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On the existence of Luenberger reduced order observer*

by

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Abstract: In this paper the explicit necessary and sufficient conditions for the existence of Luenberger reduced order observer are established. In particular, it is proven that for the given linear time-invariant system of order n, having p linearly independent outputs and m inputs, a Luenberger observer of order $(n - p)$ can be constructed if and only if the given system is detectable. Furthermore, a procedure is given for the construction of the observer. Our approach is based on the properties of real and polynomial matrices.

Keywords: Luenberger reduced order observer, necessary and sufficient conditions

1. Introduction

The work of Luenberger (1963) initiated the theory of observers for the state estimation of linear time-invariant systems. Then, in Luenberger (1964) the reduced order observer was proposed. Later on, in Luenberger (1966) and Luenberger (1971) the following result was proven: corresponding to a completely observable linear time-invariant system of order n, having p linearly independent outputs, a state observer of order $(n - p)$ can be constructed having arbitrary eigenvalues. This result can now also be found in any standard book on linear control theory.

From the above it immediately follows that observability is a sufficient condition for the existence of the Luenberger reduced order observer. Thus, it is natural to pose the following question: What are the necessary and sufficient conditions that ensure the existence of a Luenberger reduced order observer? As

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far as we know, the above question is still open. This fact motivates the present study. In this paper, using basic notions and basic results from linear systems and control theory as well as from the theory of matrices, explicit necessary and sufficient conditions for the existence of Luenberger reduced order observer are established. In particular, it is proven that for the given linear time-invariant system of order n having p linearly independent outputs and m inputs a Luenberger observer of order $(n-p)$ can be constructed if and only if the given system is detectable. Furthermore, a procedure is given for the construction of the Luenberger observer of order $(n-p)$.

2. Basic concepts and preliminary results

This section contains lemmas, which are needed to prove the main results of this paper, as well as some basic notions from linear systems and control theory and the theory of matrices that are used throughout the paper. Let $\mathbf{R}$ be the field of real numbers. Also, let $\mathbf{R}[s]$ be the ring of polynomials with coefficients in $\mathbf{R}$. Further, let C be the field of complex numbers, also let C^+ be the set of complex numbers λ with $Re(\lambda) \geq 0$. All nonzero finite real numbers are called units of $\mathbf{R}[s]$; see Mc Duffee (1946). A matrix, whose elements are polynomials over $\mathbf{R}[s]$ is termed a polynomial matrix. A polynomial matrix $\mathbf{U}(s)$ over $\mathbf{R}[s]$ of dimensions $(k\,x\,k)$ is said to be unimodular if and only if

$$det[\mathbf{U}(s)] = \mu \tag{1}$$

where μ is a unit of $\mathbf{R}[s]$; therefore, every unimodular polynomial matrix has a polynomial inverse. Every polynomial matrix $\mathbf{M}(s)$ of size $(m \times p)$ with $rank[\mathbf{M}(s)] = r$, can be expressed as, see Kucera (1991),

$$\mathbf{U}_1(s)\mathbf{M}(s)\mathbf{U}_2(s) = \begin{bmatrix} \mathbf{M}_r(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{2}$$

The non-singular polynomial matrix $\mathbf{M}_r(s)$ of size $(r \times r)$ in (2) is given by

$$\mathbf{M}_r\,(s) = diag[a_1(s), a_2(s), \cdots ., a_r(s)]. \tag{3}$$

The nonzero polynomials $a_i(s)$ for $i = 1,2,\ldots,\ r$ are termed invariant polynomials of $\mathbf{M}(s)$ and have the following property:

$$a_i(s) \text{ divides } a_{i+1}(s), \text{ for } i = 1, 2, \cdots, r-1. \tag{4}$$

The relationship (2) with $\mathbf{M}_r(s)$ given by (3) is called Smith-McMillan form of $\mathbf{M}$(s) over $\mathbf{R}[s]$. Since the matrices $\mathbf{U}_1(s)$ and $\mathbf{U}_2(s)$ are unimodular and the polynomial matrix $\mathbf{M}_r(s)$, given by (3), is non-singular, from (2) and (3) it follows that

$$rank[\mathbf{M}(s)] = rank[\mathbf{M}_r(s)] = r. \tag{5}$$

Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$ be matrices over $\mathbf{R}[s]$ of appropriate dimensions. If there is a matrix $\mathbf{Q}(s)$ over $\mathbf{R}[s]$ of appropriate size such that

$$\mathbf{A}(s) = \mathbf{B}(s)\mathbf{Q}(s), \tag{6}$$

then the matrix $\mathbf{Q}(s)$ is called a right divisor of the matrix $\mathbf{A}(s)$ and the matrix $\mathbf{A}(s)$ is called a left multiple of the matrix $\mathbf{Q}(s)$, see Wolowich (1974). Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$ be matrices over $\mathbf{R}[s]$ of appropriate dimensions. If there are matrices $\mathbf{D}$(s), $\mathbf{A}_1(s)$ and $\mathbf{B}_1(s)$ over $\mathbf{R}[s]$ of appropriate dimensions, such that

$$\mathbf{A}(s) = \mathbf{A}_1(s)\mathbf{D}(s),\ \mathbf{B}(s) = \mathbf{B}_1(s)\mathbf{D}(s), \tag{7}$$

then the polynomial matrix $\mathbf{D}(s)$ is called a common right divisor of polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$, see Wolowich (1974). The greatest common right divisor $\mathbf{D}(s)$ of two polynomial matrices $\mathbf{A}$(s) and $\mathbf{B}$(s) is a common right divisor, which is a left multiple of every common right divisor of the matrices $\mathbf{A}$(s) and $\mathbf{B}$(s), see Wolowich (1974) and Mc Duffee (1946), that is

$$\mathbf{A}(s) = \mathbf{P}(s)\mathbf{D}(s) \tag{8}$$

$$\mathbf{B}(s) = \mathbf{Q}(s)\mathbf{D}(s) \tag{9}$$

$$\mathbf{D}(s) = \mathbf{F}(s)\mathbf{C}(s) \tag{10}$$

where $\mathbf{A}(s)$, $\mathbf{B}(s)$, $\mathbf{P}(s)$, $\mathbf{Q}(s)$, $\mathbf{F}(s)$, as well as $\mathbf{C}(s)$ are matrices over $\mathbf{R}[s]$ of appropriate dimensions, with $\mathbf{C}(s)$ being any greatest common right divisor of the polynomial matrices $\mathbf{A}$(s) and $\mathbf{B}$(s).

Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$ be the matrices over $\mathbf{R}$ of size $(p \times m)$ and $(q \times m)$, with $rank[\mathbf{A}(s)] = m$. Then, there always exists a unimodular matrix $\mathbf{U}(s)$ over $\mathbf{R}[s]$, such that (see Wolowich, 1974)

$$\begin{bmatrix} \mathbf{A}(s) \\ \mathbf{B}(s) \end{bmatrix} = \mathbf{U}(s) \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{0} \end{bmatrix}. \tag{11}$$

The non-singular polynomial matrix $\mathbf{V}(s)$ of size $(m \times m)$ is a greatest common right divisor of the polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$, see Wolowich (1974).

Definition 1 *The nonzero polynomial* $c(s)$ *over* $\mathbf{R}[s]$ *is said to be strictly Hurwitz if and only if* $c(s) \neq 0$, $\forall s \in C^+$.

Definition 2 *Let* $\mathbf{V}(s)$ *be a non-singular matrix over* $\mathbf{R}[s]$, *of size* $(n\,x\,n)$. *Further, let* $c_i(s)$ *for* i =1,2, . . . ,n *be the invariant polynomials of the polynomial* *matrix* $\mathbf{V}(s)$. *The polynomial matrix* $\mathbf{V}(s)$ *is said to be strictly Hurwitz if and only if the polynomials* $c_i(s)$ *are strictly Hurwitz for every* i *=1,2,...,n, or, alternatively, if and only if* $det[\mathbf{V}(s)]$ *is a strictly Hurwitz polynomial.*

Definition 3 *The matrix* $\mathbf{A}$ *over* $\mathbf{R}$ *of size (n x n), is said to be Hurwitz stable if and only if all eigenvalues of the matrix* $\mathbf{A}$ *have negative real parts or, alternatively, if and only if the characteristic polynomial of matrix* $\mathbf{A}$ *is a strictly Hurwitz polynomial.*

Definition 4 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* $\mathbf{R}$ *of size* $(n\,x\,n)$ *and* $(p\,x\,n)$*, respectively. Then the pair* $(\mathbf{A}, \mathbf{C})$ *is said to be detectable if and only if there exists a matrix* $\mathbf{K}$ *over* $\mathbf{R}$ *of size* $(n\,x\,p)$ *such that the matrix* $[\mathbf{A}+\mathbf{K}C]$ *is Hurwitz stable, see Wonham (1968).*

Definition 5 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* $\mathbf{R}$ *matrices of size* $(n\,x\,n)$ *and* $(p\,x\,n)$*, respectively and* $\mathbf{C}$ *not zero. Then an eigenvalue* λ *of the matrix* $\mathbf{A}$ *is said to be observable, see Tredelman, Stoorvogel and Hautus (2002), if and only if the following condition holds:*

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n\lambda - \mathbf{A} \end{bmatrix} = n\,.$$

Let $\mathbf{A}$ be a real matrix of size $(n \times n)$. The spectrum of the matrix $\mathbf{A}$ is the set of all its eigenvalues and is denoted by $\sigma(\mathbf{A})$. An eigenvalue λ of $\mathbf{A}$ is called a stable eigenvalue if and only if $Re(\lambda) < 0$. Otherwise, the eigenvalue λ of the matrix $\mathbf{A}$ is said to be unstable. The following Lemma 1 is taken from Tredelman, Stoorvogel and Hautus (2002).

Lemma 1 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* $\mathbf{R}$ *of size* $(n\,x\,n)$ *and* $(p\,x\,n)$*, respectively, and* $\mathbf{C}$ *not zero. Further, let* $\sigma(\mathbf{A})$ *be the spectrum of the matrix* $\mathbf{A}$*. Then the pair* $(\mathbf{A}, \mathbf{C})$ *is observable if and only if the following condition holds:*

$$(a)\, rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n\lambda - \mathbf{A} \end{bmatrix} = n\,,\ \forall\lambda \in \sigma(\boldsymbol{A}).$$

Lemma 2 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* $\mathbf{R}$ *of size* $(n\,x\,n)$ *and* $(p\,x\,n)$*, respectively, and* $\mathbf{C}$ *not zero. Further, let* $\sigma(\mathbf{A})$ *be the spectrum of the matrix* $\mathbf{A}$*. The pair* $(\mathbf{A}, \mathbf{C})$ *is detectable, see Zhou, Doyle and Glover (1996), if and only if one of the following equivalent conditions holds:*

$$\text{(a) } rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n s - \mathbf{A} \end{bmatrix} = n\,,\ \forall s \in C^+$$

$$(b)\, rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n\lambda - \mathbf{A} \end{bmatrix} = n\,,\ \forall\lambda \in \sigma(\boldsymbol{A}) \text{ with } Re(\lambda) \geq 0.$$

From condition (b) of Lemma 2 it follows that the pair $(\mathbf{A},\ \mathbf{C})$ is detectable if and only if all unstable eigenvalues of the matrix $\mathbf{A}$ are observable, see Zhou, Doyle and Glover (1996).

