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Thin inclusion in elastic body: identification of damage parameter*

by

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Dedicated to Günter Leugering on the occasion of His 65th birthday

Abstract: In the paper, we consider an equilibrium problem for a 2D elastic body with a thin elastic inclusion crossing an external boundary. The elastic body has a defect which is characterized by a positive damage parameter. The presence of a defect means that the problem is formulated in a non-smooth domain. Non-linear boundary conditions at the defect faces are imposed to prevent the mutual penetration between the faces. Both variational and differential problem formulations are proposed, and existence of solutions is established. We study an asymptotics of solutions with respect to the damage parameter as well as with respect to a rigidity parameter of the inclusion. Identification problems for finding the damage parameter are investigated. To this end, existence of solutions of optimal control problems is proven.

Keywords: thin inclusion, defect, damage parameter, non-penetration boundary conditions, optimal control

1. Introduction

The analysis of equilibrium problems for composite materials with thin inclusions based on high-level mathematical models leads to the need of finding solutions in non-smooth domains. In the presence of delamination of thin inclusions from the matrix, and the formation of defects in this way, it is necessary to appropriately choose boundary conditions on the defect faces. From the point of view of mechanics, the best boundary conditions should be considered as those that ensure the mutual non-penetration of the opposite faces of the defect. At the same time, the nature of the interaction of the opposite faces is characterized by the damage parameter.

The paper deals with the equilibrium problem of an inhomogeneous anisotropic elastic body containing a thin inclusion, in the presence of a defect. It is assumed that the inclusion crosses the outer boundary of the elastic body. In

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this case, the thin inclusion is characterized by the rigidity parameter $\beta > 0$, while the defect is characterized by the damage parameter $\delta > 0$. The analysis of the dependence of the solutions on the specified parameters is carried out. In particular, the limiting models are investigated, corresponding to the values of the parameter $\delta = 0, \delta = \infty$. If the value of the damage parameter is unknown, then as an additional information, we can set the deflection of the outer part of the thin inclusion. Thus, to identify the damage parameter, one can investigate the optimal control problem, the solution of which allows for finding such a damage parameter value that minimizes the difference between the deflection of the outer part of the thin inclusion and the specified function. The solvability of these optimal control problems is proven both in the case of the elastic and the rigid inclusions.

General approaches to the formulation of boundary value problems for composite materials can be found in a large number of publications, see, for example, Kozlov and Maz'ya (1991), Nasser and Hassen (1987), Saccomandi and Beatty (2001), Panasenko (2005), Yao (2015). The equilibrium problem for an elastic body containing a defect with a damage parameter is investigated in Khludnev (2018), see also Almi (2017), Perelmuter (2014). We also refer the reader to the results concerning boundary problems for elastic bodies with cracks and thin inclusions, see Khludnev and Sokolowski (1997), Khludnev and Kovtunenko (2000), Khludnev (2010), Khludnev and Leugering (2011, 2014, 2015, 2016), Lazarev and Rudoy (2011, 2014, 2015), Shcherbakov (2014a, b) and the references therein.

2. Setting the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ such that $a_0 \subset \Omega$, $a_0 = (0,1) \times \{0\}, a_e = (-1,0) \times \{0\}$, $a_e \not\subset \Omega$, $a = a_0 \cup a_e \cup \{(0,0)\}$. Denote by $\nu = (0,1)$ a unit normal vector to a, and set $\Omega_0 = \Omega \setminus \bar{a}_0$, see Fig. 1.

In our considerations, the domain Ω_0 represents a region with an elastic material, and a corresponds to an elastic inclusion with specified properties crossing the external boundary Γ. In particular, we consider a as an elastic beam incorporated in the elastic body. Vertical displacements (along the axis x_2) of the inclusion should coincide with the vertical displacements of the elastic body at a_0^-. From the standpoint of geometry, the defect is located at a_0. In our model, inequality type boundary conditions will be considered to prevent mutual penetration between the defect faces $a_0^{\pm}$. Let $B = \{b_{ijkl}\}, i,j,k,l = 1,2$, be a given elasticity tensor with the usual properties of symmetry and positive definiteness, $b_{ijkl} \in L^\infty(\Omega)$.

Denote by $\delta > 0$ a damage parameter, which characterizes the defect. An equilibrium problem for the body Ω_0 and the elastic inclusion a in the presence of the defect is formulated in the following manner. For the given external forces $g = (g_1, g_2) \in L^2(\Omega)^2$, acting on the elastic body, we have to find a displacement field $u^\delta = (u_1^\delta, u_2^\delta)$, a stress tensor $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$, defined

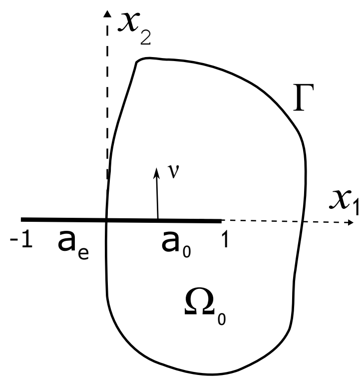

Figure 1.

