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Stabilization of multivariable linear time-invariant systems by proportional-plus-derivative state feedback*

by

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Abstract: This paper is devoted to the study of the problem of stabilization by proportional-plus-derivative state feedback for multivariable linear time-invariant systems. In particular, explicit necessary and sufficient conditions are established for the stability of a closed-loop system obtained by proportional-plus-derivative state feedback from the given multivariable linear time-invariant system. A procedure is given for the computation of stabilizing proportional-plus-derivative state feedback. Our approach is based on properties of real and polynomial matrices.

Keywords: stabilization, proportional-plus-derivative state feedback, linear time-invariant systems

1. Introduction

Proportional-plus-derivative feedback control of multivariable linear-time invariant systems has quite a long history. Seraji and Tarokh (1977) did introduce the problem of pole assignment by proportional-plus-derivative output feedback for multivariable linear time-invariant systems. In particular, in Seraji and Tarokh (1977) a method was given for placing up to max($2m$, $2p$) poles of the closed-loop system, where m and p are the numbers of inputs and outputs, respectively, of the closed-loop system. A method for assigning up to max($2m+p$-1, $2p+m$-1) poles of a closed-loop system by the proportional-plus-derivative output feedback has been presented in Seraji (1980). In Tarokh and Seraji (1977) a new class of multivariable output feedback controllers was introduced, consisting of proportional-plus multiple derivative terms. It is shown that all poles

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of the closed-loop system can be placed at desired positions provided a sufficient number of derivative terms are available, which can be calculated using the procedure, described in Tarokh and Seraji (1977).

It was demonstrated in Abdelaziz (2015) that controllability is the sufficient condition for the solution of pole assignment problem by proportional-plus-derivative feedback for single-input-single-output linear time-invariant systems.

In Kiritsis (2022) the explicit necessary and sufficient conditions have been established for a given polynomial with real coefficients to be characteristic polynomial of a closed-loop system obtained by proportional-plus-derivative state feedback from the given multivariable linear time-invariant system. Furthermore, a procedure is given for the calculation of proportional-plus-derivative state feedback, which places the poles of the closed-loop system at desired locations. It is also proven in Kiritsis (2022) that every multivariable linear time-invariant controllable system is stabilizable by proportional-plus-derivative state feedback.

In Chu and Datta (2006) and in Henrion, Sebek and Kucera (2005) the robust pole assignment problem for second order systems by proportional plus derivative state feedback has been investigated. The so-called eigenstructure assignment problem (simultaneous assignment of poles and eigenvectors of the closed-loop system) by proportional-plus-derivative feedback for multivariable linear-time invariant systems has attracted much attention over the last two decades. For more details we refer the reader to Duan and Wang (2004, 2005), Abdelaziz and Valasek (2005), Chu (2002), Duan and Liu (2003), Wang, Qiang and Duan (2006), and Abdelaziz (2017), as well as the references given therein. A control design method for multivariable linear time-invariant systems using proportional-plus-derivative output feedback is presented in Haraldsdottir, Kabamba and Ulsoy (1990). The study of Chu and Malabre (2002) concerns the row-by-row decoupling problem of multivariable linear time-invariant systems by proportional-plus-derivative state feedback and they developed a numerically reliable method for the computation of the desired feedback matrices. The proportional-plus-derivative state feedback design methods have been extensively studied over the last forty-five years and many papers have been published in this area; for a more complete list of references, we refer the reader to Abdelaziz (2017). To the best of our knowledge, the stabilization problem by proportional-plus-derivative state feedback for multivariable linear time-invariant systems, in its full generality, is an open problem yet. This motivates the present study. In this paper, the explicit necessary and sufficient conditions are established for the solution of the stabilization problem by proportional-plus-derivative state feedback for multivariable linear time-invariant systems. Furthermore, a procedure is given for the computation of stabilizing proportional-plus-derivative state feedback.

2. Problem formulation

Consider a multivariable linear time-invariant system described by the following state-space equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(\mathrm{t}) + \mathbf{B}\mathbf{u}(\mathrm{t}) \tag{1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are real matrices of sizes nxn, nxm, respectively, $\mathbf{x}(t)$ is the state vector of dimensions $nx1$ and $\mathbf{u}(t)$ is the vector of inputs of dimensions $mx1$. Without any loss of generality we assume that

$$rank[\mathbf{B}] = m. \tag{2}$$

Consider the control law

$$\mathbf{u}(t) = -\mathbf{F}\mathbf{x}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{v}(t), \tag{3}$$

where $\mathbf{F}$ and $\mathbf{D}$ are real matrices of sizes mxn and mxn, respectively, and $\mathbf{v}(t)$ is the reference input vector of size $mx1$. By applying the proportional-plus-derivative state feedback (3) to the system (1) the state-space equations of closed–loop system become

$$[\mathbf{I} - \mathbf{BD}]\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BF})\mathbf{x}(t) + \mathbf{B}\mathbf{v}(t). \tag{4}$$

Let R be the field of real numbers. Also, let R$[s]$ be the ring of polynomials with coefficients in R. The stabilization problem considered in this paper can be stated as follows:

Does there exist a proportional-plus-derivative state feedback (3) such that

$$det[(\mathbf{I} - \mathbf{BD})\mathrm{s} - \mathbf{A} + \mathbf{BF}] = \mu c(s) \tag{5}$$

where $c(s)$ is a strictly Hurwitz polynomial (it has no root λ such that $\mathrm{Re}\lambda \geq 0$) over R$[s]$ of degree n, and μ is a finite nonzero real number, and for this formulation, conditions for the existence and a procedure for the computation of matrices $\mathbf{F}$ and $\mathbf{D}$ should be given.

3. Basic concepts and preliminary results

This section contains the lemmas, which are needed to prove the main results of this paper, and also some basic notions from linear control theory that are used throughout the paper. A matrix whose elements are polynomials over R$[s]$ is termed polynomial matrix. A polynomial matrix $\mathbf{U}(s)$ over R$[s]$ of dimensions kxk is said to be unimodular if and only if its inverse exists and is also a polynomial matrix. Every polynomial matrix $\mathbf{W}(s)$ of dimensions pxm with $rank[\mathbf{W}(\mathrm{s})] = r$, can be expressed as

$$\mathbf{W}(s) = \mathbf{U}_1(s)\mathbf{M}(s)\mathbf{U}_2(s). \tag{6}$$

The polynomial matrices $\mathbf{U}_1(s)$ and $\mathbf{U}_2(s)$ are unimodular and the matrix $\mathbf{M}(s)$ is given by

$$\mathbf{M}\left(s\right) = \begin{bmatrix} \mathbf{M}_r(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{7}$$

The matrix $\mathbf{M}_r(s)$ of size rxr in (7) is given by

$$\mathbf{M}_r(\mathrm{s}) = diag[a_1(s), a_2(s), \ \ldots. \ , a_r(s)], \tag{8}$$

where the polynomials $\mathrm{a}_i(s)$ for $i = 1, 2, .., r$, are termed invariant polynomials of $\mathbf{W}(s)$ and have the following property

$$a_i(s) \ divides \ a_{i+1}(s), \ \text{for } i = 1, 2, ...r - 1. \tag{9}$$

The relationship (6) is called Smith-McMillan form of the matrix $\mathbf{W}(s)$ over R[s]. Let $\mathbf{A}(s)$ be a polynomial matrix over R[s] if there are matrices $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ over R[s] such that

$$\mathbf{A}(s) = \mathbf{P}(s)\mathbf{Q}(s). \tag{10}$$

Then, the polynomial matrix $\mathbf{P}(s)$ over R[s] is termed left divisor of $\mathbf{A}(s)$. Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$, be matrices over R[s], if

$$\mathbf{A}(s) = \mathbf{D}(s)\mathbf{M}(s) \tag{11}$$

$$\mathbf{B}(s) = \mathbf{D}(s)\mathbf{N}(s) \tag{12}$$

for polynomial matrices $\mathbf{M}(s)$, $\mathbf{N}(s)$ and $\mathbf{D}(s)$ over R[s], then $\mathbf{D}(s)$ is termed common left divisor of polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$.