LEMMA 3 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* $\mathbf{R}$ *of size* $(n \times n)$ *and* $(p \times n)$*, respectively. Then the pair* $(\mathbf{A},\ \mathbf{C})$ *is observable if and only if for every monic polynomial* $c\,(s)$ *over* $\mathbf{R}[s]$ *of degree* n *there exists a matrix* $\mathbf{K}$ *over* $\mathbf{R}$ *of size* $(n \times p)$*, such that the matrix* $[\mathbf{A}+\mathbf{KC}]$ *has the characteristic polynomial* $c(s)$*, see Kucera (1991).*

The standard decomposition of unobservable systems, given in the following lemma, was first published by Kalman (1963) and can also be found in any standard book on linear systems theory.

LEMMA 4 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* $\mathbf{R}$ *of size (n x n) and (p x n), respectively. Further, let the pair* $(\mathbf{A},\ \mathbf{C})$ *be unobservable and* $\mathbf{C}$ *not zero. Then there exists a non-singular matrix* $\mathbf{T}$ *of size* $(n\,x\,n)$ *such that*

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

$$\mathbf{CT} = [\mathbf{C}_1, \mathbf{0}].$$

The pair $(\mathbf{A}_{11},\ \mathbf{C}_1)$ *is observable and the eigenvalues of the matrix* $\mathbf{A}_{22}$ *are the unobservable eigenvalues of the pair* (A, C)*.*

LEMMA 5 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* $\mathbf{R}$ *of size (n x n), (p x n), respectively, and* $\mathbf{C}$ *not zero. Further, let*

$$\mathbf{A} = \mathbf{T}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1},\ \ \mathbf{C} = [\mathbf{C}_1, \mathbf{0}]\mathbf{T}^{-1}$$

with $\mathbf{A}_{11}$*,* $\mathbf{C}_1)$ *observable. If the pair* $(\mathbf{A},\ \mathbf{C})$ *is detectable, then the matrix* $\mathbf{A}_{22}$ *is Hurwitz stable (i.e., all eigenvalues of the matrix* $\mathbf{A}_{22}$ *are stable), see Zhou, Doyle and Glover (1996).*

The following lemma and its proof are based on the results of Kucera (1991).

LEMMA 6 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* $\mathbf{R}$ *of size (n x n), (p x n), respectively, and* $\mathbf{C}$ *not zero. Further, let the pair* $(\mathbf{A},\ \mathbf{C})$ *be detectable. Then there exists a matrix* $\mathbf{K}$ *over* $\mathbf{R}$ *of size* $(n \times p)$ *such that the matrix* $[\mathbf{A}+\mathbf{KC}]$ *is Hurwitz stable.*

PROOF Let the pair $(\mathbf{A},\ \mathbf{C})$ be detectable. Detectability of the pair $(\mathbf{A},\ \mathbf{C})$ implies that the pair $(\mathbf{A},\ \mathbf{C})$ is either observable or unobservable with stable unobservable eigenvalues. If the pair ($\mathbf{A}$, $\mathbf{C}$) is observable, then from Lemma 3 it follows that there exists a matrix $\mathbf{K}$ of appropriate size over $\mathbf{R}$ such that

$$det[\mathbf{I}_n s - \mathbf{A} - \mathbf{KC}] = c(s) \tag{12}$$

where $c(s)$ is an arbitrary monic, strictly Hurwitz polynomial over $\mathbf{R}[s]$ of degree n. Since the notion of observability is a dual of controllability (i.e., observability of the pair ($\mathbf{A}$, $\mathbf{C}$) implies controllability of the pair ($\mathbf{A}^T$, $\mathbf{C}^T$)), Luenberger (1971), the matrix $\mathbf{K}$ can be calculated using known methods for the solution of pole assignment problem by state feedback, see Kucera (1991). From the relationship (12) and Definition 3 it follows that the matrix [$\mathbf{A}$+$\mathbf{KC}$] is Hurwitz stable.

If the pair ($\mathbf{A}$, $\mathbf{C}$) is unobservable with stable unobservable eigenvalues, then from Lemmae 4 and 5 it follows that there exists a matrix $\mathbf{T}$ such that

$$\mathbf{A} = \mathbf{T}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1}, \quad \mathbf{C} = [\mathbf{C}_1, \mathbf{0}]\mathbf{T}^{-1}. \tag{13}$$

The pair ($\mathbf{A}_{11}$, $\mathbf{C}_1$) is observable and the matrix $\mathbf{A}_{22}$ is Hurwitz stable. Hurwitz stability of the matrix $\mathbf{A}_{22}$ and Definition 3 imply that the polynomial $\chi(s)$, given by

$$\det[\mathbf{I}s - \mathbf{A}_{22}] = \chi(s) \tag{14}$$

is a strictly Hurwitz polynomial. Observability of the pair ($\mathbf{A}_{11}$, $\mathbf{C}_1$) and Lemma 3 imply the existence of a matrix $\mathbf{K}_1$ over $\mathbf{R}$ of appropriate dimensions, such that

$$det[\mathbf{I}s - \mathbf{A}_{11} - \mathbf{K}_1\mathbf{C}_1] = \varphi(s) \tag{15}$$

where $\varphi(s)$ is an arbitrary monic, strictly Hurwitz polynomial over $\mathbf{R}[s]$ of appropriate degree. Since the notion of observability is a dual of controllability (i.e., observability of the pair ($\mathbf{A}_{11}$, $\mathbf{C}_1$) implies controllability of the pair ($\mathbf{A}_{11}^T$, $\mathbf{C}_1^T$)), the matrix $\mathbf{K}_1$ can be calculated using known methods for the solution of pole assignment problem by the state feedback, see Kucera (1991). Let

$$\mathbf{K} = \mathbf{T}\begin{bmatrix} \mathbf{K}_1 \\ \mathbf{0} \end{bmatrix}. \tag{16}$$

Using (13) and (16), we obtain that

$$\mathbf{A} + \mathbf{KC} = \mathbf{T}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1} + \mathbf{T}\begin{bmatrix} \mathbf{K}_1 \\ \mathbf{0} \end{bmatrix}[\mathbf{C}_1, \mathbf{0}]\mathbf{T}^{-1} =$$

$$= \mathbf{T} = \begin{bmatrix} \mathbf{A}_{11} + \mathbf{K}_1\mathbf{C}_1 & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1}. \tag{17}$$

