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On optimal and quasi-optimal controls in coefficients for multi-dimensional thermistor problem with mixed Dirichlet-Neumann boundary conditions*

by

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Dedicated to Günter Leugering on the occasion of His 65th birthday

Abstract: In this paper we deal with an optimal control problem in coefficients for the system of two coupled elliptic equations, also known as the thermistor problem, which provides a simultaneous description of the electric field $u = u(x)$ and temperature $\theta(x)$. The coefficients of the operator div $(B(x)\,\nabla\,\theta(x))$ are used as the controls in $L^\infty(\Omega)$. The optimal control problem is to minimize the discrepancy between a given distribution $\theta_d \in L^r(\Omega)$ and the temperature of thermistor $\theta \in W_0^{1,\gamma}(\Omega)$ by choosing an appropriate anisotropic heat conductivity matrix B. Basing on the perturbation theory of extremal problems and the concept of fictitious controls, we propose an "approximation approach" and discuss the existence of the so-called quasi-optimal and optimal solutions to the given problem.

Keywords: nonlinear elliptic equations, control in coefficients, $p(x)$-Laplacian, approximation approach, thermistor problem

1. Introduction

1.1. The settings

In a bounded open domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N are the disjoint parts of the boundary $\partial\Omega$ with positive $(N-1)$-dimensional measures, we consider, as a control object, a boundary value problem describing the coupling between the electric field with potential u and the temperature θ in an anisotropic thermistor, where

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its anisotropic heat conductivity is given by a matrix of positive coefficients $B = [b_{i\,j}(x)]_{i,\,j=1,\dots,N}$. This model, also known as thermistor problem, is based on rational mechanics of electrorheological fluids, which takes into account the complex interactions between the electromagnetic fields and the moving liquid. In particular, the electrorheological fluids have the interesting property that their viscosity depends on the electric field in the fluid. In this paper, we deal with the following optimization problem:

$$\text{Minimize} \quad \left\{ J(B,u,\theta) = \int_\Omega |\theta(x) - \theta_d(x)|^r \, dx \right\} \tag{1}$$

subject to the constraints

$$\begin{aligned}
&\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \operatorname{div} g \quad \text{in} \ \ \Omega,\\
&\qquad u = 0 \ \text{ on } \ \Gamma_D, \quad |\nabla u|^{p-2}\partial_\nu u = 0 \ \text{ on } \ \Gamma_N, &(2)\\
&-\operatorname{div}(B\nabla\theta) = |\nabla u|^p \quad \text{in} \ \ \Omega, \quad \theta|_{\partial\Omega} = 0, &(3)\\
&p(\cdot) = \sigma(\theta(\cdot)) \quad \text{a.e. in} \ \ \Omega, \quad B \in \mathfrak{B}_{ad}, &(4)\\
&\mathfrak{B}_{ad} =\\
&\left\{ B \in BV(\Omega)^{N\times N} \ : \ m_1 I \le B(\cdot) \le m_2 I, \ \int_\Omega |Db_{ij}| \le \mu \ \forall\, i,j = \overline{1,N} \right\}. &(5)
\end{aligned}$$

where $r \in \left(1, \frac{N}{N-2}\right)$ if $N > 2$ and $r \in (1, +\infty)$ for $N = 2$ is a given value, m_1 and m_2 are constants such that $0 < m_1 \le m_2 < +\infty$, I is the identity matrix in $\mathbb{R}^{N\times N}$, the inequalities (5) are in the sense of the quadratic forms, defined by $(B\xi,\xi)_{\mathbb{R}^N}$ for $\xi \in \mathbb{R}^N$, $\theta_d \in L^r(\Omega)$ and $g \in L^\infty(\Omega)^N$ are given distributions such that $(g,\nu)_{\mathbb{R}^N} = 0$ $\mathcal{H}^{N-1} = 0$-a.e. on Γ_N, σ is a continuous function such that $\alpha \le \sigma(y) \le \beta$ for all $y \in \mathbb{R}$, and the constants α and β satisfy the condition

$$1 < \alpha \le \beta < \alpha^* = \begin{cases} +\infty, & \text{if } \alpha \ge N,\\ \frac{\alpha N}{N-\alpha}, & \text{if } \alpha < N. \end{cases} \tag{6}$$

Here we also use the following notation: ν is the outward unit normal vector to $\partial\Omega$, $\mathcal{H}^{N-1}$ is the $(N-1)$-D Hausdorff measure on Γ_N, $\mathfrak{B}_{ad}$ stands for the class of admissible controls, μ is a given positive value, which is assumed to be large enough, Db_{ij} denotes the $\mathbb{R}^N$-valued finite Radon measure such that

$$\int_\Omega b_{ij} \operatorname{div}\varphi \, dx = -\int_\Omega (\varphi, d\,[Db_{ij}])_{\mathbb{R}^N} = -\sum_{k=1}^N \int_\Omega \varphi_k \, d\,[(Db_{ij})_k], \quad \forall\, \varphi \in C_0^1(\Omega)^N,$$

and $\int_\Omega |Db_{ij}|$ stands for the total variation of b_{ij} in Ω, which can be defined as follows

$$\int_\Omega |Db_{ij}| = \sup\left\{ \int_\Omega b_{ij} \operatorname{div}\varphi \, dx \ : \ \varphi \in C_0^1(\Omega)^N, |\varphi| \le 1 \right\}.$$