The greatest common left divisor of two polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ is a common left divisor, which is a right multiple of every common left divisor.

The material on polynomial matrices and their properties, presented in this section, was obtained primarily from the references Wolowich (1974), Antsaklis and Michel (2006), Kucera (1991) and Rosenbrock (1970).

Definition 1 *Let* $\mathbf{V}(s)$ *be a non-singular matrix over* $R[s]$, *of size* mxm. *Also let* $c_i(s)$ *for* $i = 1, 2, \cdots, m$ *be the invariant polynomials of polynomial matrix* $\mathbf{V}(s)$. *Then the zeros of the polynomial matrix* $\mathbf{V}(s)$ *are the roots of the polynomials* $c_i(s)$ *for* $i = 1, 2, ..., m$ *taken all together, see Rosenbrock (1973).*

Definition 2 *Let* $\mathbf{V}(s)$ *be a non-singular matrix over* $R[s]$*, of size mxm. The polynomial matrix* $\mathbf{V}(s)$ *is said to be strictly Hurwitz if and only det[*$\mathbf{V}(s)$*] is a strictly Hurwitz polynomial or, alternatively, if and only if all zeros of the matrix* $\mathbf{V}(s)$ *have negative real parts.*

Remark 1 *Let* $\mathbf{V}(s)$ *be a non-singular and strictly Hurwitz matrix over* $R[s]$*, of size* $m \times m$*. Also let* Φ *be a non-singular matrix over R of size* $m \times m$*. Since, by assumption, the matrix* $\mathbf{V}(s)$ *over* $R[s]$*, is strictly Hurwitz, from Definition 2 it follows that the polynomial det[*$\mathbf{V}(s)$*] is strictly Hurwitz. Since det[*$\mathbf{V}(s)$*] is a strictly Hurwitz polynomial,* $det[\,\Phi\mathbf{V}(s)] = det[\Phi]det[\mathbf{V}(s)]$ *is also a strictly Hurwitz polynomial and therefore, by Definition 2, the matrix* $\Phi\mathbf{V}(s)$ *is strictly Hurwitz.*

Definition 3 *The matrix* $\mathbf{A}$ *over R matrices of size nxn, is said to be Hurwitz stable if and only if all eigenvalues of the matrix* $\mathbf{A}$ *have negative real parts or, alternatively, if and only if the characteristic polynomial of matrix* $\mathbf{A}$ *is a strictly Hurwitz polynomial.*

Definition 4 *Let* $\mathbf{A}$ *and* $\boldsymbol{B}$ *be matrices over R matrices of size nxn, nxm, respectively. Then, an eigevalue* λ *of* $\mathbf{A}$ *is called a controllable eigenvalue of the pair (*$\mathbf{A}\ \mathbf{B}$*), see Tredelman, Stoorvogel and Hautus (2002), if and only if*

$$rank[\mathbf{I}\lambda - \mathbf{A}, \mathbf{B}] = n.$$

The following definition follows directly from Definition 4:

Definition 5 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices over R matrices of size nxn, nxm, respectively. Then an eigevalue* λ *of* $\mathbf{A}$ *is called an uncontrollable eigenvalue of the pair* $(\mathbf{A},\, \mathbf{B})$ *if and only if*

$$rank[\mathbf{I}\lambda - \mathbf{A}, \mathbf{B}] < n.$$

Definition 6 *Let* $\mathbf{A}$ *and* $\boldsymbol{B}$ *be matrices over R matrices of size nxn, nxm, respectively. Then the pair* $(\mathbf{A},\, \mathbf{B})$ *is said to be stabilizable if and only if there exists a matrix* $\mathbf{F}$ *over R of size mxn such that the matrix* $[(\mathbf{A}+\mathbf{BF}]$ *is Hurwitz stable, see Wonham (1967).*

Lemma 1 *Let* $\mathbf{V}(s)$ *be a non-singular matrix over* $R[s]$*, of size mxm. A complex number* ξ *is a zero of polynomial matrix* $\mathbf{V}(s)$ *if and only if the following condition holds:*

$$rank[\mathbf{V}(\xi)] < m.$$

Proof See Pugh and Ratcliffe (1979). ■

LEMMA 2 *Let* $\mathbf{A}(s)$ *and* $\mathbf{B}(s)$ *be matrices over* $R[s]$ *of sizes* n x n, n x m, *respectively. Let* $\mathbf{K}(s)$ *be a unimodular matrix such that*

$$[\mathbf{A}(s), \mathbf{B}(s)] = [\mathbf{V}(s)0]\mathbf{K}(s).$$

Then the polynomial matrix $\mathbf{V}(s)$ *is the greatest common left divisor of polynomial matrices* $\mathbf{A}(s)$ *and* $\mathbf{B}(s)$.

PROOF See Antsaklis and Michel (2006). ■

The following lemma was first published by Wonham (1967), and can be also found in any standard book on linear control theory, see, e.g., Kucera (1991).

LEMMA 3 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices over* R *matrices of size nxn, nxm, respectively. Then the pair* $(\mathbf{AB})$ *is controllable if and only if for every monic polynomial* $c(s)$ *over* $R[s]$ *of degree* n *there exists a matrix* $\mathbf{F}$ *over* R *of size* m x n, *such that the matrix* $[\mathbf{A} + \mathbf{BF}]$ *has characteristic polynomial* $c(s)$.

The standard decomposition of uncontrollable systems, which is given in the subsequent lemma was first published by Kalman (1963), and can be also found in any standard book on linear control theory, see, e.g., Antsaklis and Michel (2006) or Kucera (1991),

LEMMA 4 *Let the pair* $(\mathbf{AB})$ *be uncontrollable and* $\mathbf{B}$ *not zero. Then there exists a non-singular matrix* $\mathbf{T}$ *such that*

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}, \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}.$$

The pair $(\mathbf{A}_{11}\mathbf{B}_1)$ *is controllable and the eigenvalues of the matrix* $\mathbf{A}_{22}$ *are the uncontrollable eigenvalues of the pair* $(\mathbf{AB})$.

REMARK 2 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices over* R *matrices of size nxn, nxm, respectively. Further, let* $\mathbf{V}(s)$ *be the greatest common left divisor of polynomial matrices* $[\mathbf{I}s - \mathbf{A}]$ *and* $\mathbf{B}$. *In Rosenbrock (1970), the zeros of the polynomial matrix* $\mathbf{V}(s)$ *are termed input decoupling zeros. In Rosenbrock (1974) it is stated that the input decoupling zeros are the uncontrollable eigenvalues of the pair* $(\mathbf{AB})$. *The Lemma 5 that follows gives an alternative proof of this fundamental result.*

LEMMA 5 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices over* $\mathbf{R}$ *matrices of size nxn, nxm, respectively. Further, let the pair* $(\mathbf{AB})$ *be uncontrollable and* $\mathbf{B}$ *not zero. Let also* $\mathbf{V}(s)$ *be the greatest common left divisor of matrices* $[\mathbf{I}s - \mathbf{A}]$, $\mathbf{B}$. *Then the following condition holds:*

the zeros of the polynomial matrix $\mathbf{V}(s)$ *are the uncontrollable eigenvalues of the pair* $(\mathbf{AB})$.