in Ω_0, and a thin inclusion displacement v^δ defined on a such that

$$-\text{div }\sigma = g,\ \sigma - B\varepsilon(u^\delta) = 0 \text{ in } \Omega_0, \tag{1}$$

$$v^\delta_{,1111} = [\sigma_{22}] \text{ on } a_0;\ v^\delta_{,1111} = 0 \text{ on } a_e, \tag{2}$$

$$u^\delta = 0 \text{ on } \Gamma;\ v^\delta_{,11} = v^\delta_{,111} = 0 \text{ for } x_1 = -1, 1, \tag{3}$$

$$[u^\delta_2] \geq 0, -\sigma^+_{22} + \frac{1}{\delta}[u^\delta_2] \geq 0,\ -\sigma^\pm_{12} + \frac{1}{\delta}[u^\delta_1] = 0 \text{ on } a_0, \tag{4}$$

$$v^\delta = u^{\delta-}_2,\ [u^\delta_2](-\sigma^+_{22} + \frac{1}{\delta}[u^\delta_2]) = 0 \text{ on } a_0, \tag{5}$$

$$v^\delta(0) = 0,\ [v^\delta_{,1}(0)] = [v^\delta_{,11}(0)] = 0. \tag{6}$$

Here, $[p] = p^+ - p^-$ is a jump of a function p on a_0, where $p^\pm$ are the traces of p on the defect faces $a_0^\pm$. The signs $\pm$ correspond to positive and negative directions of ν; $v_{,1} = \frac{dv}{dx_1}$, $(x_1, x_2) \in \Omega$; $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is the strain tensor, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $i, j = 1, 2$. Functions defined on a we identify with functions of the variable x_1.

Relations (1) are the equilibrium equations for the elastic body and Hooke's law, respectively; (2) are equilibrium equations for the elastic inclusion parts a_0, a_e. The right-hand side $[\sigma_{22}]$ in (2) corresponds to forces acting on the inclusion from the surrounding elastic body. According to the first relation of (5), the vertical displacement of the elastic body at a_0^- coincides with the inclusion displacement. The second group of boundary conditions (3) is related to zero moments and zero transverse forces at the tip points of the inclusion a. The first inequality in (4) ensures the mutual non-penetration between the defect faces. Contact points between the defect faces are unknown a priori. If there is no contact at a given point x_0, i.e. $[u^\delta_2(x_0)] > 0$, then by (5), we get $-\sigma^+_{22}(x_0) + \frac{1}{\delta}[u^\delta_2(x_0)] = 0$. On the other hand, if $-\sigma^+_{22}(x_0) + \frac{1}{\delta}[u^\delta_2(x_0)] > 0$, we obtain a contact condition $[u^\delta_2(x_0)] = 0$. Relations (6) are junction conditions at the point $(0, 0)$.

The problem (1)-(6) can be written in a variational form. To this end, introduce the Sobolev spaces

$$H^1_\Gamma(\Omega_0) = \{\phi \in H^1(\Omega_0) \,|\, \phi = 0 \text{ on } \Gamma\},$$

$$H^{2,0}(a) = \{w \in H^2(a) \,|\, w(0) = 0\}$$

and a set of admissible displacements

$$P = \{(u, v) \in H^1_\Gamma(\Omega_0)^2 \times H^{2,0}(a) \mid u = (u_1, u_2);\ [u_2] \geq 0,\ u_2^- = v \text{ on } a_0\}.$$

Consider the energy functional with $\sigma(u) = B\varepsilon(u)$,

$$\pi(u, v) = \frac{1}{2}\int\limits_{\Omega_0} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_0} gu + \frac{1}{2}\int\limits_{a} v_{,11}^2 + \frac{1}{2\delta}\int\limits_{a_0} [u]^2.$$

Here and below, for simplicity, we write $\sigma(u)\varepsilon(u), gu$ instead of $\sigma_{ij}(u)\varepsilon_{ij}(u), g_i u_i$, respectively. Summation convention over repeated indices is used.

For any fixed δ, the minimization problem:

$$\text{find } (u^\delta, v^\delta) \in P \text{ such that } \pi(u^\delta, v^\delta) = \inf_P \pi \tag{7}$$

has a solution satisfying the variational inequality

$$(u^\delta, v^\delta) \in P, \tag{8}$$

$$\int\limits_{\Omega_0} \sigma(u^\delta)\varepsilon(\bar{u} - u^\delta) - \int\limits_{\Omega_0} g(\bar{u} - u^\delta) + \tag{9}$$

$$+ \int\limits_{a} v^\delta_{,11}(\bar{v}_{,11} - v^\delta_{,11}) + \frac{1}{\delta}\int\limits_{a_0} [u^\delta][\bar{u} - u^\delta] \geq 0 \ \ \forall\, (\bar{u}, \bar{v}) \in P.$$

A coercivity of the functional π can be proven as that in Khludnev and Leugering (2014). The set P is weakly closed, hence, the problem (7) indeed has a solution.

The following statement takes place.

Theorem 1 *Problem formulations* (1)-(6) *and* (8)-(9) *are equivalent, provided that the solutions are quite smooth.*

We omit the proof of this statement, since the proof basically reminds that given in Khludnev (2015), where the equilibrium problem for an elastic body with a thin inclusion is analyzed (without a damage parameter).

3. Passage to the limit in (8)-(9) as $\delta \to \infty$

This section is concerned with a justification of the passage to the limit as $\delta \to \infty$ in the problem (8)-(9). For any given δ, there exists a solution of the problem (8)-(9). From (8)-(9) it follows that

$$\int_{\Omega_0} \sigma(u^\delta)\varepsilon(u^\delta) - \int_{\Omega_0} g u^\delta + \frac{1}{\delta}\int_{a_0} [u^\delta]^2 + \int_a (v^\delta_{,11})^2 \pm \alpha \int_{a_0} (v^\delta)^2 = 0 \tag{10}$$

with $\alpha > 0$. Thus, by Korn's inequality, imbedding theorems and the first boundary condition of (5), we have for a small α

$$\frac{1}{2}\int_{\Omega_0} \sigma(u^\delta)\varepsilon(u^\delta) - \alpha \int_{a_0} (v^\delta)^2 \geq 0.$$