From (14), (15) and (17) we obtain

$$\det[(\mathbf{I}s - \mathbf{A} - \mathbf{KC}] = \varphi(s)\chi(s). \tag{18}$$

Since by (14) and (15) the polynomials $\chi(s)$ and φ(s) are strictly Hurwitz, the polynomial $\varphi(s)\chi(s)$ is also a strictly Hurwitz polynomial and therefore, from (18) and Definition 3 it follows that matrix [**A**+**KC**] is Hurwitz stable. This completes the proof. ■

LEMMA 7 *Let* **A** *and* **C** *be matrices over* **R** *matrices of size (n x n) and (p x n), respectively, and* **C** *not zero. Further, let the pair* (**A**, **C**) *be detectable. Also, let* **V**(*s*) *be a greatest common right divisor of polynomial matrices* [**Is** − **A**] *and* **C** *of size* (*n x n*)*. Then the following condition holds:*

(a) The polynomial matrix **V**(*s*) *is strictly Hurwitz.*

PROOF Let the pair (**A**, **C**) be detectable. Then, from Definition 4 it follows that the matrix [**A** + **KC**] is Hurwitz stable, that is

$$det\,[\mathbf{I}\text{s} - \mathbf{A} - \mathbf{KC}] = c(s) \tag{19}$$

where $c(s)$ is a monic and strictly Hurwitz polynomial over **R**[s] of degree n. Since, by the assumption, the polynomial matrix **V**(s) is the greatest common right divisor of the polynomial matrices [**Is**-**A**] and **C**, from (8) and (9) it follows that

$$[\mathbf{I}s - \mathbf{A}] = \mathbf{X}(s)\mathbf{V}(s) \tag{20}$$

$$\mathbf{C} = \mathbf{Y}(s)\mathbf{V}(s) \tag{21}$$

for polynomial matrices **X**(s) and **Y**(s) over **R**[s] of appropriate dimensions. Using (20) and (21) and after simple algebraic manipulations, the relationship (19) can be rewritten as

$$det\,[\mathbf{I}s - \mathbf{A} - \mathbf{KC}] = det[\mathbf{X}(s) - \mathbf{KY}(s)]\,det\,\mathbf{V}(s) = c(s). \tag{22}$$

From relationship (22) it follows that

$$det[\mathbf{V}(s)]\,divides\ (c\,(s)). \tag{23}$$

Since, by assumption, $c\,(s)$ is a monic and strictly Hurwitz polynomial over **R**[s] of degree n, from (23) it follows that $det[\mathbf{V}(s)]$ is a strictly Hurwitz polynomial over **R**[s]; therefore, by Definition 2, the polynomial matrix **V**(s) is strictly Hurwitz. This is condition (a) of the Lemma and the proof is complete. ■

LEMMA 8 *Let* V*(s) be a non-singular and strictly Hurwitz matrix over* **R***[s] of size* ($m \times m$)*. Then, for any matrix* ***X****(s) over* **R***[s] of size* (*p x m*)*, all invariant polynomials of the polynomial matrix* $\begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix}$ *are strictly Hurwitz.*

Proof Let $\mathbf{U}_1(s)$, $\mathbf{U}_2(s)$ be unimodular matrices such that

$$\begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix} = \mathbf{U}_1(s) \begin{bmatrix} \mathbf{V}_m(s) \\ \mathbf{0} \end{bmatrix} \mathbf{U}_2(s). \tag{24}$$

The non-singular polynomial matrix $\mathbf{V}_m\,(s)$ of size $(m \times m)$ in (24) is given by

$$\mathbf{V}_m(s) = diag[v_1(s), v_2(s), \cdots, v_m(s)]. \tag{25}$$

The nonzero polynomials $v_i(s)$ for $i = 1, 2, ..., m$ are the invariant polynomials of $\begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix}$ and the relationship (24) with $\mathbf{V}_m\,(s)$, given by (25), is the Smith-McMillan form of $\begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix}$ over $\mathbf{R}[s]$. Since, by assumption, the polynomial matrix $\mathbf{V}(s)$ is a non-singular and strictly Hurwitz matrix and the polynomial matrix $\mathbf{U}_1(s)$ is unimodular, from (11) and (24) it follows that the matrix $[\mathbf{V}_m\,(s)\mathbf{U}_2(s)]$ is a greatest common right divisor of the polynomial matrices $\mathbf{V}(s)$ and $\mathbf{X}(s)$; therefore, according to (8), there is a polynomial matrix $\mathbf{P}(s)$ such that

$$\mathbf{V}(s) = \mathbf{P}(s)[\mathbf{V}_m\,(s)\,\mathbf{U}_2(s)]. \tag{26}$$

From (25) and (26) we have:

$$\begin{aligned} &det[\mathbf{V}(s)] = \\ &= det[\mathbf{P}(s)]det[\mathbf{V}_m\,(s)]det[\mathbf{U}_2(s)] = det[\mathbf{P}(s)]det[\mathbf{U}_2(s)][\Pi_{\iota=1}^{m} v_i(s)] \end{aligned} \tag{27}$$

From (27) it follows that

$$[\Pi_{\iota=1}^{m} v_i(s)] \text{ divides } \{det[\mathbf{V}(s)]\}. \tag{28}$$

Since, by assumption, matrix $\mathbf{V}(s)$ is a strictly Hurwitz polynomial matrix, from Definition 2 it follows that the polynomial $det[\mathbf{V}(s)]$ is strictly Hurwitz polynomial over $\mathbf{R}[s]$; therefore, from (28) it follows that the polynomial $[\Pi_{\iota=1}^{m} v_\text{i}(s)]$ is also a strictly Hurwitz polynomial over $\mathbf{R}[s]$. Since $[\Pi_{\iota=1}^{m} v_\text{i}(s)]$ is a strictly Hurwitz polynomial over $\mathbf{R}[s]$, all polynomials $v_i(s)$ for $i = 1, 2, ..., m$ must be strictly Hurwitz.