PROOF Let λ be an uncontrollable eigenvalue of the pair $(\mathbf{A}, \mathbf{B})$. Then, from Definition 5 it follows that

$$rank[\mathbf{I}\lambda - \mathbf{A}, \mathbf{B}] < n. \tag{13}$$

Let $\mathbf{V}(s)$ be the greatest common left divisor of polynomial matrices $[\mathbf{I}s–\mathbf{A}]$ and $\mathbf{B}$. From Lemma 2 it follows that there exists a unimodular matrix $\mathbf{U}(s)$ such that

$$[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = [\mathbf{V}(s)\mathbf{0}]\mathbf{U}(s). \tag{14}$$

For $s = \lambda$, from (14) we have that

$$[\mathbf{I}\lambda - \mathbf{A}, \mathbf{B}] = [\mathbf{V}(\lambda), \mathbf{0}]\mathbf{U}(\lambda). \tag{15}$$

Since the matrix $\mathbf{U}(s)$ is unimodular, the matrix $\mathbf{U}(\lambda)$ is non-singular and therefore from (13) and (15) it follows that

$$rank[\mathbf{V}(\lambda)] < n. \tag{16}$$

Relationship (16) and Lemma 1 imply that the complex number λ is a zero of the polynomial matrix $\mathbf{V}(s)$.

On the other hand, let λ be a zero of the polynomial matrix $\mathbf{V}(s)$. Then, from Lemma 1 it follows that

$$rank[\mathbf{V}(\lambda)] < n. \tag{17}$$

Since the matrix $\mathbf{U}(s)$ is unimodular, the matrix $\mathbf{U}(\lambda)$ is non-singular and therefore from (15) and (17) it follows that

$$rank[\mathbf{I}\lambda - \mathbf{A}, \mathbf{B}] < n. \tag{18}$$

From (18) and Definition 5 it follows that the complex number λ, which is, by assumption, a zero of the polynomial matrix $\mathbf{V}(s)$, is an uncontrollable eigenvalue of the pair $(\mathbf{A}, \mathbf{B})$. This proves the claim and the proof is complete. ■

REMARK 3 *Let us point out that in Rosenbrock (1970) it is proven that the pair* $(\mathbf{A}, \mathbf{B})$ *is controllable if and only if the matrices* $[\mathbf{I}s - \mathbf{A}]$ *and* $\mathbf{B}$ *are relatively left prime over* $R[s]$ *or, alternatively, if and only if the greatest common left divisor* $\mathbf{V}(s)$*, of matrices* $[\mathbf{I}s - \mathbf{A}]$*,* $\mathbf{B}$*, is a unimodular matrix. From the above and Lemma 2 it follows that controllability of the pair* $(\mathbf{A}, \mathbf{B})$ *implies the existence of a unimodular matrix* $\mathbf{U}(s)$ *such that*

$$[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = [\mathbf{I}, \mathbf{0}]\mathbf{U}(s).$$

From the above and the proof of Lemma 5 it follows directly that the pair ($\mathbf{A}$, $\mathbf{B}$) is uncontrollable if and only if the greatest common left divisor $\mathbf{V}(s)$ of the polynomial matrices [$\mathbf{I}$s - $\mathbf{A}$] and $\mathbf{B}$ is not a unimodular matrix.

Lemma 6 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices over* R *matrices of size* nxn, nxm, *respectively and* $\mathbf{B}$ *not zero. Further, let* $\mathbf{V}(s)$ *be the greatest common left divisor of the polynomial matrices* $[\mathbf{I}s - \mathbf{A}]$ *and* $\mathbf{B}$. *The pair* ($\mathbf{A}$, $\mathbf{B}$) *is stabilizable if and only if any of the following equivalent conditions hold*

(a) If

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}, \ \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$$

with $(\mathbf{A}_{11}\mathbf{B}_1)$ *controllable, then* $\mathbf{A}_{22}$ *is Hurwitz stable.*

(b) The polynomial matrix $\mathbf{V}(s)$ *is strictly Hurwitz.*

Proof We shall first prove that stabilizability of the pair ($\mathbf{A}$ $\mathbf{B}$) is equivalent to (a), see Kucera (1991). From the statement of the Lemma we have that

$$\mathbf{A} = \mathbf{T}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1}, \ \mathbf{B} \ = \ \mathbf{T}\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}. \tag{19}$$

with $(\mathbf{A}_{11}, \mathbf{B}_1)$ controllable. If the pair ($\mathbf{A}$, $\mathbf{B}$) is stabilizable, then from Definition 6 it follows that there exists a matrix $\mathbf{F}$ such that the matrix $[\mathbf{A} - \mathbf{B}\mathbf{F}]$ is Hurwitz stable. Using (19) we have that

$$\begin{aligned} \mathbf{A} - \ \mathbf{B}\mathbf{F} \ &= \mathbf{T}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1} - \mathbf{T}\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}\mathbf{F} \ = \\ &= \mathbf{T}\{\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} - \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}\mathbf{F}\mathbf{T}\}\mathbf{T}^{-1}. \end{aligned} \tag{20}$$

Let

$$\mathbf{F}\mathbf{T} = [\mathbf{F}_1, \mathbf{F}_2]. \tag{21}$$

By substituting (21) into (20) and performing simple algebraic manipulations we obtain that

$$\mathbf{A} - \ \mathbf{B}\mathbf{F} \ = \mathbf{T}\begin{bmatrix} \mathbf{A}_{11} - \mathbf{B}_{11}\mathbf{F}_1 & \mathbf{A}_{12} - \mathbf{B}_1\mathbf{F}_2 \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1}. \tag{22}$$

From (22) it follows that

$$[\mathbf{A} - \mathbf{BF}], \begin{bmatrix} \mathbf{A}_{11}-\mathbf{B}_1\mathbf{F}_1 & \mathbf{A}_{12}-\mathbf{B}_1\mathbf{F}_2 \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}$$

are similar; therefore, Hurwitz stability of $[\mathbf{A} - \mathbf{BF}]$ implies Hurwitz stability of $\mathbf{A}_{22}$. On the other hand, controllability of the pair $(\mathbf{A}_{11}\ \mathbf{B}_{11})$ and Lemma 3 imply the existence of a matrix $\mathbf{F}_1$ of appropriate size such that the matrix

$$[\mathbf{A}_{11}-\mathbf{B}_1\mathbf{F}_1] \tag{23}$$

is Hurwitz stable. The matrix $\mathbf{F}_1$ can be computed using known methods for the solution of pole assignment problem by state feedback, see Kucera (1991).

Let

$$\mathbf{F} = [\mathbf{F}_1,\ \mathbf{0}]\mathbf{T}^{-1}. \tag{24}$$

By substituting (24) into (20) we obtain that

$$\mathbf{A} - \mathbf{BF} = \mathbf{T} \begin{bmatrix} \mathbf{A}_{11}-\mathbf{B}_1\mathbf{F}_1 & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \mathbf{T}^{-1}. \tag{25}$$

Using (23), from (25) it follows that Hurwitz stability of $\mathbf{A}_{22}$ implies Hurwitz stability of $[\mathbf{A} - \mathbf{BF}]$ with $\mathbf{F}$ given by (24) and therefore stabilizability of the pair $(\mathbf{A},\ \mathbf{B})$. Hence, (a) is complete. We shall show the equivalence of (a) and (b). From Lemma 4 we have that the eigenvalues of the matrix $\mathbf{A}_{22}$ are the uncontrollable eigenvalues of the pair $(\mathbf{A}, \mathbf{B})$. Since, by condition (a), the matrix $\mathbf{A}_{22}$ is Hurwitz stable, Hurwitz stability of $\mathbf{A}_{22}$ and Definition 3 imply that all eigenvalues of the matrix$\mathbf{A}_{22}$ and therefore all uncontrollable eigenvalues of the pair $(\mathbf{A}, \mathbf{B})$ have negative real parts. Since the uncontrollable eigenvalues of the pair $(\mathbf{A}, \mathbf{B})$ have negative real parts, from Lemma 5 it follows that all zeros of the polynomial matrix $\mathbf{V}(s)$ have negative real parts. Therefore, according to Definition 2 the polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz. Hence (a) implies (b).