Moreover, it is possible to prove that there exists a constant $c_0 > 0$ such that

$$\int_a (v^\delta_{,11})^2 + \alpha \int_{a_0} (v^\delta)^2 \geq c_0 \|v^\delta\|^2_{H^{2,0}(a)}.$$

Hence, from (10), the following a priori estimate is obtained being uniform in δ,

$$\|u^\delta\|_{H^1_\Gamma(\Omega_0)^2} + \|v^\delta\|_{H^{2,0}(a)} \leq c.$$

Thus, by imbedding theorems,

$$\int_{a_0} [u^\delta]^2 \leq c.$$

By these estimates, choosing a sequence, if necessary, we assume that as $\delta \to \infty$

$$u^\delta \to u^\infty \text{ weakly in } H^1_\Gamma(\Omega_0)^2,\ [u^\delta] \to [u^\infty] \text{ weakly in } L^2(a_0)^2, \qquad v^\delta \to v^\infty \text{ weakly in } H^{2,0}(a). \tag{11}$$

In view of (11), passing to the limit in (8)-(9) as $\delta \to \infty$ we obtain

$$(u^\infty, v^\infty) \in P, \int_{\Omega_0} \sigma(u^\infty)\varepsilon(\bar{u} - u^\infty) - \int_{\Omega_0} g(\bar{u} - u^\infty) + \int_a v^\infty_{,11}(\bar{v}_{,11} - v^\infty_{,11}) \geq 0 \ \ \forall (\bar{u}, \bar{v}) \in P. \tag{12}$$

We can provide an equivalent (for smooth solutions) differential formulation of the problem (12). Namely, find a displacement field $u^\infty = (u_1^\infty, u_2^\infty)$ and a stress tensor $\sigma = \{\sigma_{ij}\}, i,j = 1,2$, defined in Ω_0, as well as a function v^∞, defined on a such that

$$-\text{div}\ \sigma = g,\ \sigma - B\varepsilon(u^\infty) = 0\ \text{ in }\ \Omega_0, \tag{13}$$

$$v^\infty_{,1111} = [\sigma_{22}] \text{ on } a_0;\ v^\infty_{,1111} = 0 \text{ on } a_e, \tag{14}$$

$$u^\infty = 0 \text{ on } \Gamma;\ v^\infty_{,11} = v^\infty_{,111} = 0 \text{ for } x_1 = -1, 1, \tag{15}$$

$$[u_2^\infty] \geq 0,\ \ \sigma_{22}^+ \leq 0,\ \sigma_{12}^\pm = 0,\ \sigma_{22}^+[u_2^\infty] = 0 \text{ on } a_0, \tag{16}$$

$$v^\infty = u_2^{\infty -} \text{ on } a_0, \tag{17}$$

$$v^\infty(0) = 0,\ [v^\infty_{,1}(0)] = [v^\infty_{,11}(0)] = 0. \tag{18}$$

The problem (12) or (13)-(18) describes an equilibrium state of the elastic body with the thin delaminated inclusion a and zero friction between $a_0^\pm$. This model was analyzed in Khludnev (2017).

Thus, we have proven the following statement.

Theorem 2 *Solutions* (u^δ, v^δ) *of the problems* (8)-(9) *converge in the sense* (11) *to the solution* (u^∞, v^∞) *of the problem* (12) *as* $\delta \to \infty$.

4. Passage to the limit in (8)-(9) as $\delta \to 0$

In this section, we analyze a passage to the limit as $\delta \to 0$ in the problem (8)-(9). For any fixed $\delta > 0$, consider the problem (8)-(9). The solutions of this problem satisfy the equality

$$\int\limits_{\Omega_0} \sigma(u^\delta)\varepsilon(u^\delta) - \int\limits_{\Omega_0} g u^\delta + \frac{1}{\delta}\int\limits_{a_0} [u^\delta]^2 + \int\limits_{a} (v^\delta_{,11})^2 \pm \tag{19}$$

$$\pm\alpha \int\limits_{a_0} (v^\delta)^2 = 0$$

with $\alpha > 0$. Taking small α, it follows from (19) that the following estimates hold

$$\|u^\delta\|_{H^1_\Gamma(\Omega)^2} + \|v^\delta\|_{H^{2,0}(a)} \leq c,\ \int\limits_{a_0} [u^\delta]^2 \leq c\delta$$

and they are uniform in δ. By these estimates, we can assume that as $\delta \to 0$,

$$u^\delta \to u^0 \text{ weakly in } H^1_\Gamma(\Omega_0)^2,\ [u^\delta] \to [u^0] = 0\ \text{ in } L^2(a_0)^2, \tag{20}$$

$$v^\delta \to v^0 \text{ weakly in } H^2(a).$$

It is clear that $u^0 \in H^1_0(\Omega)^2$, where

$$H^1_0(\Omega) = \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \Gamma\}.$$

Denote

$$W = \{(u,v) \in H_0^1(\Omega)^2 \times H^{2,0}(a) \mid u = (u_1,u_2);\ u_2 = v \text{ on } a_0\}.$$

We substitute in (9) a test function $(\bar{u},\bar{v}) = (u^\delta, v^\delta) \pm (\tilde{u},\tilde{v})$, $(\tilde{u},\tilde{v}) \in W$. It provides the relation

$$\int_{\Omega_0} \sigma(u^\delta)\varepsilon(\tilde{u}) - \int_{\Omega_0} g\tilde{u} + \int_a v^\delta_{,11}\tilde{v}_{,11} + \frac{1}{\delta}\int_{a_0} [u^\delta][\tilde{u}] = 0. \tag{21}$$

