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A note on failure rate of the mixture of an exponential distribution and distribution with increasing quadratic failure rate function*

by

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Abstract: We show that a unimodal failure rate function can be obtained as a mixture of two increasing failure rate functions. Specifically, we study the failure rate of the mixture of an exponential distribution and an IFR (increasing failure rate) distribution with increasing quadratic failure rate function. At the end of the paper we show a numerical example of the modified unimodal failure rate function.

Keywords: mixture of distributions, quadratic failure rate function, failure rate function

1. Introduction

The distributions with non-monotone failure rate function are being considered in the theory and practice of reliability. The distributions with the bathtub and the upside-down bathtub shaped failure functions belong among such distributions. In reliability theory, the models with a bathtub shape failure function are very useful. There are many known applications of failure rate functions with the upside-down bathtub shape. For example, in the papers by Jiang and Murthy (1998), Mudholkar, Sirvastava and Trimer (1995), and Xie, Tang and Goh (2002) the failure rate functions of this shape are used. One method of generating a distribution with non-monotone failure rate function is the mixing of standard distributions (see the review by Lai and Xie, 2005). There is a well-known result showing that a mixture of distributions with decreasing failure rate functions (DFR) has a decreasing failure rate function (see Barlow, Marshall and Prochan, 1963). Gurland and Sethurman (1995) have given a condition under which a mixture of an exponential and an IFR distributions is a DFR distribution. In this paper, we divide the set of shapes of failure rate functions into the following six categories:

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(a) monotonic failure rate if it is either increasing (IFR) or decreasing (DFR);

(b) bathtub type failure rate if λ(t) is of bathtub shape (BT);

(c) modified bathtub failure rate if λ(t) is increasing and then bathtub (MBT);

(d) upside-down bathtub (UBT) shape or unimodal shape;

(e) modified upside-down bathtub if λ(t) is first decreasing and then upside-down bathtub (MBUT);

(f) roller coaster (decreasing, increasing, decreasing and increasing or increasing, decreasing, increasing and decreasing).

Klutke, Kiessler and Wortman (2003) studied the mixture of Weibull and exponential distributions, and suggested that the mixture can be a distribution with unimodal shape failure rate function. However, Wondmagegnehu, Navarro and Hernandez (2005) showed that this failure rate function has a decreasing initial period. It is shown in the above-cited work that the failure rate function of the mixture of exponential and Weibull distributions takes one of three shapes: UBT, MUBT and roller-coaster (in this case increasing, decreasing and increasing). Jiang and Murthy (1998) categorized the possible shapes of the failure rate function for a mixture of any two Weibull distributions in terms of five parameters. The failure rate can be shaped as eight different types, including IFR, DFR, MBT, UTB and roller-coaster. It is shown by many authors (see, for instance, Jiang and Xiao, 2003, or Wondmagegnehu, 2004) that this mixture distribution cannot have a BT failure rate. They also stated that the mixture of the failure rates from two strictly IFR Weibull distributions with the same shape parameter can be either MTB or IFR.

Wondmagegnehu (2004) developed over the work of Jiang and Murthy (1998) and assumed the two Weibull distributions involved to be strictly IFR. He also used several examples to illustrate the possible shapes that the mixture failure rate can take on when the two Weibull distributions have different shapes and scale parameters. In Gupta and Warren (2001) the shape of the failure rate function of the mixture of two gamma distributions was studied. In this case, the possible shapes are IFR, DFR, BT and MBT.

Block, Savits and Wondmagegnehu (2003) gave explicit conditions, which delineate the possible shapes of the failure rate function for the mixture of two IFR linear distributions. Let the two increasing linear failure rates be given by, respectively,

$\lambda_1(t) = c_1 t + d_1,$ and $\lambda_2(t) = c_1 t + d_2,$

where $c_1 \geq c_2 > 0$ and $d_{1,} d_2 \geq 0$. The failure rate shape can be of three different types: IFR, BT and MTB. This mixture depends on five parameters: c_1, c_2, $d_{1,}$ d_2, and the mixing parameter p. In this paper we derive a mixture of exponential, $\lambda_1(t) = \lambda$, and increasing quadratic, $\lambda_2(t) = at^2 + bt + c$, where

a ¿ 0, $b \geq 0$, $c \geq 0$, failure rate functions. This mixture depends also on five parameters: a, b, c, λ and p. The failure rate shape can be of three different types: DFR, UBT and MUTB.

2. The model

We consider a mixture, involving lifetimes T_1 and T_2, with the densities $f_1(t)$ and $f_2(t)$, the reliability functions $R_1(t)$ and $R_2(t)$, the failure rate functions $\lambda_1(t)$, $\lambda_2(t)$, and weights p and $q = 1 - p$, where $0 < p < 1$. The mixed density function is then written as:

$$f(t) = pf_1(t) + (1 - p)f_2(t),$$

and the reliability function is:

$$R(t) = pR_1(t) + (1 - p)R_2(t).$$

The failure rate function of the mixture can be written as the following mixture:

$$\lambda(t) = w(t)\lambda_1(t) + (1 - w(t))\lambda_2(t),$$

where $w(t) = pR_1(t)/R(t)$.

