = Y$,

$$J_i(y^*, u^*) - J_i(\bar{y}^{i,s}, \bar{u}^{i,s}) \le d(\nu)(c_0 + \rho_{i,s-1})^2 r_i^{1-2\nu} \tag{57}$$

is valid.

Proof By inserting $y = y^*$, $u = u^*$ into the inequalities (30), (31), which hold true also if $\mathcal{U}_{ad}$ is unbounded, we obtain

$$\frac{1}{r_i}\left\|\frac{d\bar{y}^{i,s}}{dt} + A\bar{y}^{i,s} - \bar{u}^{i,s}\right\|^2_{0,Q} + \|\bar{y}^{i,s}(T) - y_d\|^2_{0,\Omega}$$
$$+ \frac{\chi_i}{2}\|\bar{u}^{i,s} - u^{i,s-1}\|^2_{0,Q} \le c_0^2 + \rho^2_{i,s-1},$$

hence

$$\left\|\frac{d\bar{y}^{i,s}}{dt} + A\bar{y}^{i,s} - \bar{u}^{i,s}\right\|_{0,Q} \le (c_0 + \rho_{i,s-1}) r_i^{1/2}, \tag{58}$$

$$\|\bar{u}^{i,s} - u^{i,s-1}\|_{0,Q} \le \sqrt{\frac{2}{\chi_i}}(c_0 + \rho_{i,s-1}), \tag{59}$$

$$\|\bar{y}^{i,s}(T) - y_d\|_{0,\Omega} \le c_0 + \rho_{i,s-1} \tag{60}$$

and

$$\|\bar{u}^{i,s} - u^*\|_{0,Q} \le \rho_{i,s-1} + \sqrt{\frac{2}{\chi_i}}(c_0 + \rho_{i,s-1}). \tag{61}$$

For $\hat{f}^{i,s} = \mathcal{T}\bar{u}^{i,s}$, $\hat{z}^{i,s} = (\hat{y}^{i,s}, \bar{u}^{i,s})$, estimate (58) yields

$$\|\bar{f}^{i,s} - \hat{f}^{i,s}\|_Y \le (c_0 + \rho_{i,s-1}) r_i^{1/2} \tag{62}$$

and

$$\left\|\bar{z}^{i,s} - \hat{z}^{i,s}\right\|_3 \le (c_0 + \rho_{i,s-1}) r_i^{1/2}. \tag{63}$$

Due to $r_i < 1$ and (58), the inequality

$$\left\|\frac{d\bar{y}^{i,s}}{dt} + A\bar{y}^{i,s} - \frac{dy^*}{dt} - Ay^*\right\|_{0,Q} - \|\bar{u}^{i,s} - u^*\|_{0,Q} \le \left\|\frac{d\bar{y}^{i,s}}{dt} + A\bar{y}^{i,s} - \bar{u}^{i,s}\right\|_{0,Q} \le c_0 + \rho_{i,s-1}$$

holds. Together with $\underline{\chi} \le \chi_i \le 2$ and (61) this leads to

$$\left\|\frac{d\bar{y}^{i,s}}{dt} + A\bar{y}^{i,s} - \frac{dy^*}{dt} - Ay^*\right\|_{0,Q} < 3\sqrt{\frac{2}{\underline{\chi}}}(c_0 + \rho_{i,s-1})$$

and

$$\|\bar{z}^{i,s} - z^*\|_3 \le \frac{6}{\sqrt{\underline{\chi}}}(c_0 + \rho_{i,s-1}). \tag{64}$$

Now, from (63), (64) and the inequality

$$\|z^{i,s} - z(\bar{u})\|_3 \le \|z^{i,s} - \bar{z}^{i,s}\|_3 + \|\bar{z}^{i,s} - z^*\|_3 + \|z^* - z(\bar{u})\|_3$$

we obtain

$$\|z^{i,s} - z(\bar{u})\|_3 \le \|z^* - z(\bar{u})\|_3 + (c_0 + \rho_{i,s-1}),$$

and regarding the choice of c_0,

$$\|z^{i,s} - z(\bar{u})\|_3 \le \frac{10}{\sqrt{\underline{\chi}}}(c_0 + \rho_{i,s-1}). \tag{65}$$

Let us define

$$h^{i,s} \in \{z(\bar{u}) + \lambda (z^{i,s} - z(\bar{u})) : \lambda \ge 0\} \cap \{\partial G \times U_{ad}\}$$

and

$$w^{i,s} = \arg\min_{v \in G \times U_{ad}} \|z^{i,s} - v\|_3 .$$

In case $z^{i,s} \notin G$ and $w^{i,s} \ne h^{i,s}$, the inequalities (65), (29) and Remark 4.1 imply that

$$\|z^{i,s} - h^{i,s}\|_3 \le \frac{1}{\tau_{min}} \frac{10}{\sqrt{\underline{\chi}}}(c_0 + \rho_{i,s-1}) \|z^{i,s} - w^{i,s}\|_3 . \tag{66}$$