The last term of the left-hand side of (21) is zero, thus, by (20), it is possible to pass to the limit in (21) as $\delta \to 0$. This yields

$$\int_{\Omega_0} \sigma(u^0)\varepsilon(\tilde{u}) - \int_{\Omega_0} g\tilde{u} + \int_a v^0_{,11}\tilde{v}_{,11} = 0 \ \ \forall(\tilde{u},\tilde{v}) \in W,$$

and, by $[u^0] = 0$ on a_0, we can replace Ω_0 by Ω, which provides the identity

$$(u^0,v^0) \in W, \ \int_\Omega \sigma(u^0)\varepsilon(\tilde{u}) - \int_\Omega g\tilde{u} + \tag{22}$$
$$+ \int_a v^0_{,11}\tilde{v}_{,11} = 0 \ \ \forall(\tilde{u},\tilde{v}) \in W.$$

Consequently, the limit problem can be written in the form (22). Its equivalent differential formulation is as follows. We have to find functions $u^0 = (u_1^0, u_2^0)$, $\sigma = \{\sigma_{ij}\}, i,j = 1,2$, defined in Ω, as well as a function v^0, defined on a such that

$$-\text{div}\ \sigma = g,\ \sigma - B\varepsilon(u^0) = 0 \ \text{ in } \Omega_0, \tag{23}$$

$$v^0_{,1111} = [\sigma_{22}] \text{ on } a_0;\ v^0_{,1111} = 0 \text{ on } a_e, \tag{24}$$

$$u^0 = 0 \text{ on } \Gamma;\ v^0_{,11} = v^0_{,111} = 0 \text{ for } x_1 = -1, 1, \tag{25}$$

$$[u^0] = 0,\ v^0 = u_2^0,\ [\sigma_{12}] = 0 \text{ on } a_0, \tag{26}$$

$$v^0(0) = 0,\ [v^0_{,1}(0)] = [v^0_{,11}(0)] = 0. \tag{27}$$

We see that the limit problem (23)-(27) corresponds to the elastic body with the thin inclusion a and without any defects. Remark that we find the functions u^0, σ in the smooth domain Ω. Nevertheless, the equilibrium equation is fulfilled in the non-smooth domain Ω_0.

To conclude, we formulate the result of this section.

THEOREM 3 *Solutions* (u^δ, v^δ) *of the problems* (8)-(9) *converge in the sense* (20) *to the solution* (u^0, v^0) *of the problem* (22) *as* $\delta \to 0$.

5. Optimal control problem: case of elastic inclusion

This section concerns the analysis of the optimal control problem related to the models (8)-(9), (12), (22) with the damage parameter being a control function. For a given parameter $\delta > 0$, we can find the solution of the problem (8)-(9). For $\delta = \infty, \delta = 0$, it is also possible to find suitable solutions from (12), (22).

Let $v_* \in H^2(a_e)$ be a given function. For each $\delta \in (0, \infty)$ we find a solution (u^δ, v^δ) of the problem (8)-(9), and for $\delta = \infty$ we find the solution (u^∞, v^∞) of (12); for $\delta = 0$ we find the solution (u^0, l^0) of (22). Define a cost functional for $\delta \in [0, \infty]$,

$$G(\delta) = \|v^\delta - v_*\|_{H^2(a_e)}.$$

The optimal control problem is formulated as follows

$$\inf_{\delta \in [0,\infty]} G(\delta). \tag{28}$$

The cost functional characterizes a displacement of the inclusion part a_e. We see that a solution of the optimal control problem (28) allows us to find a damage parameter that minimizes a difference on a_e, in a suitable sense, between the solution v^δ and the given function v_*. Note that we compare solutions of different models corresponding to finite and infinite values of δ.

THEOREM 4 *There exists a solution of the optimal control problem* (28).

PROOF Let $\delta_n \in [0, \infty]$ be a minimizing sequence. For any δ_n we can find a unique solution of the problem like (8)-(9) provided δ_n is finite, or of the problems (12), (22) for $\delta = \infty, \delta = 0$, respectively. We can assume that the sequence is converging. There are three possible cases:

1. $\delta_n \to \delta_*,\ n \to \infty,\ \delta_n \in (0, \infty),\ \delta_* \in \mathbb{R}$;
2. $\delta_n \to \infty,\ n \to \infty,\ \delta_n \in (0, \infty)$;
3. $\delta_n \to 0,\ n \to \infty,\ \delta_n \in (0, \infty)$.

If $\delta_n = +\infty$ for $n \geq n_0$, or $\delta_n = 0$ for $n \geq m_0$, with some n_0, m_0, then a solution of the problem (28) exists. We consider the three cases separately.

Case 1. Consider the case when $\delta_n \to \delta_*,\ n \to \infty,\ \delta_n \in (0, \infty),\ \delta_* \in \mathbb{R}$. For every n we find a solution of the problem

$$(u^n, v^n) \in P, \tag{29}$$

$$\int_{\Omega_0} \sigma(u^n)\varepsilon(\bar{u} - u^n) - \int_{\Omega_0} g(\bar{u} - u^n) + \tag{30}$$

$$+ \int_a v^n_{,11}(\bar{v}_{,11} - v^n_{,11}) + \frac{1}{\delta_n}\int_{a_0} [u^n][\bar{u} - u^n] \geq 0 \ \ \forall\, (\bar{u}, \bar{v}) \in P.$$

We check that it is possible to pass to the limit in (29)-(30) as $n \to \infty$. Similarly as in Section 3, from (29)-(30) it follows uniformly in n,

$$\|u^n\|^2_{H^1_\Gamma(\Omega_0)^2} + \|v^n\|^2_{H^{2,0}(a)} \leq c. \tag{31}$$

Consequently, by (31), we can assume that as $n \to \infty$

$$u^n \to u \text{ weakly in } H^1_\Gamma(\Omega_0)^2,\ v^n \to v \text{ weakly in } H^{2,0}(a).$$