Proposition 1 *For the first derivative of w(t), we have:*

$$w'(t) = w(t)(1 - w(t))(\lambda_2(t) - \lambda_1(t)).$$

Proposition 2 *The first derivative of $\lambda(t)$ is:*

$$\lambda'(t) = (1 - w(t))((1 - w(t)(\lambda_2(t) - \lambda_1(t))^2 + \lambda_2'(t)) + w(t)\lambda_1'(t)).$$

Proposition 3 *If $\lambda_1(t) = \lambda$ then*

$$\lambda'(t) = (1 - w(t))((1 - w(t)(\lambda_2(t) - \lambda)^2 + \lambda_2'(t)).$$

3. Mixture of an exponential distribution and a distribution with increasing quadratic failure rate function

Let

$$\lambda_1(t) = \lambda,\ \lambda_2(t) = at^2 + bt + c,\ g_1(t) = w(t)(at^2 + bt + c - \lambda)^2,\ g_2(t) = 2at + b,$$

where $a > 0, b \geq 0, c \geq 0$.

The equation $\lambda'(t) = 0$ is equivalent to the equation:

$$g_1(t) = g_2(t). \tag{1}$$

For the ratio $u(t) = g_1(t)\ /\ g_2(t)$, we have:

$$\lim_{t\to\infty} u(t) = \infty. \qquad (2)$$

If $c < \lambda$, then there is t_0 such that $g_2(t_0) = 0$. For the first derivative of

$$u_1(t) = (\lambda_2(t) - \lambda)^2/g_1(t),$$

we calculate:

$$u'(t) = \frac{2(at^2 + bt + c - \lambda)}{(2at + b)^2}\{3a^2t^2 + 3abt + b^2 - a(c - \lambda)\}. \qquad (3)$$

We will now consider three cases:

Case A: $c \leq \lambda$.

If $c = \lambda$, then $u(t) = u_1(t)w(t)$, increasing from $u(0) = 0$ to $u(\infty) = \infty$, and the equation $u(t) = 1$ has exactly one solution. In this case, $\lambda(t)$ is UBT.

If $c < \lambda$, the function $g_1(t)$ is an increasing function, and $g_2(t)$ is decreasing on the interval $(0, t_0)$. If p $(c - \lambda)^2 \leq b$, then equation (1) has no solution on $(0, t_0)$. If $p(c - \lambda)^2 > b$, then equation (1) has exactly one solution on $(0, t_0)$.

If c ¡ λ, then, by (3), u'$_1(t)$ ¿ 0 on $(t_0,\ \infty)$. By Proposition 1, the function $w(t)$ is increasing. The functions $u_1(t)$ and $w(t)$ are continuous and increasing on $(t_0,\ \infty)$. The equation $u(t) = 1$ has exactly one solution.

If $p(c - \lambda)^2 \leq b$, then the failure rate function $\lambda(t)$ is UBT. If $p(c - \lambda)^2 > b$, then $\lambda(t)$ assumes one minimum and one maximum (MUBT).

Corollary 1 *If $c \leq \lambda$, then $\lambda(t)$ is UBT or MUBT.*

Case B: $c > \lambda,\ \ b^2 - a(c - \lambda) \geq 0$.

By (3) and Proposition 1, the function $u(t) = u_1(t)w(t)$ increases from $u(0) = p(c - \lambda)^2/\ b$ to $u(\infty) = \infty$.

If $p(c - \lambda)^2/\ b$ ¡ 1, then $\lambda(t)$ is UBT, and if $p(c - \lambda)^2/\ b \geq 1$, then $\lambda(t)$ is DFR.

Corollary 2 *If $c > \lambda$ and $b^2 - a(c - \lambda) \geq 0$ then $\lambda(t)$ is DFR or UBT.*

Case C: $c > \lambda$, $b^2 - a(c - \lambda) < 0$.

By (3) we conclude that there is t_1 such that $t_1 > 0$ and $u_1'(t_1) = 0$. The function $w(t)$ is increasing for $t \in (0, \infty)$, this being the consequence of the fact that $c - \lambda > 0$. Since $u(t) = u_1(t)w(t)$, this function is increasing for $t \in (t_1, \infty)$. From the above, and from the fact that $u(0) = p(c - \lambda)^2/b$, we have the thesis of the next corollary:

Corollary 3 *If $c > \lambda, b^2 - a(c - \lambda) < 0$ and $p(c - \lambda)^2/b \leq 1$, then the equation $u(t) = 1$ for $t > t_1$ has exactly one solution.*

We calculate the first derivative $u'(t)$:

$$u'(t) = \frac{w(t)(at^2 + bt + c - \lambda)}{(2at + b)^2} \{z_1(t) + z_2(t)\}$$

where
$z_1(t) = (1 - w(t))\ (at^2 + bt + c - \lambda)^2\ (2at + b)$,
$z_2(t) = 2(3a^2t^2 + 3abt + b^2 - a\ (c - \lambda))$.