We shall show the equivalence of (b) and (a). Since the polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz, from Definition 2 we have that all zeros of polynomial matrix $\mathbf{V}(s)$ have negative real parts. Since all zeros of the polynomial matrix $\mathbf{V}(s)$ have negative real parts, from Lemma 5 it follows that all uncontrollable eigenvalues of the pair $(\mathbf{A}, \mathbf{B})$ have negative real parts and therefore, according to Lemma 4, all eigenvalues of matrix $\mathbf{A}_{22}$ have negative real parts. Since all eigenvalues of matrix $\mathbf{A}_{22}$ have negative real parts, from Definition 3 it follows that the matrix $\mathbf{A}_{22}$ is Hurwitz stable. Hence, (b) implies (a) and the proof is complete. ■

The proof of the following lemma is taken from Kiritsis (2022).

Lemma 7 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices over R matrices of size nxn and nxm, respectively. Suppose that the pair (*$\mathbf{A}$*,* $\mathbf{B}$*) is controllable. Further, let* $\mathbf{D}$ *be an arbitrary matrix over R of appropriate dimensions, such that det*$[\mathbf{I}-\mathbf{BD}] \neq \mathbf{0}$. *Then the following condition holds:*

the pair $[(\mathbf{I}-\mathbf{BD})^{-1}\ \mathbf{A},\ (\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}]$ *is controllable.*

Proof Since, by assumption, the pair $(\mathbf{A},\mathbf{B})$ is controllable, from Remark 3 it follows that there exists a unimodular matrix $\mathbf{U}(s)$ such that

$$[\mathbf{I}s-\mathbf{A},\mathbf{B}] = [\mathbf{I},\mathbf{0}]\mathbf{U}(s). \tag{26}$$

We rewrite the polynomial matrix $[(\mathbf{I}-\mathbf{BD})s-\mathbf{A},\mathbf{B}]$ as

$$[(\mathbf{I}-\mathbf{BD})s-\mathbf{A},\mathbf{B}] = [\mathbf{I}s-\mathbf{A},\mathbf{B}]\begin{bmatrix} \mathbf{I}_\mathrm{n} & \mathbf{0} \\ -\mathbf{D}s & \mathbf{I}_\mathrm{m} \end{bmatrix}. \tag{27}$$

Since, by assumption, $det[\mathbf{I}-\mathbf{BD}] \neq \mathbf{0}$, the matrix $[\mathbf{I}-\mathbf{BD}]$ is non-singular and therefore from (26) and (27) it follows that

$$\begin{aligned}&[\mathbf{I}-\mathbf{BD}][\mathbf{I}\mathrm{s}-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A},(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}] = \\ &= [\mathbf{I},\mathbf{0}]\mathbf{U}(s)\begin{bmatrix} \mathbf{I}_\mathrm{n} & \mathbf{0} \\ -\mathbf{D}\mathrm{s} & \mathbf{I}_\mathrm{m} \end{bmatrix}.\end{aligned} \tag{28}$$

Since, again by assumption, the matrix $[\mathbf{I}-\mathbf{BD}]$ is non-singular, from (28) it follows that

$$\begin{aligned}&[\mathbf{I}\mathrm{s}-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A},(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}] = \\ &= [(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{,}\ \mathbf{0}]\mathbf{U}(s)\begin{bmatrix} \mathbf{I}_\mathrm{n} & \mathbf{0} \\ -\mathbf{D}\mathrm{s} & \mathbf{I}_\mathrm{m} \end{bmatrix}.\end{aligned} \tag{29}$$

Then, since the following matrix over R$[s]$

$$\begin{bmatrix} \mathbf{I}_\mathrm{n} & \mathbf{0} \\ -\mathbf{D}s & \mathbf{I}_\mathrm{m} \end{bmatrix}\mathbf{U}(s) \tag{30}$$

is a unimodular matrix, it follows from (29) and Lemma 2 that the unimodular matrix $(\mathbf{I}-\mathbf{BD})^{-1}$ is the greatest common left divisor of the polynomial matrices $[\mathbf{I}s-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}]$, $[\mathbf{I}-\mathbf{BD}]^{-1}\mathbf{B}$. This implies, according to Remark 3, that the pair $[(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A},\ (\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}]$ is controllable. This is the condition (a) of the lemma and the proof is complete. ∎

LEMMA 8 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices over* R *matrices of size nxn and nxm, respectively. Suppose that the pair* $(\mathbf{A}, \mathbf{B})$ *is uncontrollable and stabilizable. Further, let* $\mathbf{D}$ *be an arbitrary matrix over* R *of appropriate dimensions such that* $det[\mathbf{I} - \mathbf{BD}] \neq 0$ *Then the following condition holds:*

the pair $[(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{B}]$ *is uncontrollable and stabilizable.*

PROOF We rewrite the polynomial matrix $[(\mathbf{I} - \mathbf{BD})s - \mathbf{A}, \mathbf{B}]$ as

$$[(\mathbf{I} - \mathbf{BD})s - \mathbf{A}, \mathbf{B}] = [\mathbf{I}s - \mathbf{A}, \mathbf{B}] \begin{bmatrix} \mathbf{I}_{\mathrm{n}} & \mathbf{0} \\ -\mathbf{D}s & \mathbf{I}_m \end{bmatrix}. \tag{31}$$

Let $\mathbf{V}(s)$ be the greatest common left divisor of the polynomial matrices $[\mathbf{I}s - \mathbf{A}]$ and $\mathbf{B}$. Since, by assumption, the pair $(\mathbf{A}, \mathbf{B})$ is uncontrollable, from Lemma 2 it follows that there exists a unimodular matrix $\mathbf{U}(s)$ such that

$$[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = [\mathbf{V}(s)\mathbf{0}]\mathbf{U}(s). \tag{32}$$

From (31) and (32) it follows that

$$[(\mathbf{I} - \mathbf{BD})s - \mathbf{A}, \mathbf{B}] = [\mathbf{V}(s), \mathbf{0}]\mathbf{U}(s) \begin{bmatrix} \mathbf{I}_{\mathrm{n}} & \mathbf{0} \\ -\mathbf{D}s & \mathbf{I}_m \end{bmatrix}. \tag{33}$$

Since, by assumption, $det[\mathbf{I} - \mathbf{BD}] \neq 0$, the matrix $[\mathbf{I} - \mathbf{BD}]$ is non-singular and therefore the polynomial matrix $[(\mathbf{I} - \mathbf{BD})s - \mathbf{A}, \mathbf{B}]$ can be rewritten in the following manner:

$$[(\mathbf{I} - \mathbf{BD})s - \mathbf{A}, \mathbf{B}] =$$
$$[\mathbf{I} - \mathbf{BD}][\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}, (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{B}]. \tag{34}$$