From the above it follows that polynomials $v_i(s)$ for $i = 1, 2, ..., m$, which are the invariant polynomials of $\begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix}$, are all strictly Hurwitz. This proves the claim and so the proof is complete. ■

Let $\mathbf{A}$ and $\mathbf{C}$ be matrices over $\mathbf{R}$ matrices of size $(n \times n)$ and $(p \times n)$, respectively, with $rank[\mathbf{C}] = p$. Then there exists a non-singular matrix $\mathbf{L}$ over $\mathbf{R}$ of size $(n \times n)$ such that

$$\mathbf{CL} = \mathbf{C}_1 \tag{29}$$

$$\mathbf{L}^{-1}\mathbf{AL} = \mathbf{A}_1. \tag{30}$$

The matrices $\mathbf{C}_1$ and $\mathbf{A}_1$ are given by

$$\mathbf{C}_1 = [\mathbf{I}_p, \mathbf{0}] \tag{31}$$

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{32}$$

where $\mathbf{I}_p$ is the identity matrix of size $(p \times p)$, and $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ are matrices over $\mathbf{R}$ of dimensions

$(p \times p)$, $(p \times (n-p))$, $((n-p) \times p)$ and $((n-p) \times (n-p))$,

respectively.

The following Lemma plays a central role in the proof of the main results of this paper.

Lemma 9 *Let* A *and* C *be matrices over* R *of size* $(n \times n)$, $(p \times n)$*, respectively, with* $rank[\mathbf{C}] = p$. *The pair* (A, C) *is detectable if and only if the following condition holds:*

(a) The pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$ *is detectable.*

Proof Let the pair $(\mathbf{A}, \mathbf{C})$ be detectable. Then, from condition (a) of Lemma 2 it follows that

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n s - \mathbf{A} \end{bmatrix} = n\,,\ \forall s \in C^{+\cdot} \tag{33}$$

From (29), (30), (31) and (32) we have that

$$\mathbf{C} = [\mathbf{I}_{\mathrm{p}}, \mathbf{0}]\mathbf{L}^{-1} = \mathbf{C}_1\mathbf{L}^{-1} \text{ and } \mathbf{A} = \mathbf{L}\mathbf{A}_1\mathbf{L}^{-1} = \mathbf{L}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\mathbf{L}^{-1}. \tag{34}$$

Using (34) and performing some simple algebraic manipulations, (33) can be expressed as follows:

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n s - \mathbf{A} \end{bmatrix} = rank\,[diag[\mathbf{I}_p\mathbf{L}] \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{I}_n s - \mathbf{A}_1 \end{bmatrix} \mathbf{L}^{-1}] =$$

$$= rank\,[diag[\mathbf{I}_p, \mathbf{L}] \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{I}_p s - \mathbf{A}_{11} & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & \mathbf{I}_{\mathrm{n-p}} s - \mathbf{A}_{22} \end{bmatrix} \mathbf{L}^{-1}] = n\,, \forall s \in C^{+\cdot} \tag{35}$$

Since the matrices $diag[\mathbf{I}_n, \mathbf{L}]$ and $\mathbf{L}^{-1}$ are non-singular, from (35) it follows that

$$rank \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{I}_p s - \mathbf{A}_{11} & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & \mathbf{I}_{\mathrm{n-p}} s - \mathbf{A}_{22} \end{bmatrix} = n, \quad \forall s \in C^+. \tag{36}$$

Since the n columns of the matrix on the left-hand side of (36) are linearly independent over C, $\forall s \in C^+$, a subset of these columns, consisting of the last $(n-p)$ columns must also be linearly independent over C, $\forall s \in C^+$; and therefore

$$rank \begin{bmatrix} 0 \\ -\mathbf{A}_{12} \\ \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} = rank \begin{bmatrix} -\mathbf{A}_{12} \\ \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} = rank \begin{bmatrix} \mathbf{A}_{12} \\ \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} =$$

$$= (n-p), \ \ \forall s \in C^+. \tag{37}$$

Relationship (37) and condition (a) of Lemma 2 imply that the pair ($\mathbf{A}_{22}$, $\mathbf{A}_{12}$) is detectable. This is the condition (a) of the Lemma.

Now, to prove sufficiency, we assume that the pair ($\mathbf{A}_{22}$, $\mathbf{A}_{12}$) is detectable. Let $\mathbf{\Gamma}(s)$ be a greatest common right divisor of polynomial matrices $[\mathbf{I}_{n-p} s - \mathbf{A}_{22}]$ and $\mathbf{A}_{12}$. Then, from (11) it follows that there exists a unimodular matrix $\mathbf{U}(s)$ of size $(n \times n)$ such that

$$\begin{bmatrix} \mathbf{A}_{12} \\ \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} = \mathbf{U}(s) \begin{bmatrix} \mathbf{\Gamma}(s) \\ \mathbf{0} \end{bmatrix}. \tag{38}$$

We define the following matrices

$$\begin{bmatrix} \mathbf{I}_p s - \mathbf{A}_{11} \\ -\mathbf{A}_{21} \end{bmatrix} = \mathbf{U}(s) \begin{bmatrix} \mathbf{E}(s) \\ \mathbf{Z}(s) \end{bmatrix} \tag{39}$$

$$\mathbf{V}(s) = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{E}(s) & \mathbf{\Gamma}(s) \end{bmatrix} \tag{40}$$

$$\mathbf{X}(s) = [\mathbf{Z}(s), \mathbf{0}] \tag{41}$$

where $\mathbf{V}(s)$, $\mathbf{E}(s)$ and $\mathbf{Z}(s)$ are polynomial matrices of sizes, respectively, $(n \times n)$, $((n-p) \times p), (p \times p)$. Using (31), (32), (38), (39), (40) and (41) we obtain:

$$\begin{bmatrix} \mathbf{C}_1 \\ \mathbf{I}s - \mathbf{A}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{I}_p s - \mathbf{A}_{11} & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} = diag[\mathbf{I}_p, \mathbf{U}(s)] \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix}. \tag{42}$$