For the function $z(t) = z_1(t) + z_2(t)$ we calculate:

$$z(0) = (1 - p)(c - \lambda)^2 b + 2(b^2 - a(c - \lambda).$$

The sign of the term $z(0)$ can be " + " or " − ". The first derivative $z_1^{'}(t)$ can by written down as:

$$z_1'(t) = (1 - w(t))(at^2 + bt + c - \lambda)((-w(t)(at^2 + bt + c - \lambda)^2(2at + b) + \\ +2(2at + b)^2 + 2a(at^2 + bt + c - \lambda)^2).$$

The inequality $z_1'(t)$ ¿ 0 holds if and only if
$w(t) < 2(2at + b)/(at^2 + bt + c - \lambda)^2 + 2a/((at^2 + bt + c - \lambda)(2at + b))$.
The last inequality is equivalent to

$u(t) < 2 + 2a(at^2 + bt + c - \lambda)/(2at + b)^2$.

The function $w(t)$ is increasing for $t > 0$ and $u_1'(t) > 0$ for $t > t_1$, therefore $u(t)$ is increasing for $t > t_1$. We can see that if $u(t) < 2$, then $z(t)$ is increasing on $(0, t_1)$. The function $z_2(t)$ is increasing and changes the sign from " − " to " + " at the point t_1. If $z(0) \geq 0$, then the function $u(t)$ is increasing, and $\lambda(t)$ is UBT or increasing (DFR). If $z(0) < 0$, then $z(t)$ changes the sign from " − " to " + ", since $u(t)$ assumes a minimum at a point $t_2 \leq t_1$. In this case, the function $u(t)$ has exactly one minimum, and if $p(c - \lambda)^2 / b > 1$, then $\lambda(t)$ is DFR or MUBT. By this and Corollary 3, we have:

COROLLARY 4 *If $c > \lambda, b^2 - a(c - \lambda) < 0$, then $\lambda(t)$ is DFR or UBT or MUBT.*

Summing up the above, we can formulate the following theorem:

THEOREM 1 *We assume that $\lambda_1(t) = \lambda$ and $\lambda_2(t) = at^2 + bt + c$, where $a > 0$, $b \geq 0$, $c \geq 0$. The shape of the failure rate function of the mixture of an exponential distribution with parameter λ and a distribution with failure rate function $\lambda_2(t)$ can be as follows:*
if $c \leq \lambda$ then λ (t) is UBT or MUBT,
if $c > \lambda$ and $b^2 - a(c - \lambda) \geq 0$ then $\lambda(t)$ is DFR or UBT,
if $c > \lambda$ and $b^2 - a(c - \lambda) < 0$ then $\lambda(t)$ is UBT or MUBT.

4. Numerical example

In this example, we assume that the mixing parameter $p \in \{0.5, 0.6, 0.7\}$, and the other parameters are: $a = 2$, $b = 0.1$, $c = 2$, $\lambda = 1$. We have:

$$c - \lambda > 0, \; b^2 - a(c - \lambda) < 0, \; \text{and } p(c - \lambda)^2/b > 1$$

and we will consider the case C, when $\lambda(t)$ has one minimum and one maximum. The shape of the failure rate function of the mixture is MUBT. The respective illustration is provided in Fig. 1.

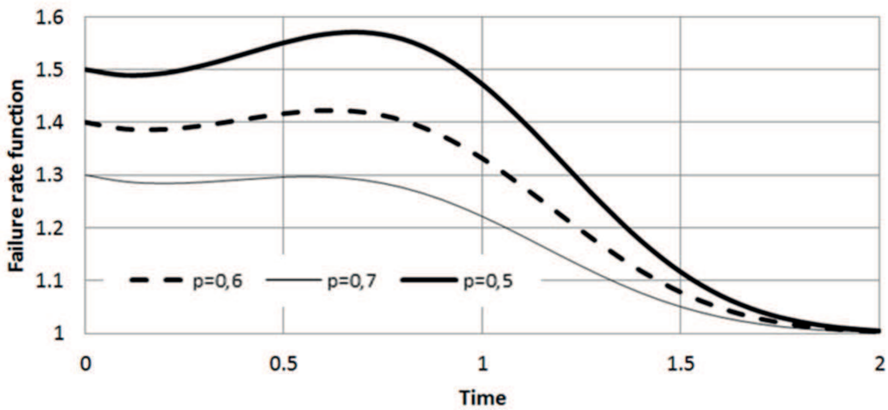

Figure 1. The failure rate function of the mixture of an exponential distribution and a distribution with increasing quadratic failure rate function (case C)

5. Conclusion

The paper considers a lifetime model with a simple failure rate function. The failure rate function has three shapes. In this paper, it is shown that the distribution of lifetime with the unimodal failure rate can be obtained from a mixture of an exponential distribution and a distribution with a (IFR) quadratic failure rate function.

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