This convergence allows us to pass to the limit in (29)-(30) as $n \to \infty$, which yields

$$(u, v) \in P, \tag{32}$$

$$\int\limits_{\Omega_0} \sigma(u)\varepsilon(\bar u - u) - \int\limits_{\Omega_0} g(\bar u - u) + \tag{33}$$

$$+ \int\limits_a v_{,11}(\bar v_{,11} - v_{,11}) + \frac{1}{\delta_*} \int\limits_{a_0} [u][\bar u - u] \ge 0 \ \ \forall (\bar u, \bar v) \in P.$$

Relations (32)-(33) mean that $(u, v) = (u^{\delta_*}, v^{\delta_*})$. Thus, we have

$$\inf_{\delta \in [0,\infty]} G(\delta) = \liminf G(\delta_n) = \liminf \|v^n - v_*\|_{H^2(a_e)} \ge$$

$$\ge \|v - v_*\|_{H^2(a_e)} = G(\delta_*) \ge \inf_{\delta \in [0,\infty]} G(\delta),$$

and the existence proof is complete in the case 1.

Case 2. Consider the second case, when $\delta_n \to \infty,\ n \to \infty,\ \delta_n \in (0, \infty)$. In this situation, for any n, the solution (u^n, v^n) satisfies (29)-(30). From (29)-(30) we obtain the relation

$$\int\limits_{\Omega_0} \sigma(u^n)\varepsilon(u^n) - \int\limits_{\Omega_0} g u^n + \int\limits_a (v^n_{,11})^2 + \frac{1}{\delta_n} \int\limits_{a_0} [u^n]^2 = 0. \tag{34}$$

By the arguments used to derive the estimates (31), from (34) we derive uniformly in n

$$\|u^n\|^2_{H^1_\Gamma(\Omega_0)^2} \le c,\ \|v^n\|^2_{H^{2,0}(a)} + \|[u^n]\|_{L^2(a_0)^2} \le c. \tag{35}$$

In view of estimates (35), we assume that as $n \to \infty$

$$u^n \to u^\infty \text{ weakly in } H^1_\Gamma(\Omega_0)^2,\ v^n \to v^\infty \text{ weakly in } H^{2,0}(a), \tag{36}$$

$$[u^n] \to [u^\infty] \text{ weakly in } L^2(a_0)^2. \tag{37}$$

By (36)-(37), we pass to the limit in (29)-(30). It provides

$$(u^\infty, v^\infty) \in P,$$

$$\int\limits_{\Omega_0} \sigma(u^\infty)\varepsilon(\bar u - u^\infty) - \int\limits_{\Omega_0} g(\bar u - u^\infty) + \int\limits_a v^\infty_{,11}(\bar v_{,11} - v^\infty_{,11}) \ge 0 \ \ \forall (\bar u, \bar v) \in P.$$

Thus, the limit functions u^∞, v^∞ from (36)-(37) correspond to the damage parameter $\delta = \infty$; see Section 3. By this, we derive

$$\inf_{\delta\in[0,\infty]} G(\delta) = \liminf G(\delta_n) = \liminf \|v^n - v_*\|_{H^2(a_e)} \geq$$

$$\geq \|v^\infty - v_*\|_{H^2(a_e)} = G(\infty) \geq \inf_{\delta\in[0,\infty]} G(\delta).$$

Case 3. Consider the last case, when $\delta_n \to 0$, $n \to \infty$, $\delta_n \in (0,\infty)$. Like before, for any n, we find the solution (u^n, v^n) of the problem (29)-(30). Again, from the relation

$$\int_{\Omega_0} \sigma(u^n)\varepsilon(u^n) - \int_{\Omega_0} gu^n + \int_a (v^n_{,11})^2 + \frac{1}{\delta_n}\int_{a_0} [u^n]^2 = 0. \tag{38}$$

we derive the estimate uniform in n

$$\|u^n\|^2_{H^1_\Gamma(\Omega_0)^2} \leq c,\ \|v^n\|^2_{H^{2,0}(a)} \leq c. \tag{39}$$

Moreover, from (38) it follows that

$$\int_{a_0} [u^n]^2 \leq c\delta_n. \tag{40}$$

Choosing a subsequence, if necessary, by (39), (40), we can assume that as $n \to \infty$

$$u^n \to u^0 \text{ weakly in } H^1_\Gamma(\Omega_0)^2,\ v^n \to v^0 \text{ weakly in } H^{2,0}(a), \tag{41}$$

$$[u^0] = 0 \text{ on } a_0. \tag{42}$$

Choosing a test function $(\bar{u}, \bar{v})$ in (29)-(30) in the form $(\bar{u}, \bar{v}) = (u^n, v^n) \pm (\tilde{u}, \tilde{v})$, where $(\tilde{u}, \tilde{v}) \in W$, and passing to the limit as $n \to \infty$, we obtain

$$(u^0, v^0) \in W,\ \int_{\Omega_0} \sigma(u^0)\varepsilon(\tilde{u}) - \int_{\Omega_0} g\tilde{u} +$$

$$+ \int_a v^0_{,11}\tilde{v}_{,11} = 0\ \ \forall (\tilde{u}, \tilde{v}) \in W.$$

Thus, the limit functions u^0, v^0 from (41), (42) correspond to the zero value of the damage parameter; see Section 4. Consequently,

$$\inf_{\delta\in[0,\infty]} G(\delta) = \liminf G(\delta_n) = \liminf \|v^n - v_*\|_{H^2(a_e)} \geq$$

$$\geq \|v^0 - v_*\|_{H^2(a_e)} = G(0) \geq \inf_{\delta\in[0,\infty]} G(\delta).$$

The proof of Theorem 4 is complete. □

6. Rigidity parameter of thin inclusion tends to infinity

When analyzing the model (1)-(6) we assumed that the parameter, responsible for the rigidity properties of the elastic inclusion, is equal to 1. In this section, this parameter will be introduced into the model, and a passage to infinity will be analyzed as the rigidity parameter goes to infinity. In so doing, we fix the damage parameter δ. Moreover, we assume that a given force $h \in L^2(a)$ is applied to the inclusion a.