Let $\mathbf{U}_1(s)$, $\mathbf{U}_2(s)$ be unimodular matrices such that

$$\begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix} = \mathbf{U}_1(s) \quad \begin{bmatrix} \mathbf{V}_n(s) \\ \mathbf{0} \end{bmatrix} \; \mathbf{U}_2(s). \tag{43}$$

The non-singular polynomial matrix $\mathbf{V}_n\,(s)$ of size $(n\,x\,n)$ in (43) is given by

$$\mathbf{V}_n(s) = diag[v_1(s), v_2(s), \cdots, v_n(s)]. \tag{44}$$

The nonzero polynomials $v_i(s)$ for $i = 1, 2, ..., n$ are the invariant polynomials of $\begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix}$. The relationship (43) with $\mathbf{V}_n\,(s)$, given by (44), is the Smith-McMillan form of $\begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix}$ over $\mathbf{R}[s]$. By substituting (43) into (42), we obtain:

$$\begin{bmatrix} \mathbf{C}_1 \\ \mathbf{I}s{-}\mathbf{A}_1 \end{bmatrix} = diag[\mathbf{I}_p,\ \mathbf{U}\,(s)]\ \mathbf{U}_1(s) \begin{bmatrix} \mathbf{V}_n(s) \\ \mathbf{0} \end{bmatrix} \ \mathbf{U}_2(s). \tag{45}$$

Since the polynomial matrices $[diag[\mathbf{I}_p,\ \mathbf{U}\,(s)]\mathbf{U}_1(s)]$ and $\mathbf{U}_2(s)$ are unimodular matrices, from (45) it follows that

$$rank \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{I}s{-}\mathbf{A}_1 \end{bmatrix} = rank \begin{bmatrix} \mathbf{V}_n(s) \\ \mathbf{0} \end{bmatrix} = rank\,[\mathbf{V}_n\,(s)] = n. \tag{46}$$

Since, by assumption, the pair $(\mathbf{A}_{22},\ \mathbf{A}_{12})$ is detectable, from Lemma 7 it follows that the matrix $\mathbf{\Gamma}(s)$ in (38) is a strictly Hurwitz polynomial matrix. From (40) it follows that

$$det[\mathbf{V}(s)] = det[\mathbf{\Gamma}(s)]. \tag{47}$$

Since the matrix $\mathbf{\Gamma}(s)$ is strictly Hurwitz, from Definition 2 it follows that the $det[\,\mathbf{\Gamma}(s)]$ is a strictly Hurwitz polynomial and therefore from (47) and Definition 2 it follows that the polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz. Since $\mathbf{V}(s)$ is a strictly Hurwitz matrix over $\mathbf{R}[s]$, from Lemma 8 it follows that the invariant polynomials (i.e., the polynomials $v_i(s)$ for $i = 1, 2, ..., m$) of the polynomial matrix $\begin{bmatrix} \mathbf{V}(s) \\ \mathbf{X}(s) \end{bmatrix}$ are all strictly Hurwitz; therefore, from Definition 1 it follows that

$$v_i(s)) \neq 0,\ \forall s \in C^+,\ \forall\ i = 1, 2, \cdots, n. \tag{48}$$

From (44) and (48) it follows that

$$rank[\mathbf{V}_n(s)] = rank[diag[v_1(s), v_2(s), \cdots, v_n(s)]\} = n,\ \forall s \in C^+. \tag{49}$$

From (46) and (49) it follows that

$$rank \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{I}s - \mathbf{A}_1 \end{bmatrix} = n, \forall s \in C^+. \tag{50}$$

Relationship (50) and condition (a) of Lemma 2 imply that the pair ($\mathbf{A}_1$, $\mathbf{C}_1$) is detectable. Since the pair ($\mathbf{A}_1$, $\mathbf{C}_1$) is detectable and the detectability is invariant under similarity transformation, see Zhou, Doyle and Glover (1996), from (34) it follows that the pair ($\mathbf{A}$, $\mathbf{C}$) is detectable. This completes the proof. ■

3. Main results

Theorem 1, provided in this section, is the main result of this paper and it gives the explicit necessary and sufficient conditions for the existence of a Luenberger reduced order observer.

Consider a linear time-invariant system described by the following state-space equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{51}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{52}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are real matrices of size $(n \times n), (n \times m)$ and $(p \times n)$, respectively, $\mathbf{x}(t)$ is the state vector of dimensions $(n \times 1)$, $\mathbf{u}(t)$ is the vector of inputs of size $(m \times 1)$, and $\mathbf{y}(t)$ is the vector of outputs of size $(p \times 1)$. In what follows, we assume without any loss of generality that

$$rank[\mathbf{C}] = p. \tag{53}$$

Relationship (53) implies the existence of a non-singular matrix $\mathbf{M}$ over $\mathbf{R}$, having size $(n \times n)$, such that

$$\mathbf{C}\mathbf{M} = [\mathbf{I}_p, \mathbf{0}]. \tag{54}$$

By introducing the following similarity transformation

$$\mathbf{x}(t) = \mathbf{M}\mathbf{z}(t) \tag{55}$$

and taking into account (54) and (55), the state-space equations (51) and (52) of the given system can be expressed as

$$\begin{bmatrix} \dot{\mathbf{z}}_1(t) \\ \dot{\mathbf{z}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} \mathbf{u}(t) \tag{56}$$

$$\mathbf{y}(t) = [\mathbf{I}_p, \mathbf{0}] \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix} \tag{57}$$

where

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}, \ \ \mathbf{M}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}$$

and

$$\mathbf{z}(t) = \mathbf{M}^{-1}\mathbf{x}\,(\mathrm{t}) = \begin{bmatrix} \mathbf{z}_1(\mathrm{t}) \\ \mathbf{z}_2(\mathrm{t}) \end{bmatrix} \tag{58}$$