Because $z^{i,s} \in G \times U_{ad}$, estimate (63) yields

$$\|w^{i,s} - z^{i,s}\|_3 \le (c_0 + \rho_{i,s-1}) r_i^{1/2}. \tag{67}$$

Let, as before, $\mathcal{Z} = \{z = (Tu,u): z \in G \times U_{ad}\}$,

$$\bar z^{i,s} = \arg\min\{\|\hat z^{i,s} - z\|_\Xi : z \in \mathcal{Z}\}. \tag{68}$$

If $y^{i,s} \notin G$ and $w^{i,s} \neq h^{i,s}$, then in view of $h^{i,s} \in \mathcal{Z}$, we obtain from (63), (66), (67), $\chi_i \le 2$ and $\tau_{\min} \le c_0$ that

$$\begin{aligned}
\|\hat z^{i,s} - \bar z^{i,s}\|_\Xi &\le \|\hat z^{i,s} - h^{i,s}\|_\Xi \le \|\hat z^{i,s} - \tilde h^{i,s}\|_\Xi + \|\tilde z^{i,s} - \hat z^{i,s}\|_\Xi \\
&\le \frac{1}{\tau_{\min}}\Big(\frac{10}{\gamma K}(c_0 + \rho_{i,s-1})^2 r_i\Big)^{1/2} + (c_0 + \rho_{i,s-1})h^{1/2} \\
&= \frac{1}{\tau_{\min}}\Big[\Big(\frac{10}{\gamma K}(c_0 + \rho_{i,s-1})^2 r_i\Big)^{1/2} + \tau_{\min}(c_0 + \rho_{i,s-1})r_i^{1/2}\Big] \\
&\le \frac{1}{\tau_{\min}}\Big(\frac{12}{\gamma K}(c_0 + \rho_{i,s-1})^2 r_i\Big)^{1/2}.
\end{aligned} \tag{69}$$

If $y^{i,s} \notin G$ but $w^{i,s} = h^{i,s}$, then

$$\|\hat z^{i,s} - \bar z^{i,s}\|_\Xi \le \|\hat z^{i,s} - w^{i,s}\|_\Xi \le \|\hat z^{i,s} - \tilde z^{i,s}\|_\Xi + \|\tilde z^{i,s} - w^{i,s}\|_\Xi,$$

and regarding $\chi_i \le 2$ and $\tau_{\min} \le c_0$, estimate (69) is a consequence of (63) and (67). Finally, if $\hat\xi^{i,s} \in G$ then the inequality

$$\|\hat z^{i,s} - \bar z^{i,s}\|_\Xi < (c_0 + \rho_{i,s-1})r_i^{1/2} \tag{70}$$

follows immediately from (63) and the definitions of $\hat z^{i,s}$ and $\bar z^{i,s}$, proving (69), as well.

From inequality (42) (see Remark 4.1), in view of $\chi_i \le 2$ and the embedding $Y \hookrightarrow C([0,T];H_0^1(D))$, one can conclude that

$$\begin{aligned}
&\frac{2}{\chi_i}\Big\|\frac{d\hat y^{i,s}}{dt} + A\hat y^{i,s} - \hat u^{i,s}\Big\|^2_{0,Q} \\
&\le c_1\|\hat y^{i,s} - \hat y^{i,s}\|_Y\big(c_1\|\hat y^{i,s} - y^{i,s}\|_Y + 2\|\hat y^{i,s}(T) - y_d\|_{0,\Omega}\big) \\
&\quad + 2\|\hat u^{i,s} - u^{i,s}\|_{0,Q}\|u^{i,s} - u^{i,s-1}\|_{0,Q}.
\end{aligned} \tag{71}$$

In case $G = Y$ (Problem (P) without state constraints), due to the inclusion $\hat\xi^{i,s} \in G$ we can use estimate (70), and together with (59), (60) and (71) this yields

$$\Big\|\frac{d\hat y^{i,s}}{dt} + A\hat y^{i,s} - \hat u^{i,s}\Big\|_{0,Q} \le c_2(c_0 + \rho_{i,s-1})r_i^{1/4}. \tag{72}$$

Now, taking into account the inequality

$$\|\hat z^{i,s} - \bar z^{i,s}\|_\Xi \le \|\hat z^{i,s} - \bar z^{i,s}\|_\Xi,$$

which follows from ff^{s} E G and the definition of $\bar{y}^{s}$, and continuing as in the proof of Lemma 4.4 (starting below inequality (43)), estimate (57) can be established.