Consider the equilibrium problem for the elastic body Ω_0 with the elastic inclusion a assuming that $\beta > 0$ is the rigidity parameter of the inclusion. We have to find a displacement field $u^\beta = (u_1^\beta, u_2^\beta)$, a stress tensor $\sigma = \{\sigma_{ij}\},\ i,j = 1,2,$ defined in Ω_0, and a thin inclusion displacement v^β, defined on a, such that

$$-\text{div}\ \sigma = g,\ \sigma - B\varepsilon(u^\beta) = 0 \text{ in } \Omega_0, \tag{43}$$

$$\beta v^\beta_{,1111} = [\sigma_{22}] + h \text{ on } a_0;\ \beta v^\beta_{,1111} = h \text{ on } a_e, \tag{44}$$

$$u^\beta = 0 \text{ on } \Gamma;\ v^\beta_{,11} = v^\beta_{,111} = 0 \text{ for } x_1 = -1, 1, \tag{45}$$

$$[u_2^\beta] \geq 0, -\sigma_{22}^+ + \frac{1}{\delta}[u_2^\beta] \geq 0,\ -\sigma_{12}^\pm + \frac{1}{\delta}[u_1^\beta] = 0 \text{ on } a_0, \tag{46}$$

$$v^\beta = u_2^{\beta -},\ [u_2^\beta](-\sigma_{22}^+ + \frac{1}{\delta}[u_2^\beta]) = 0 \ \text{ on } a_0, \tag{47}$$

$$v^\beta(0) = 0,\ [v^\beta_{,1}(0)] = [v^\beta_{,11}(0)] = 0. \tag{48}$$

The problem (43)-(48) admits the variational formulation

$$(u^\beta, v^\beta) \in P, \tag{49}$$

$$\int\limits_{\Omega_0} \sigma(u^\beta)\varepsilon(\bar{u} - u^\beta) - \int\limits_{\Omega_0} g(\bar{u} - u^\beta) - \int\limits_a h(\bar{v} - v^\beta) + \tag{50}$$

$$+\beta \int\limits_a v^\beta_{,11}(\bar{v}_{,11} - v^\beta_{,11}) + \frac{1}{\delta}\int\limits_{a_0} [u^\beta][\bar{u} - u^\beta] \geq 0 \qquad \forall\,(\bar{u}, \bar{v}) \in P.$$

To analyze the passage to the limit as $\beta \to \infty$ in (49)-(50) we first obtain a priori estimates. The relations (49)-(50) imply with a small $\alpha > 0$

$$\int\limits_{\Omega_0} \sigma(u^\beta)\varepsilon(u^\beta) - \int\limits_{\Omega_0} g u^\beta + \frac{1}{\delta}\int\limits_{a_0} [u^\beta]^2 + \beta \int\limits_a (v^\beta_{,11})^2 - \tag{51}$$

$$- \int\limits_a h v^\beta \pm \alpha \int\limits_{a_0} (v^\beta)^2 = 0.$$

Taking into account the arguments used in Section 3, from (51) we obtain, uniformly in $\beta \geq \beta_0 > 0$

$$\|u^\beta\|_{H^1_\Gamma(\Omega_0)^2} + \|v^\beta\|_{H^{2,0}(a)} \leq c$$

and, moreover,

$$\int_a (v^\beta_{,11})^2 \le \frac{c}{\beta}.$$

Choosing a subsequence, if necessary, we assume that as $\beta \to \infty$,

$$u^\beta \to u \text{ weakly in } H^1_\Gamma(\Omega_0)^2,\ v^\beta \to v \text{ weakly in } H^{2,0}(a), \qquad (52)$$
$$v_{,11} = 0 \text{ on } a,$$

thus

$$v(x_1) = c_0 + c_1 x_1,\ x_1 \in (-1,1),\ c_0, c_1 \in \mathbb{R}.$$

In view of the boundary condition $v(0) = 0$ this provides

$$v(x_1) \equiv l_0(x_1) = c_1 x_1,\ x_1 \in (-1,1).$$

Introduce the set of admissible displacements for the limit problem

$$P_\infty = \{(u,l) \in H^1_\Gamma(\Omega_0)^2 \times L(a) \mid u = (u_1,u_2);\ [u_2] \ge 0,\ u_2^- = l \text{ on } a_0\},$$

where

$$L(a) = \{l \mid l(x_1) = cx_1,\ x_1 \in (-1,1),\ c \in \mathbb{R}\}.$$

In the definitions of $P_\infty, L(a)$, a function l and a constant c are arbitrary, respectively.

Now we take $(\bar u, \bar l) \in P_\infty$. Then $(\bar u, \bar l) \in P$. Substituting $(\bar u, \bar l)$ as a test function in (50), by (52), we can provide the passage to the limit as $\beta \to \infty$. The limit relation takes the form

$$(u, l_0) \in P_\infty, \qquad (53)$$

$$\int_{\Omega_0} \sigma(u)\varepsilon(\bar u - u) - \int_{\Omega_0} g(\bar u - u) - \qquad (54)$$

$$- \int_a h(\bar l - l_0) + \frac{1}{\delta}\int_{a_0} [u][\bar u - u] \ge 0 \ \ \forall (\bar u, \bar l) \in P_\infty.$$

Thus, we have proven the following statement.