From (33) and (34) it follows that

$$[\mathbf{I} - \mathbf{BD}][\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}, (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{B}] =$$
$$= [\mathbf{V}(s)\mathbf{0}]\mathbf{U}(s) \begin{bmatrix} \mathbf{I}_{\mathrm{n}} & \mathbf{0} \\ -\mathbf{D}s & \mathbf{I}_m \end{bmatrix}, \tag{35}$$

or, equivalently,

$$[\mathbf{I}\mathrm{s} - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}, (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{B}] =$$
$$= [(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{V}(s), \mathbf{0}]\mathbf{U}(s) \begin{bmatrix} \mathbf{I}_{\mathrm{n}} & \mathbf{0} \\ -\mathbf{D}s & \mathbf{I}_m \end{bmatrix}. \tag{36}$$

Since the matrix $\mathbf{U}(s)$ is unimodular, the following matrix

$$\mathbf{U}(\mathrm{s}) \begin{bmatrix} \mathbf{I}_{\mathrm{n}} & \mathbf{0} \\ -\mathbf{D}s & \mathbf{I}_m \end{bmatrix} \tag{37}$$

is also unimodular and therefore, in accordance with Lemma 2, the polynomial matrix $(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{V}(s)$ is the greatest common left divisor of the polynomial matrices [$\mathbf{I}s$ - $(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}$] and $(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}$. Since, by assumption, the pair $(\mathbf{A},\mathbf{B})$ is uncontrollable, from Remark 3 it follows that the polynomial matrix $\mathbf{V}(s)$, and therefore also the matrix $(\mathbf{I}-\mathbf{BD})^{-1}$ $\mathbf{V}(s)$ are not unimodular. Hence, the pair $[(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A},\ (\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}]$ is uncontrollable. This is a part of the condition from the lemma. Since, by assumption, the pair $(\mathbf{A},\mathbf{B})$ is stabilizable, from condition (b) of Lemma 6 it follows that the matrix $\mathbf{V}(s)$ is strictly Hurwitz. Since the matrix $(\mathbf{I}-\mathbf{BD})^{-1}$is non-singular, from Remark 1 it follows that the matrix $(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{V}(s)$ over R[s] is also strictly Hurwitz and therefore, according to condition (b) of Lemma 6, the pair $[(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}$, $(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}]$ is stabilizable. Hence, condition of the lemma is complete and the proof of the lemma is also complete. ■

4. Main results

The theorems, provided in the following, are the main results of this paper and give explicit necessary and sufficient conditions for the solution of the stabilization problem by proportional-plus-derivative state feedback for multivariable linear time-invariant systems.

Theorem 1 *Let the pair* $(\mathbf{A},\mathbf{B})$ *be controllable. Then there exists a proportional-plus-derivative state feedback (3) such that the closed-loop system (4) is stable.*

Proof From relationship (2) it follows that there exists a non-singular matrix $\mathbf{S}$ such that

$$\mathbf{B} \ = \mathbf{S}\begin{bmatrix} \mathbf{I}_{\mathrm{m}} \\ \mathbf{0} \end{bmatrix}. \tag{38}$$

We form the matrix

$$\mathbf{D} \ = [(\mathbf{X}+\mathbf{I}_m),\mathbf{0}]\mathbf{S}^{-1} \tag{39}$$

where $\mathbf{X}$ is an arbitrary non-singular matrix over R of size $m\ x\ m$ with $\mathbf{X}\neq-\mathbf{I}_m$. From (38) and (39) it follows that the matrix

$$\begin{aligned}[]
[\mathbf{I}-\mathbf{BD}] &= \mathbf{S}diag[\mathbf{I}_m\ ,\ \mathbf{I}_{n-m}]\ \mathbf{S}^{-1} - \mathbf{S}\begin{bmatrix} \mathbf{I}_{\mathrm{m}} \\ \mathbf{0} \end{bmatrix}[(\mathbf{X}+\mathbf{I}_m),\mathbf{0}]\mathbf{S}^{-1} = \\
&= \mathbf{S}diag[-\mathbf{X},\mathbf{I}_{n-m}]\mathbf{S}^{-1}
\end{aligned} \tag{40}$$

is non-singular. Let $\mathbf{A_1}$ and $\mathbf{B}_1$ be real matrices of appropriate dimensions, given by

$$\mathbf{A}_1 = [\mathbf{I} - \mathbf{BD}]^{-1}\mathbf{A} \tag{41}$$

$$\mathbf{B}_1 = [\mathbf{I} - \mathbf{BD}]^{-1}\mathbf{B}. \tag{42}$$

Since, by assumption, the pair $(\mathbf{A}\ \mathbf{B})$ is controllable, from Lemma 7 it follows that the pair $(\mathbf{A}_1, \mathbf{B}_1)$ or, equivalently, the pair $[(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}, (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{B}]$ with $\mathbf{D}$ given by (40), is also controllable.

Controllability of the pair $[(\mathbf{I} - \mathbf{BD})^{-1}\ \mathbf{A}, (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{B}]$ and Lemma 3 imply the existence of a matrix $\mathbf{F}$ such that

$$\begin{aligned} &det[\mathbf{I}\mathrm{s} - \mathbf{A}_1 + \mathbf{B}_1\mathbf{F}] = \\ &= \det[\mathbf{I}\mathrm{s} - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A} + (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{BF}] = c(s) \end{aligned} \tag{43}$$

where $c(s)$ is an arbitrary monic and strictly Hurwitz polynomial over $\mathrm{R}[s]$ of degree n. The matrix $\mathbf{F}$ can be computed using known methods for the solution of pole assignment problem by state feedback, see Kucera (1991). Since the matrix $(\mathbf{I} - \mathbf{BD})$ with $\mathbf{D}$ given by (40) is, by construction, non-singular, the relationship (43), after simple algebraic manipulations, can be rewritten as follows

$$\begin{aligned} &det[\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A} + (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{BF}] = \\ &= det[(\mathbf{I} - \mathbf{BD})^{-1}] det[(\mathbf{I} - \mathbf{BD})\mathrm{s} - \mathbf{A} + \mathbf{BF}] = c(s). \end{aligned} \tag{44}$$

From (44) we have that

$$det[(\mathbf{I} - \mathbf{BD})s - \mathbf{A} + \mathbf{BF}] = \mu c(s), \tag{45}$$

where μ is a finite nonzero real number given as

$$\mu = 1/(det[(\mathbf{I} - \mathbf{BD})^{-1}). \tag{46}$$

Since $c\,(s)$ is a strictly Hurwitz polynomial, from (45) and (5) it follows that the closed-loop system (4) is stable. This completes the proof. ∎

It should be pointed out that Theorem 1 was first proven in Kiritsis (2022), and in this paper we present an alternative proof of it.