In the general case (Problem (P) with state constraints), from (47) and the inequality

$$\|y^*(T) - y_d\|_{0,\Omega} \leq \|\bar{y}^{i,s}(T) - y_d\|_{0,\Omega},$$

we obtain

Ji(z*) - Ji(zi,s) S ll:if^3(T) - Ydll6,n - llff^3(T) -ydll6,n

S llzf^3(T) - yi,s(T)llo,n (11Yi,s(T) - Yi,s(T)llo,n + 211:ili,s(T) - Ydllo,n),

and due to (60),

Ji(z*) - Ji(zi,s)

S IIYi,s (T) - Yi,s (T) llo,n (11Yi,s (T) - Yi,s (T) llo,n + 2(co + Pi,s-i)) •

Now, estimate (56) results immediately from (69), the embedding Y .-- C([0, T]; HJ(D)) and ri < 1. D

In the following statement the parameter v E (0, ¼) is arbitrarily chosen and the constants d and $d(v)$ are defined according to Lemma 4.5.

THEOREM 4.2 *Suppose that condition (54) is fulfilled. Let* u^0 E Uad, $z^{1,0}$ = (Tu^0,u^0), p1 > lz^{1,0} - z*I, *and* a E (0, ¼)- *Assume that the sequence {pi} is defined recursively by*

Pi+l = P i + $\sqrt{\frac{2d}{Xi}}$ [(co+ P i h$^{1/2}$ + 1] (co+ Pi)^3 ri$^{1/4}$ + v3 $\frac{Ei}{Xi}$ (73)

(in the general case), or

$$\rho_{i+1} = \rho_i + \sqrt{\frac{2d(\nu)}{\chi_i}}\,(c_0 + \rho_i)\, r_i^{1/2-\nu} + \sqrt{3}\frac{\epsilon_i}{\chi_i} \tag{74}$$

(if G = Y). Moreover, assume that

(i) *in the general case the controlling parameters of the RP-method satisfy the conditions*

supiri < 1, infixi > 0, SUPiXi S 2, supir; Pi S d2 < oo (75)

and

$\sum_{i=1}^{\infty}$ Lrl- < oo, $\sum_{i=1}^{\infty}$ LEi < oo, (76)

and for each i,

2,- [! (co+ Pi)^3 ((co+ Pi)rY^2 + 1) r;^{12} - (Di - ;:) $^{2]}$ + V3;: 0, (77)

8i > .s__. Xi'

(ii) if $G = Y$, *the controlling parameters satisfy*

$$supiri < 1, \quad infiXi > 0, \quad SUPiXi \ S\ 2, \tag{78}$$

and

$$\sum_{i=1}^{\infty} r_i^{1}{;2\text{-}v} < \infty, \quad \sum_{i=1}^{\infty} Ei < \infty, \tag{79}$$

and for each i,

$$\frac{2}{2pi}\ [\frac{2d(v)}{Xi}\ (co+ p_i)^2\ \frac{r\}\text{-}2v}{Xi}\ -\ (8_i\ -\ \frac{2)}{Xi}^2\ ] + \frac{J32}{Xi} < 0, \quad Di> \frac{2}{Xi} \tag{80}$$

Then, s(i) $< \infty$ *holds for each* i; *the sequences* $\{u^{i,s}\}, \{y^{i,s}\}$ *converge weakly in* $L_2(Q), Y$ *to* u, fj *respectively, where* (y, u) *is an optimal process for Problem (P);* $\{IIY^{i,s}(T) \text{-} Ydl16,o\}$ *converges to* $J(u)$.

Proof We show that the statement of the theorem holds true if {pi} is defined by (73) and condition (i) is valid. The proof is quite similar for the case of (74) and (ii).

Suppose that

$$P_k > lz^* - z^{k,o} l \tag{81}$$

for some k. At first, let us prove that $s(k) < \infty$ and

$$P_k > lz^* - z^{k,s} l \text{ for } 1 \ S\ s < s(k).$$

Applying Lemma 4.5 (with $i = k$, $s = 1$ and $P_{k\ o} = P_k$) and Lemma 4.3 (with the same data as in Theorem 4.1, see the proof of Theorem 4.1 starting after inequality (50)), we obtain from (24), (56) and (81) that for $s(k) > 1$

$$lz^{k,1} - z^*I$$

$$< lz^{k0} - z^*I + \frac{1}{2p_k}\ [2d(c_o + P_k)_3 ((c_o + P_k)r_k^{1/2} + 1)\ \frac{r!^{12}}{X_k}(8_k - \frac{Bk}{X_k})^2] .$$