Theorem 5 *Solutions* (u^β, v^β) *of the problems* (49)-(50) *converge in the sense* (52) *to the solution* (u, l_0) *of the problem* (53)-(54) *as* $\beta \to \infty$.

The problem (53)-(54) can be written in the differential form: find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, defined in Ω_0, and a thin inclusion displacement $l_0 \in L(a)$ such that

$$-\text{div } \sigma = g,\ \sigma - B\varepsilon(u) = 0 \text{ in } \Omega_0, \tag{55}$$

$$u = 0 \text{ on } \Gamma;\ l_0 = u_2^- \text{ on } a_0, \tag{56}$$

$$[u_2] \geq 0, -\sigma_{22}^+ + \frac{1}{\delta}[u_2] \geq 0,\ -\sigma_{12}^\pm + \frac{1}{\delta}[u_1] = 0 \text{ on } a_0, \tag{57}$$

$$[u_2](-\sigma_{22}^+ + \frac{1}{\delta}[u_2]) = 0 \text{ on } a_0, \tag{58}$$

$$\int\limits_{a_0} [\sigma_{22}]l + \int\limits_a hl = 0\ \forall l \in L(a). \tag{59}$$

Note that the displacement l_0 of the inclusion a corresponds to a thin rigid inclusion.

It can be proven that problem formulations (53)-(54) and (55)-(59) are equivalent for smooth solutions. We omit the arguments.

7. Passage to the limit in (53)-(54) as $\delta \to \infty$

Denote by (u^δ, l^δ) a solution of the problem (53)-(54) for a given δ, i.e.

$$(u^\delta, l^\delta) \in P_\infty, \tag{60}$$

$$\int\limits_{\Omega_0} \sigma(u^\delta)\varepsilon(\bar{u} - u^\delta) - \int\limits_{\Omega_0} g(\bar{u} - u^\delta) - \tag{61}$$

$$- \int\limits_a h(\bar{l} - l^\delta) + \frac{1}{\delta}\int\limits_{a_0} [u^\delta][\bar{u} - u^\delta] \geq 0\ \ \forall (\bar{u}, \bar{l}) \in P_\infty.$$

The aim of the arguments below is a justification of the passage to the limit as $\delta \to \infty$ in the problem (60)-(61). First note that the solution u^δ of (60)-(61) satisfies the following relation with $\alpha > 0$,

$$\int\limits_{\Omega_0} \sigma(u^\delta)\varepsilon(u^\delta) - \int\limits_{\Omega_0} gu^\delta + \frac{1}{\delta}\int\limits_{a_0} [u^\delta]^2 - \int\limits_a hl^\delta \pm \alpha \int\limits_{a_0} (l^\delta)^2 = 0. \tag{62}$$

By Korn's inequality, the boundary condition $u_2^{\delta -} = l^\delta$ on a_0 and the embedding theorems, we have for a small α

$$\frac{1}{2}\int\limits_{\Omega_0} \sigma(u^\delta)\varepsilon(u^\delta) - \alpha \int\limits_{a_0} (l^\delta)^2 \geq 0.$$

Then, from (62) it follows that

$$\frac{1}{2}\int\limits_{\Omega_0}\sigma(u^\delta)\varepsilon(u^\delta)-\int\limits_{\Omega_0}gu^\delta+\frac{1}{\delta}\int\limits_{a_0}[u^\delta]^2-\int\limits_{a}hl^\delta+\alpha\int\limits_{a_0}(l^\delta)^2\le 0. \tag{63}$$

Note that there exists a constant $c_0>0$ such that

$$\|l\|_{L^2(a_0)}\ge c_0\|l\|_{L^2(a)}\ \ \forall l\in L(a).$$

Consequently, from (63) we have uniformly in δ

$$\|u^\delta\|_{H^1_\Gamma(\Omega_0)^2}\le c,\ \|l^\delta\|_{L^2(a)}\le c,$$

and by the embedding theorems, it follows that

$$\int\limits_{a_0}[u^\delta]^2\le c.$$

By these estimates, choosing a sequence, we can assume that as $\delta\to\infty$,

$$\begin{aligned}u^\delta\to u^\infty \text{ weakly in } H^1_\Gamma(\Omega_0)^2,\ [u^\delta]\to[u^\infty] \text{ weakly in } L^2(a_0)^2,\\ l^\delta\to l^\infty \text{ weakly in } L^2(a).\end{aligned} \tag{64}$$

Passing to the limit as $\delta\to\infty$ in (60)-(61) on the basis of (64), we derive

$$(u^\infty,l^\infty)\in P_\infty, \tag{65}$$

$$\int\limits_{\Omega_0}\sigma(u^\infty)\varepsilon(\bar u-u^\infty)-\int\limits_{\Omega_0}g(\bar u-u^\infty)-\int\limits_{a}h(\bar l-l^\infty)\ge 0\ \ \forall\,(\bar u,\bar l)\in P_\infty. \tag{66}$$

The following statement is proven.

Theorem 6 *Solutions* (u^δ,v^δ) *of the problems* (60)-(61) *converge in the sense* (64) *to the solution* (u^∞,l^∞) *of the problem* (65)-(66) *as* $\delta\to\infty$.