Theorem 2 *Let the pair* $(\mathbf{AB})$ *be uncontrollable. Then the problem of stabilization by proportional-plus-derivative state feedback for multivariable linear time-invariant systems has solution over R if and only if the following condition holds:*

the pair $(\mathbf{A}, \mathbf{B})$ *is stabilizable.*

Proof Suppose that the problem of stabilization by proportional-plus-derivative state feedback has a solution over R. From (5) it follows that

$$det[(\mathbf{I}-\mathbf{BD})s-\mathbf{A}+\mathbf{BF}]{=}\mu c(s), \tag{47}$$

where μ is a finite nonzero real number and $c\,(s)$ is monic strictly Hurwitz polynomial over R$[s]$ of degree n. Let $\mathbf{V}(s)$ be the greatest common left divisor of polynomial matrices $[\mathbf{I}\mathrm{s}-\mathbf{A}]$ and $\mathbf{B}$. Then, from (11) and (12) it follows that

$$[\mathbf{I}s-\mathbf{A}]\ = \mathbf{V}(s)\mathbf{X}(s) \tag{48}$$

$$\mathbf{B} = \mathbf{V}(s)\mathbf{Y}(s) \tag{49}$$

for polynomial matrices $\mathbf{X}(s)$ and $\mathbf{Y}(s)$ over R$[s]$ of appropriate dimensions. We rewrite the polynomial matrix $[(\mathbf{I}-\mathbf{BD})s-\mathbf{A}+\mathbf{BF}]$ as

$$[(\mathbf{I}-\mathbf{BD})s-\mathbf{A}+\mathbf{BF}] = [\mathbf{I}s-\mathbf{A},\mathbf{B}]\begin{bmatrix}\mathbf{I}\\ -\mathbf{D}s+\mathbf{F}\end{bmatrix}. \tag{50}$$

Using (48), (49) and (50), after simple algebraic manipulations, the relationship (47) can be rewritten as

$$\begin{aligned}&\det\left[(\mathbf{I}-\mathbf{BD})\,\mathrm{s}-\mathbf{A}\ +\mathbf{BF}\,\right] =\\ &= det[\mathbf{V}(s)]det\{[\mathbf{X}(s),\mathbf{Y}(s)]\begin{bmatrix}\mathbf{I}\\ -\mathbf{D}s+\mathbf{F}\end{bmatrix}\} = \mu c\,(s)\,.\end{aligned} \tag{51}$$

From the relationship (51) it follows that

$$det[\mathbf{V}(s)]\,divides\,[\mu \mathrm{c}\,(s)]. \tag{52}$$

Since, by assumption, $c\,(s)$ is a monic strictly Hurwitz polynomial over R$[s]$ of degree n, from (52) it follows that det$[\mathbf{V}(s)]$ is a strictly Hurwitz polynomial over R$[s]$; therefore, by Definition 2, the polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz. Since $\mathbf{V}(s)$ is strictly Hurwitz, from condition (b) of Lemma 6 it follows that the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable. This is the condition of the theorem.

In order to prove sufficiency, we assume that the condition of the theorem holds. From (2) it follows that there exists a non-singular matrix $\mathbf{S}$ such that

$$\mathbf{B}\ = \mathbf{S}\begin{bmatrix}\mathbf{I}_m\\ \mathbf{0}\end{bmatrix}. \tag{53}$$

We form the matrix

$$\mathbf{D}\ = [(\mathbf{X}+\mathbf{I}_m),\mathbf{0}]\mathbf{S}^{-1} \tag{54}$$

where $\mathbf{X}$ is an arbitrary non-singular matrix over R of size mxm with $\mathbf{X}\neq-\mathbf{I}_m$. From (53) and (54) it follows that the matrix

$$[\mathbf{I}-\mathbf{BD}] = \mathbf{S}diag[\mathbf{I}_m,\mathbf{I}_{n-m}]\ \mathbf{S}^{-1}-\mathbf{S}\begin{bmatrix}\mathbf{I}_\mathrm{m}\\ \mathbf{0}\end{bmatrix}[(\mathbf{X}+\mathbf{I}_m),\mathbf{0}]\mathbf{S}^{-1} =$$
$$= \mathbf{S}diag[-\mathbf{X},\mathbf{I}_{n-m}]\mathbf{S}^{-1} \quad (55)$$

is non-singular. Let $\mathbf{A_1}$ and $\mathbf{B}_1$ be real matrices of appropriate dimensions, given by

$$\mathbf{A}_1 = [\mathbf{I}-\mathbf{BD}]^{-1}\mathbf{A} \quad (56)$$
$$\mathbf{B}_1 = [\mathbf{I}-\mathbf{BD}]^{-1}\mathbf{B}. \quad (57)$$

Since, by assumption, the pair ($\mathbf{A}$, $\mathbf{B}$) is uncontrollable, it follows from Lemma 8 that the pair ($\mathbf{A}_1$, $\mathbf{B}_1$) or, equivalently, the pair $[(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A},$ $(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}]$, with $\mathbf{D}$ given by (55), is uncontrollable and therefore from Lemma 4 it follows that there exists a non-singular matrix $\mathbf{L}$ such that

$$\mathbf{A}_1 = \mathbf{L}\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{0} & \mathbf{M}_{22}\end{bmatrix}\mathbf{L}^{-1},\ \mathbf{B}_1 = \ \mathbf{L}\begin{bmatrix}\mathbf{N}_1\\ \mathbf{0}\end{bmatrix}. \quad (58)$$

The pair ($\mathbf{M}_{11}$ $\mathbf{N}_1$) is controllable and the eigenvalues of the matrix $\mathbf{M}_{22}$ are the uncontrollable eigenvalues of the pair ($\mathbf{A}_1$, $\mathbf{B}_1$). Since, by the condition of the theorem, the pair ($\mathbf{A}$, $\mathbf{B}$) is stabilizable, it follows from Lemma 8 that the pair ($\mathbf{A}_1$, $\mathbf{B}_1$) is stabilizable and therefore, by condition (a) of Lemma 6, the matrix $\mathbf{M}_{22}$ is Hurwitz stable. Hurwitz stability of the matrix $\mathbf{M}_{22}$ and Definition 3 imply that

$$det\,[\mathbf{I}\mathrm{s}-\mathbf{M}_{22}] = \chi\,(s)\,, \quad (59)$$

where $\chi\,(\mathrm{s})$ is the characteristic polynomial of the matrix $\mathbf{M}_{22}$, which is strictly Hurwitz. On the other hand, controllability of the pair ($\mathbf{M}_{11}$ $\mathbf{N}_1$) and Lemma 3 imply the existence of matrix $\mathbf{F}_1$ of appropriate size, such that the matrix

$$det[\mathbf{I}\mathrm{s}-\mathbf{M}_{11}+\mathbf{N}_1\mathbf{F}_1] = \varphi(s), \quad (60)$$

where $\varphi(s)$ is an arbitrary monic and strictly Hurwitz polynomial of appropriate degree. The matrix $\mathbf{F}_1$ can be computed using known methods for the solution of pole assignment problem by the state feedback, see Kucera (1991).

From the relationships (56), (57) and (58) we deduce that

$$\mathbf{A}_1-\ \mathbf{B}_1\mathbf{F}\ = (\mathbf{I}-\mathbf{BD})^{-\mathbf{1}}\mathbf{A}-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{BF}\ =$$
$$= \mathbf{L}\begin{bmatrix}\mathbf{M_{11}} & \mathbf{M_{12}}\\ \mathbf{0} & \mathbf{M_{22}}\end{bmatrix}\mathbf{L}^{-1}-\mathbf{L}\begin{bmatrix}\mathbf{N}_1\\ \mathbf{0}\end{bmatrix}\mathbf{F}\ =$$
$$=\ \mathbf{L}\{\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{0} & \mathbf{M}_{22}\end{bmatrix}-\begin{bmatrix}\mathbf{N}_1\\ \mathbf{0}\end{bmatrix}\mathbf{FL}\}\mathbf{L}^{-1}. \quad (61)$$

Let

$$\mathbf{F} = [\mathbf{F}_1,\mathbf{0}]\mathbf{L}^{-1}. \tag{62}$$

By substituting (62) into (61) we obtain that

$$(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{BF} = \\ = \mathbf{L}\begin{bmatrix} \mathbf{M}_{11}-\mathbf{N}_1\mathbf{F}_1 & \mathbf{M}_{12} \\ \mathbf{0} & \mathbf{M}_{22} \end{bmatrix}\mathbf{L}^{-1}. \tag{63}$$