while $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ are matrices over $\mathbf{R}$ of dimensions, respectively, $(p \times p)$, $(p \times (n-p))$, $((n-p) \times p)$ and $((n-p) \times (n-p))$, and $\mathbf{z}_1$ (t) and $\mathbf{z}_2$(t) are vectors of dimensions $(p \times 1)$ and $((n-p) \times 1)))$, respectively. From (54) it is obvious that $\mathbf{z}_1$(t) denotes the states that are measurable and $\mathbf{z}_2$(t) denotes the states that are not measurable. Using (58), equations (56) and (57) can be rewritten as, see Shafai and Saif (2015):

$$\dot{\mathbf{z}}_1(t) = \mathbf{A}_{11}\mathbf{z}_1(t) + \mathbf{A}_{12}\mathbf{z}_2(t) + \mathbf{G}_1\mathbf{u}(t) \tag{59}$$

$$\dot{\mathbf{z}}_2(t) = \mathbf{A}_{21}\mathbf{z}_1(\mathrm{t}) + \mathbf{A}_{22}\mathbf{z}_2\,(t) + \mathbf{G}_2\,\mathbf{u}(t) \tag{60}$$

$$\mathbf{y}(t) = \mathbf{z}_1(t). \tag{61}$$

From (61) we have that

$$\dot{\mathbf{y}}(t) = \dot{\mathbf{z}}_1(t). \tag{62}$$

Upon substituting (61) and (62) into (59) and (60) and after some algebraic manipulations we obtain, see Shafai and Saif (2015),

$$\dot{\mathbf{z}}_2(t) = \mathbf{A}_{22}\mathbf{z}_2(t) + \mathbf{D}\mathbf{u}_1(t) \tag{63}$$

$$\mathbf{y}_1(t) \quad = \quad \mathbf{A}_{12}\mathbf{z}_2(t). \tag{64}$$

The vectors $\mathbf{u}_1(t)$, $\mathbf{y}_1(t)$ and the matrix $\mathbf{D}$ are given by

$$\mathbf{u}_1(t) = \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{u}(t) \end{bmatrix} \tag{65}$$

$$\mathbf{y}_1(t) = \dot{y}(t)\mathbf{A}_{11}\mathbf{y}(t)\mathbf{G}_1\mathbf{u}(t) \tag{66}$$

$$\mathbf{D} \;= [\mathbf{A}_{21}, \mathbf{G}_2]. \tag{67}$$

Consider also a linear time-invariant system, described by the following state-space equations

$$\dot{\hat{\mathbf{z}}}_2(t) = [\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]\hat{\mathbf{z}}_2(t) + \mathbf{K}\mathbf{y}_1\,(t) + \mathbf{D}\,\mathbf{u}_1(t). \tag{68}$$

THEOREM 1 *The system (68) is a Luenberger observer of order* $(n{-}p)$ *of system (51) and (52) if and only if the following condition holds:*

(a) The pair $(\mathbf{A}, \mathbf{C})$ *is detectable.*

Proof Let the system (68) be a Luenberger observer of order $(n-p)$ of system (51) and (52). Further, let $\hat{\mathbf{z}}_2(t)$ be an estimate of $\mathbf{z}_2\,(t)$. We define the state estimation error $\mathbf{e}$(t) as follows

$$\mathbf{e}(t) = \mathbf{z}_2\,(t) - \hat{\mathbf{z}}_2(t). \tag{69}$$

Then, by taking the derivative of (69) and using (63), (64), (68) and (69), we obtain

$$\begin{aligned}\dot{\mathbf{e}}\,(t) &= [\mathbf{A}_{22}\mathbf{z}_2(t) + \mathbf{D}\mathbf{u}_1(t)] - \{[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]\hat{\mathbf{z}}_2(t) + \mathbf{K}\mathbf{y}_1(t) + \mathbf{D}\mathbf{u}_1(t)\}\\ &= [\mathbf{A}_{22}\mathbf{z}_2(t) + \mathbf{D}\mathbf{u}_1(t)] - \{[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]\hat{\mathbf{z}}_2(t) + \mathbf{K}\mathbf{A}_{12}\mathbf{z}_2(t) + \mathbf{D}\mathbf{u}_1(t)\}\\ &= [\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]\mathbf{z}_2\,(t) - [\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]\hat{\mathbf{z}}_2(t)\\ &= [\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}][\mathbf{z}_2(t) - \hat{\mathbf{z}}_2(t)] = [\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]\mathbf{e}(t).\end{aligned} \tag{70}$$

The solution of the system of differential equations (70) is given by

$$\mathbf{e}(t) = \mathbf{e}^{[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]\mathbf{t}}\mathbf{e}(0). \tag{71}$$

The estimation error $\mathbf{e}$(t), given by (71), approaches zero in the sense of $lim_{t\mapsto+\infty}\,\mathbf{e}(\mathrm{t}) = \mathbf{0}$ if and only if all the eigenvalues of the matrix $[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]$ are stable, equivalently, according to Definition 3, if and only if the matrix $[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]$ is Hurwitz stable, see Trinh and Fernando (2012). Hurwitz stability of the matrix $[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]$ and Definition 4 imply the detectability of the pair $(\mathbf{A}_{22},\ \mathbf{A}_{12})$. Detectability of the pair $(\mathbf{A}_{22},\ \mathbf{A}_{12})$ and Lemma 9 imply the detectability of the pair $(\mathbf{A},\ \mathbf{C})$. This is condition (a) of the Theorem.