An equivalent differential formulation of the problem (65)-(66) is as follows: find a displacement field $u^\infty=(u_1^\infty,u_2^\infty)$, a stress tensor $\sigma=\{\sigma_{ij}\},\ i,j=1,2$, defined in Ω_0, and a thin inclusion displacement $l^\infty\in L(a)$ such that

$$-\text{div}\ \sigma=g,\ \sigma-B\varepsilon(u^\infty)=0 \text{ in } \Omega_0, \tag{67}$$

$$u^\infty=0 \text{ on } \Gamma;\ l^\infty=u_2^{\infty-} \text{ on } a_0, \tag{68}$$

$$[u_2^\infty]\ge 0,\sigma_{22}^+\le 0,\ \sigma_{12}^\pm=0,\ [u_2^\infty]\sigma_{22}^+=0 \text{ on } a_0, \tag{69}$$

$$\int\limits_{a_0}[\sigma_{22}]\bar l+\int\limits_{a}h\bar l=0\ \ \forall\bar l\in L(a). \tag{70}$$

The model (67)-(70) describes an equilibrium problem for the elastic body Ω_0 with the thin delaminated rigid inclusion a; the friction between the defect faces $a_0^\pm$ is zero.

8. Passage to the limit in (53)-(54) as $\delta \to 0$

In this section, we analyze the passage to the limit in (53)-(54) as $\delta \to 0$. Again, denote by (u^δ, l^δ) a solution of the problem (53)-(54) corresponding to the parameter δ. Hence, we consider the problem (60)-(61). Like in the previous section, the following estimates take place, being uniform in δ,

$$\|u^\delta\|_{H^1_\Gamma(\Omega_0)^2} \le c, \ \|l^\delta\|_{L^2(a)} \le c.$$

Moreover, from (63) it follows uniformly in δ that

$$\int\limits_{a_0} [u^\delta]^2 \le c\delta.$$

Consequently, we can assume that as $\delta \to 0$

$$u^\delta \to u^0 \text{ weakly in } H^1_\Gamma(\Omega_0)^2, \ [u^\delta] \to [u^0] = 0 \text{ weakly in } L^2(a_0)^2, \qquad l^\delta \to l^0 \text{ weakly in } L^2(a). \tag{71}$$

Introduce the set of admissible displacements for the limit problem

$$W_0 = \{(u,l) \in H^1_\Gamma(\Omega_0)^2 \times L(a) \mid u = (u_1, u_2);\ [u] = 0,\ u_2 = l \text{ on } a_0\}.$$

The convergence (71) allows us to pass to the limit in (60)-(61) as $\delta \to 0$, which yields

$$(u^0, l^0) \in W_0, \tag{72}$$

$$\int\limits_{\Omega_0} \sigma(u^0)\varepsilon(\bar u) - \int\limits_{\Omega_0} g\bar u - \int\limits_a h\bar l = 0 \ \ \forall\, (\bar u, \bar l) \in W_0. \tag{73}$$

Since $[u^0] = 0$ on a_0 we can replace the domain Ω_0 by Ω in (73). In fact, in the definition of W_0 it is possible to replace Ω_0 by Ω.

Hence, the following statement is proven.

Theorem 7 *Solutions* (u^δ, v^δ) *of the problems* (60)-(61) *converge in the sense* (71) *to the solution* (u^0, l^0) *of the problem* (72)-(73) *as* $\delta \to 0$.

To conclude this section, we write down an equivalent differential formulation of the problem (72)-(73): find a displacement field $u^0 = (u^0_1, u^0_2)$, a stress tensor $\sigma = \{\sigma_{ij}\},\ i,j = 1,2,$ defined in Ω_0, and a thin inclusion displacement $l^0 \in L(a)$ such that

$$-\text{div}\ \sigma = g,\ \sigma - B\varepsilon(u^0) = 0 \text{ in } \Omega_0, \tag{74}$$

$$u^0 = 0 \text{ on } \Gamma;\ [u^0] = 0,\ l^0 = u^0_2,\ [\sigma_{12}] = 0 \text{ on } a_0, \tag{75}$$

$$\int\limits_{a_0} [\sigma_{22}]\bar l + \int\limits_a h\bar l = 0 \ \ \forall \bar l \in L(a). \tag{76}$$

The problem (74)-(76) describes an equilibrium state of the elastic body Ω_0 with the rigid inclusion a and without any defects.

Remark that the Hooke's law from (74) holds in the smooth domain Ω. Meanwhile, the equilibrium equation holds in the non-smooth domain Ω_0.

9. Optimal control problem: the case of rigid inclusion

As we know, it is possible to find a solution (u^δ, l^δ) of the problem (60)-(61) for any fixed $\delta > 0$. On the other hand, we have a solution (u^∞, l^∞) of the problem (65)-(66), corresponding to $\delta = \infty$, and a solution (u^0, l^0) of the problem (72)-(73), corresponding to $\delta = 0$. Let $l^* \in L(a)$ be a given function. For any parameter $\delta \in [0, \infty]$, define a cost functional

$$G(\delta) = \|l^\delta - l^*\|_{L^2(a_e)},$$

where (u^δ, l^δ) are solutions of the problems mentioned above, $\delta \in [0, \infty]$. Consider the optimal control problem

$$\inf_{\delta \in [0,\infty]} G(\delta). \tag{77}$$

A solution of the problem (77) minimizes the difference at a_e between the displacement l^δ of the thin inclusion and the given function l^*. We should underline that the problem formulations for finding l^δ are different for $\delta \in (0, \infty), \delta = \infty, \delta = 0$.

The following statement takes place in this context.

Theorem 8 *There exists a solution of the optimal control problem* (77).

We omit the proof of this theorem since it basically reminds that of Theorem 4.

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