Using (59) and (60) from (63) we obtain that

$$det[\mathbf{I}\mathrm{s}-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A} \ +(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{BF}] = \\ = det[\mathbf{I}\mathrm{s}-\mathbf{M}_{11}+\mathbf{N}_1\mathbf{F}_1]det[\mathbf{I}s-\mathbf{M}_{22}] \ = \varphi(s)\,\chi\,(s)\,. \tag{64}$$

Since the matrix $(\mathbf{I}-\mathbf{BD})$ with $\mathbf{D}$ given by (55) is a non-singular matrix over R, then from (64) and after some algebraic manipulations we obtain that

$$det[\mathbf{I}s-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A} \ +(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{BF}] = \\ = det[(\mathbf{I}-\mathbf{BD})^{-1}]det[(\mathbf{I}-\mathbf{BD})\mathrm{s}-\mathbf{A}+\mathbf{BF}]=\varphi(s)\chi(s), \tag{65}$$

or, equivalently,

$$det[(\mathbf{I}-\mathbf{BD})\mathrm{s}-\mathbf{A}+\mathbf{BF}]=\mu\varphi(s)\chi(s) = \mu c(s), \tag{66}$$

where $c(s) = \varphi(s)\chi(\mathrm{s})$ and μ is a finite nonzero real number, which is given by

$$\mu = 1/(det[(\mathbf{I}-\mathbf{BD})^{-1}). \tag{67}$$

Since by (59) and (60) the polynomials $\chi\,(s)\ \varphi(s)$ are strictly Hurwitz, the polynomial $\mathrm{c}(s)$ is also a strictly Hurwitz polynomial and therefore from (66) and (5) it follows that the closed-loop system (4) is stable. This proves the sufficiency of the condition of the theorem and the proof is complete. ∎

The sufficiency parts of the proofs of Theorem 1 and Theorem 2 suggest simple procedures to compute the matrices **F** and **D** of stabilizing proportional-plus-derivative state feedback. These are provided in the sequel.

CONSTRUCTION 1

Given: **A**, **B**

Assumption: (**A**, **B**) is controllable.

Find: **F** and **D**

Step 1: Find a non-singular matrix **S** such that

$$\mathbf{B} = \mathbf{S}\begin{bmatrix} \mathbf{I}_\mathrm{m} \\ \mathbf{0} \end{bmatrix}.$$

Form the matrix

$$\mathbf{D} = [(\mathbf{X}+\mathbf{I}_m),\ \mathbf{0}]\mathbf{S}^{-1}$$

where **X** is an arbitrary non-singular matrix over R of size mxm with $\mathbf{X}\neq -\mathbf{I}_m$.

Step 2: Calculate the matrices

$$[\mathbf{I} - \mathbf{BD}] = \mathbf{S}\mathrm{diag}[-\mathbf{X},\ \mathbf{I}_{n-m}]\mathbf{S}^{-1}$$

$$\mathbf{A}_1 = [\mathbf{I} - \mathbf{BD}]^{-1}\mathbf{A}$$

$$\mathbf{B}_1 = [\mathbf{I} - \mathbf{BD}]^{-1}\mathbf{B}.$$

Step 3: Using known methods for the solution of pole assignment problem by state feedback, see Kucera (1991), find a matrix **F** over R such that

$$det[\mathbf{I}s - \mathbf{A}_1 + \mathbf{B}_1\mathbf{F}] = det[\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A} + (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{BF}] = c(s)$$

where $c(s)$ is an arbitrary monic and strictly Hurwitz polynomial over R$[s]$ of degree n.

CONSTRUCTION 2

Given: **A**, **B**

Assumption: (**A**, **B**) is uncontrollable.

Find: **F** and **D**

Step 1: Using Lemma 6, check the condition of Theorem 2. If this condition is satisfied, then go to *Step 2.* If the condition is not satisfied, then go to *Step 7.*

Step 2: Find a non-singular matrix **S** such that

$$\mathbf{B} = \mathbf{S}\begin{bmatrix} \mathbf{I}_\mathrm{m} \\ \mathbf{0} \end{bmatrix}.$$

Form the matrix

$$\mathbf{D} = [(\mathbf{X}+\mathbf{I}_m),\ \mathbf{0}]\mathbf{S}^{-1}$$

where $\mathbf{X}$ is an arbitrary non-singular matrix over R of size mxm with $\mathbf{X}\neq-\mathbf{I}_m$.

Step 3: Calculate the matrices

$$[\mathbf{I}-\mathbf{BD}] = \mathbf{S}\ diag[-\mathbf{X},\ \mathbf{I}_{n-m}]\mathbf{S}^{-1}$$

$$\mathbf{A}_1 = [\mathbf{I}-\mathbf{BD}]^{-1}\mathbf{A}$$

$$\mathbf{B}_1 = [\mathbf{I}-\mathbf{BD}]^{-1}\mathbf{B}.$$

Step 4: Compute, according to Lemma 4, a non-singular matrix $\mathbf{L}$ such that

$$\mathbf{A}_1 = [\mathbf{I}-\mathbf{BD}]^{-1}\mathbf{A} = \mathbf{L}\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{0} & \mathbf{M}_{22} \end{bmatrix}\mathbf{L}^{-1}$$
$$\mathbf{B}_1 = [\mathbf{I}-\mathbf{BD}]^{-1}\mathbf{B} = \mathbf{L}\begin{bmatrix} \mathbf{N}_1 \\ \mathbf{0} \end{bmatrix}$$

with $(\mathbf{M}_{11}\ \mathbf{N}_1)$ controllable.

Step 5: Using known methods for the solution of pole assignment by state feedback, see Kucera (1991), find a matrix $\mathbf{F}_1$ over R of appropriate dimensions such that

$$det[\mathbf{I}s-\mathbf{M}_{11}+\mathbf{N}_1\mathbf{F}_1] = \varphi(s),$$

where $\varphi(s)$ is an arbitrary strictly Hurwitz polynomial over R$[s]$ of appropriate degree.

Step 6: Put

$$\mathbf{F}= [\mathbf{F}_1,\ \mathbf{0}]\mathbf{L}^{-1}.$$

Step 7: The solution of the stabilization problem by proportional-plus-derivative state feedback is impossible.

Example 1 Consider a linear system (1) specified by:

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

i.e. $n = 2$ and $m = 1$.

The task is to check if the problem of stabilization by the proportional-plus-derivative state feedback has a solution. The eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$ are given by

$$\lambda_1 = -2, \quad \lambda_2 = 2.$$

Let I_2 be the identity matrix of size (2 x 2). We form the matrices

$$[(\mathrm{I}_2\lambda_1\ \mathrm{A}),\ \mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -4 & 0 \end{bmatrix}$$

$$[(\mathrm{I}_2\lambda_2)\ \mathrm{A},\ \mathbf{B}] = \begin{bmatrix} 4 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

We have that

$$\mathrm{rank}[(\mathrm{I}_2\lambda_1\mathrm{A}),\ \mathbf{B}] = \mathrm{rank} \begin{bmatrix} 0 & 0 & 1 \\ -1 & -4 & 0 \end{bmatrix} = 2$$

$$\mathrm{rank}[(\mathrm{I}_2\lambda_2)\mathrm{A},\ \mathbf{B}] = \mathrm{rank} \begin{bmatrix} 4 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} = 2.$$

The last relationships and the Definition 4 imply that the eigenvalues λ_1 and λ_2 are controllable; therefore the given system is controllable, see Tredelman, Stoorvogel and Hautus (2002). Since the given system is controllable, then, according to Theorem 1, the problem of stabilization by proportional-plus-derivative state feedback has a solution over the field of real numbers. For the computation of matrices $\mathbf{F}$ and $\mathbf{D}$ of stabilizing proportional-plus-derivative state feedback we shall follow the steps of Construction 1.