But, in view of (13), (16) and (77), this leads to

$$lz^{k,l} - z^*I - lz^{k,O} - z^*I < 0.$$

Thus, $lz^{k,l} \_ z^*I < P_k$, and continuing in the same manner for $2 \ S\ s < s(k)$, one can establish the inequalities

$$lz^{k,s} - z^*I - lz^{k,s-1} - z^*I$$

$$< \frac{1}{2p_k}\ [2d(c_o + P_k)_3 ((c_o + P_k)r_k^{1/2} + l)\ \frac{r!^2}{X_k}(8_k - \frac{Bk}{X_k})^2] \tag{82}$$

$$+ \frac{V3}{X_k} < 0$$

and

$$lz^{k,s} - z^*I < P_k . \tag{83}$$

Inequality (82) guarantees that $s(k) < \infty$. For $s = s(k)$, the Lemmas 4.3 (relation (23)) and 4.5 yield

$$|z^{k,s(k)} - z^*| - |z^{k,s(k)-1} - z^*|$$
$$< \sqrt{\frac{2d}{\chi_k}\left((c_0+\rho_k)r_k^{1/2}+1\right)(c_0+\rho_k)^3\, r_k^{1/4}} + \sqrt{3}\frac{\epsilon_k}{\chi_k}, \qquad (84)$$

hence,

$$|z^{k,s(k)} - z^*| < \rho_k + \sqrt{\frac{2d}{\chi_k}\left((c_0+\rho_k)r_k^{1/2}+1\right)(c_0+\rho_k)^3\, r_k^{1/4}} + \sqrt{3}\frac{\epsilon_k}{\chi_k},$$

and (73) implies

$$|z^{k+1,0} - z^*| < \rho_{k+1}.$$

Therefore, $s(i) < \infty$ for all i, and

$$|z^{i,s} - z^*| < \rho_i \text{ for each } i \text{ and } 0 \le s < s(i)$$

is valid.

From (75) and (73) it follows

$$\rho_{i+1} \le \rho_i + c_3(c_0+\rho_i)r_i^{\frac{1}{4}-\alpha} + \sqrt{3}\frac{\epsilon_i}{\chi_i},$$

and due to (76) and $\inf_i \chi_i > 0$, Lemma 2.2.2 in Polyak (1987) guarantees that $\rho_i \to \bar{\rho} < \infty$ monotonously. Therefore,

$$|z^{i,s} - z^*| < \bar{\rho} \text{ for all } (i,s).$$

Replacing ρ_k by $\bar{\rho}$ in (84), one can obtain for each i

$$|z^{i,s(i)} - z^*| - |z^{i,s(i)-1} - z^*| < \sqrt{\frac{2d}{\chi_i}\left((c_0+\bar{\rho})+1\right)(c_0+\bar{\rho})^3\, r_i^{1/4}} + \sqrt{3}\frac{\epsilon_i}{\chi_i}. \qquad (85)$$

Now, taking into account (76), (81) and (85), convergence of the sequence $\{|z^{i,s} - z^*|\}$ follows from the lemma just mentioned.
The rest of the proof is the same as for Theorem 4.1. □

The question which optimal solution is obtained in the limit by using the RP-method (in correspondence with the Theorems 4.1 and 4.2) is open even in the case of $\epsilon_i \equiv 0$.

REMARK 4.2 *The conditions (73), (75)-(77), defining the controlling parameters according to Theorem 4.2, are compatible. In particular, these parameters can be chosen as follows:*

- *take {xi}, {Ei}, {ra according to {75), {76), {except for the condition supirf^a Pi < d2);*
- *choose r1 r_ such that rf^a P1 d2;*
- *with E1, r1 and p1 given as above, calculate 81 from {77) and p2 via {73);*
- *define r2 r such that r ^a p2 < d2 etc.*

REMARK 4.3 *The conditions of Theorem 4.1 as well as Theorem 4.2 {in case G = Y) permit a slow change of the parameters Ei, ri and Xi: for instance, it is possible to take*

$$0 < \quad Xi \quad 2 \text{ and } ri = qf, \; E = q \quad \text{with arbitrary } q1, \; q2 \; E \; (0,1),$$

and then to choose Di according to {49) or (74), {80). However, the calculation of d(v) may raise difficulties.

There are no principal problems in ex.tending these considerations to other objective functions of the form *J(u)* = JJCy$_u$ - Yd|1i (where 1i is a Hilbert space on n, *Q* or , *C* E *£(Y,* 1i) and Yd E H) if JJCyJJrt constJJyJJy for all y E *Y.*

In a similar manner convergence of the RP-method for Problem (P) controlled by (1) with nonhomogeneous initial and Dirichlet conditions can be established, provided the solution of the parabolic equation with u E £$_2$(0, *T;* £$_2$(0)) is sufficiently smooth (like in the space *W)* and satisfies the L_p-estimates in Ladyshenskaja, Solonnikov, Ural'zewa (1968).

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