In order to prove sufficiency, we assume that the pair $(\mathbf{A},\ \mathbf{C})$ is detectable. Detectability of the pair $(\mathbf{A},\ \mathbf{C})$ and Lemma 9 imply detectability of the pair $(\mathbf{A}_{22},\ \mathbf{A}_{12})$. Since the pair $(\mathbf{A}_{22},\ \mathbf{A}_{12})$ is detectable, from Lemma 6 it follows that there exists a real matrix $\mathbf{K}$ of appropriate dimensions such that the matrix $[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]$ is Hurwitz stable. The matrix $\mathbf{K}$ can be calculated as in the proof of Lemma 6. Now, since the matrix $[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]$ is by construction Hurwitz stable, from (71) we obtain

$$lim_{t\mapsto+\infty}\mathbf{e}(t) = 0 \tag{72}$$

for any $\mathbf{e}(0)$. Hence, from (69) and (72) it follows that $\hat{\mathbf{z}}_2(t)$ is an estimate of $\mathbf{z}_2(t)$. The vector $\hat{\mathbf{z}}_2(t)$ can be calculated by solving the system of differential equations, given by (68) (i.e., the state-space equations of the Luenberger observer of order $(n-p)$).

Using relationships (55) and (61) one can always find an estimate $\hat{\mathbf{x}}\,(t)$ of the state vector $\mathbf{x}(t)$ of the system (51) and (52) as follows

$$\hat{\mathbf{x}}\,(t) = \mathbf{M}\begin{bmatrix} \mathbf{y}(t) \\ \hat{\mathbf{z}}_2(t) \end{bmatrix} \tag{73}$$

This completes the proof. ■

The sufficiency part of the proof of Theorem 1 provides a construction of the matrices **K** and **D** of Luenberger observer of order $(n-p)$ (68). The major steps of this construction are given below.

Construction

Given: **A**, **B** and **C**

Find: **K** and **D**

Step 1: Check condition (a) of Theorem 1. If this condition is satisfied, go to *Step 2.* If condition (a) is not satisfied, then construction of the Luenberger observer of order $(n-p)$ is impossible.

Step 2: Find a non-singular matrix **M** over **R** of dimensions $(n \times n)$ such that

$$\mathbf{CM} = [\mathbf{I}_{\mathrm{p}}, \mathbf{0}].$$

Step 3: Calculate the following real matrices

$$\mathbf{M}^{-1}\mathbf{AM} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} = \mathbf{M}^{-1}\mathbf{B}, \ \mathbf{D} = [\mathbf{A}_{21}, \mathbf{G}_2 \,].$$

Step 4: Detectability of the pair (**A**, **C**) and Lemma 9 imply detectability of the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$. Since the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$ is detectable, from Lemma 6 it follows that there exists a real matrix **K** such that the matrix

$$[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]$$

is Hurwitz stable. The matrix **K** can be calculated as in the proof of Lemma 6.

4. Computational examples

Example 1 Consider a linear system with state-space equations given by (51) and (52), specified by:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$,

with $n=4$, $m=2$, $p=2$ and $rank[\mathbf{C}]=2$.

The task is to find the matrices **K** and **D** of the Luenberger reduced observer of order $(n-p)$, which estimates the state vector of the given system.

We shall follow the steps of the construction procedure, provided in the preceding section. The eigenvalues of the matrix **A** of the given system are: $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = 0$. In order to execute Step 1 of the construction procedure, we form the following matrices

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_2\lambda_1 - \mathrm{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_2\lambda_2 - \mathrm{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

We have the following:

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_2\,\lambda_1 - \mathrm{A} \end{bmatrix} = rank \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4$$

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_2\lambda_3 - \mathrm{A} \end{bmatrix} = rank \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 4.$$

The last relationships and the Definition 5 imply that the unstable eigenvalues λ_1, λ_2, λ_3 and λ_4 are observable; therefore, according to the condition (b) of Lemma 2, the given system is detectable. Detectability of the pair (**A**, **C**) and Theorem 1 imply the existence of Luenberger reduced observer of order $(n - p)$, which estimates the state vector of the given system.

In order to carry out Step 2 set

$$\mathbf{M} = \mathbf{I}_4$$

where $\mathbf{I}_4$ is the identity matrix of size (4 x 4).

Step 3, for $\mathbf{M} = \mathbf{I}_4$, yields

$$\mathbf{A}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{A}_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{G}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{G}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Detectability of the pair ($\mathbf{A}$, $\mathbf{C}$) and Lemma 9 imply detectability of the pair ($\mathbf{A}_{22}$, $\mathbf{A}_{12}$). The matrix $\mathbf{K}$, given by

$$\mathbf{K} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

produces $[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]$ in companion form, see Kucera (1991),

$$[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}] = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

with characteristic polynomial $c(s) = s^2 + 2s + 1$. Since the roots $s_{1,2} = -1$ of the polynomial $c(s)$ have negative real parts, the polynomial $c(s)$ is strictly Hurwitz and therefore, according to Definition 3, the matrix $[\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}]$ is Hurwitz stable. This completes Step 4.

Example 2 Consider a linear system with state-space equations given by (51) and (52), specified by:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

with $n = 3$, $m = 2$, $p = 2$ and $rank[\mathbf{C}] = 2$.

The task is to find the matrices $\mathbf{K}$ and $\mathbf{D}$ of the Luenberger reduced observer of order $(n - p)$, which estimates the state vector of the given system.

We shall follow the steps of the construction procedure, given in the preceding section. For executing Step 1, we form the following matrix

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_3 s - \mathrm{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s-1 & 0 & 0 \\ 0 & s-1 & 0 \\ 0 & -1 & s-1 \end{bmatrix}.$$

For $s = 1$ we have that

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_3 - \mathrm{A} \end{bmatrix} = rank \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = 2 < 3.$$

The last relationship and condition (a) of Lemma 2 imply that the given system is not detectable; therefore, according to Theorem 1, the construction of Luenberger reduced observer of order $(n-p)$ is impossible.

5. Conclusions

In this paper, the explicit necessary and sufficient conditions for the existence of Luenberger reduced order observer are established. The proof of the main results is constructive and furnishes a procedure for the construction of the Luenberger reduced order observer. The main results obtained for linear continuous-time systems also hold for linear discrete-time systems.

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