To carry out Step 1 set

$$\mathbf{S} = \mathrm{I}_2.$$

Since $m = 1$, put

$$\mathbf{X} = 2.$$

We have that

$$\mathbf{D} = [(\mathbf{X} + \mathbf{I}_m), \mathbf{0}]\mathbf{S}^{-1} = [3, 0]\mathbf{I}_2 = [3, 0].$$

In order to execute Step 2, we first calculate the matrix $[\mathbf{I}_2 - \mathbf{BD}]$ as follows

$$[\mathbf{I}_2 - \mathbf{BD}] = \mathbf{S}\, diag[-\mathbf{X}, \mathbf{I}_{n-m}]\mathbf{S}^{-1} = \mathbf{I}_2 \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{I}_2 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have that

$$[\mathbf{I} - \mathbf{BD}]^{-1} = \begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ are given by

$$\mathbf{A}_1 = [\mathbf{I} - \mathbf{BD}]^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{B}_1 = [\mathbf{I} - \mathbf{BD}]^{-1}\mathbf{B} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}.$$

In order to execute Step 3, we first calculate the following rational matrix

$$[\mathbf{I}_2 s - \mathbf{A}_1]^{-1}\mathbf{B}_1 = 1/(s^2 - 3s + 2) \begin{bmatrix} -(\frac{s}{2}) + 1 \\ -1/2 \end{bmatrix}.$$

Let $\mathbf{D}(s)$ and $\mathbf{N}(s)$ be polynomial matrices given by, see Kucera (1991),

$$\mathbf{D}(s) = s^2 - 3s + 2 \text{ and } \mathbf{N}(s) = \begin{bmatrix} -(\frac{s}{2}) + 1 \\ -1/2 \end{bmatrix}.$$

Further, let $c(s)$ be a strictly Hurwitz polynomial, given by

$$c(s) = s^2 + 4s + 4.$$

Then the equation

$$\mathbf{D}(s) + \mathbf{FN}(s) = s^2 - 3s + 2 + \mathbf{F} \begin{bmatrix} (\frac{s}{2}) + 1) \\ -1/2 \end{bmatrix} = s^2 + 4s + 4$$

has a unique solution for $\mathbf{F}$ over R, given by

$$\mathbf{F} = [-14, -32].$$

The matrices $[\mathbf{D}(s) + \mathbf{FN}(s)]$ and $[\mathbf{I}s - \mathbf{A}_1 + \mathbf{B}_1\mathbf{F}]$ have the same non-unit invariant polynomials and the same determinant, see Kucera (1991); therefore

$$det[\mathbf{D}(s) + \mathbf{FN}(s)] = det[\mathbf{I}s - \mathbf{A}_1 + \mathbf{B}_1\mathbf{F}] = c(s) = s^2 + 4s + 4.$$

Example 2 Consider a linear system (1) specified by:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

i.e. $n = 3$ and $m = 2$.

The task is to check if the problem of stabilization by proportional-plus-derivative state feedback has a solution.

Let I_3 be the identity matrix of size (3 x 3). From Lemma 4 and for $\mathbf{T} = I_3$ we have that

$$\mathbf{A}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{A}_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{A}_{22} = [-3]$$

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From Lemma 4 it follows that the given system is uncontrollable. Since the matrix $\mathbf{A}_{22}$ is Hurwitz stable, it follows from Lemma 6 that the given system is stabilizable; therefore, according to Theorem 2, the problem of stabilization by proportional-plus-derivative state feedback has a solution over the field of real numbers. For the computation of matrices $\mathbf{F}$ and $\mathbf{D}$ of stabilizing proportional-plus-derivative state feedback we shall follow the steps of Construction 2.

To carry out Step 2 set

$$\mathbf{S} = I_3.$$

Since $m = 2$, put

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have that

$$\mathbf{D} = [(\mathbf{X}{+}\mathbf{I}_2),\ \mathbf{0}]\mathbf{S}^{-1} \quad = \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \mathrm{I}_3 \ = \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

In order to execute Step 3, we first calculate the matrix $[\mathrm{I}_3{-}\mathbf{BD}]$ as follows

$$[\mathbf{I}_3 - \mathbf{BD}] = \mathbf{S}\, diag[-\mathbf{X}, \mathbf{I}_{n-m}]\mathbf{S}^{-1} = \mathbf{I}_3 \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{I}_3 =$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have that

$$(\mathrm{I}_3{-}\mathbf{BD})^{-1} \quad = \quad \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_1 = [(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A} \quad = \quad \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\mathbf{B}_1 = [(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{B} \quad = \quad \begin{bmatrix} -1/2 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Step 4 for $\mathbf{L}{=}\ \mathrm{I}_3$ yields

$$\mathbf{A}_1 \quad = [(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A} = \mathbf{L} \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{0} & \mathbf{M}_{22} \end{bmatrix} \mathbf{L}^{-1} \quad = \quad \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\mathbf{B}_1 = [(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{B} = \mathbf{L} \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{0} \end{bmatrix} \quad = \quad \begin{bmatrix} -1/2 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

We have that

$$\mathbf{M}_{11} = \begin{bmatrix} -1/2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\mathbf{M}_{12} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbf{M}_{22} = [-3]$$

$$\mathbf{N}_1 = \begin{bmatrix} -1/2 & 0 \\ 0 & -1 \end{bmatrix}.$$

The pair $(\mathbf{M}_{11}\ \mathbf{N}_1)$ is controllable. We begin Step 5 by forming the matrix

$$\mathbf{K} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}.$$

The characteristic polynomial $\varphi(s)$ of the matrix $\mathbf{K}$ is given by $\varphi(s) = det[\mathrm{I}_2\mathrm{s} \quad \mathbf{K}] = s^2 + 3s + 2$.

The roots of the polynomial φ(s) are (-2) and (-1); therefore, the polynomial $\varphi(s)$ is strictly Hurwitz. Since the matrix $\mathbf{N}_1$ is non-singular, the matrix $\mathbf{F}_1$, given by

$$\mathbf{F}_1 = [\mathbf{M}_{11} - \mathbf{K}]\mathbf{N}_1^{-1} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$$

yields the matrix

$$[\mathbf{M}_{11} - \mathbf{N}_1\mathbf{F}_1]{=}\mathbf{K} \ = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

with the characteristic polynomial $\varphi(s)$

$$\varphi(\mathrm{s}){=}\ det[\mathbf{I}_2 s - \mathbf{M}_{11}{+}\mathbf{N}_1\mathbf{F}_1]{=}det[\mathrm{I}_2 s - \mathbf{K}] = s^2 + 3s + 2.$$

Step 6 yields

$$\mathbf{F} = [\mathbf{F}_1, \mathbf{0}]\mathbf{L}^{-1} = [\mathbf{F}_1, \mathbf{0}]\mathrm{I}_3 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

5. Conclusions

In this paper, the problem of stabilization by the proportional-plus-derivative state feedback for multivariable linear time invariant systems is studied and completely solved. The proof of the main results of this paper is constructive and furnishes a procedure for the computation of stabilizing proportional-plus-derivative state feedback. The efficacy of the proposed method for the solution of stabilization problem by proportional-plus-derivative state feedback is illustrated with numerical examples.

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