A great deal of attention has been paid by many authors to the study of the thermistor problem during the last two decades (see Antontsev and Chipot, 1994; Howison, Rodrigues, and Shillor, 1993; Baranger and Mikelić, 1995; Růžička, 2000; Antontsev and Rodrigues, 2006). The search for the least assumptions on $\sigma(\theta)$, ensuring the (weak) solvability of the system (2)–(4), has been on the agenda of experts for decades. Earlier, existence theorems were proven only under Dirichlet boundary conditions for the potential u and some smallness conditions, e.g., in the case of a sufficiently small Lipschitz constant for the function $\sigma(\theta)$. For the survey of this kind of results, we refer to Zhikov (2008b). However, the most essential progress in the study of existence and qua-litative properties of solutions to the boundary value problem like (2)–(4) with homogeneous Dirichlet conditions for u was achieved by Zhikov (2011). It has been shown there that the solvability of these systems can be obtained in the multi-dimensional case without any smallness requirements on the function $\sigma(\theta)$ via a regularization approach and further passing to the limit over the parameter of regularization. Another extension of the thermistor problem, which is related to the solid-state devices, can be found in Kuttler, Shillor and Fernandez (2008).

However, as for the optimal control problem (1)–(6), to the best of the author's knowledge, the existence of optimal solutions for the above thermistor problem remains an open question. Only very few articles deal with optimal control for the thermistor problem (see Hömberg, Meyer, Rehberg and Ring, 2010; Hrynkiv, 2009, for the two dimensional case; Meinlschmidt, Meyer and Rehberg, 2016 for three spatial dimensions, and the recent papers of Hrynkiv and Koshkin, 2018; D'Apice, De Maio and Kogut, 2018, 2019, for the multi-dimensional case). There are several reasons for this:

- it is unknown whether the set of feasible points to the problem (1)–(6) is nonempty and weakly closed in the corresponding functional space;
- we have no a priori estimates for the weak solutions to the boundary value problem (2)–(4) under conditions (6);
- the asymptotic behaviour of a minimizing sequence to the cost functional (1) is unclear in general;
- the optimal control problem (1)–(6) is ill-posed and relations (2)–(4) require some relaxation (see, for instance, D'Apice, De Maio, and Kogut, 2010).

To circumvent the problems listed above, we propose the so-called indirect approach to the solvability of the optimal control thermistor problem in coefficients. Basing on the perturbation theory of extremal problems and the concept of fictitious controls (see, for instance, Horsin and Kogut, 2016; Kogut, 2014; Kogut, Manzo and Putchenko, 2016; Kogut and Leugering, 2011), we prove the existence of the so-called quasi-optimal and optimal solutions to the problem (1)–(6) and show that they can be attained by the optimal solutions of some appropriate approximations for the original optimal control problem. The main idea of our approach is based on the fact that weak solutions to the Dirichlet-Neumann boundary value problem (2)–(4) can be attained through a special regularization of the exponent $p = p(x)$ and an approximation of the operator $\mathcal{A}(u) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, using its perturbation by the $\varepsilon\Delta_\beta$-Laplacian, and the right-hand side of (3) by its transformation to

$$\operatorname{div}\left[\left(|\nabla u|^{\sigma(\theta)-2}\nabla u - g\right) u\right] + (g, \nabla u)_{\mathbb{R}^N}.$$

Here, by attainability of a weak solution (u, θ), we mean the existence of a sequence $\{(u_\varepsilon, \theta_\varepsilon)\}_{\varepsilon>0}$, where $(u_\varepsilon, \theta_\varepsilon)$ are the solutions of "more regular" boundary value problems, such that $(u_\varepsilon, \theta_\varepsilon) \to (u, \theta)$ in some appropriate topology as ε tends to zero.

When comparing these and other characteristic features of the optimization problem (1)–(6) and our approach in this paper with the results of the recent papers by D'Apice, De Maio and Kogut (2018, 2019), we can indicate the following differences:

- in this paper we deal with the mixed Dirichlet-Neumann quasi-linear coupling of the elliptic PDEs with BV-matrix valued controls in coefficients, whereas in D'Apice, De Maio and Kogut (2019) the thermistor problem wa considered with homogeneous Dirichlet boundary conditions and with a special class of admissible controls in coefficients, namely, a control function is admissible if it is an absolutely continuous scalar function with L^q-bounded generalized partial derivatives;
- in contrast to the paper of D'Apice, De Maio and Kogut (2019), in this paper we essentially extend the class of feasible solutions to the above optimal control problem, involving in the consideration not only duality solutions to the thermistor problem (2)–(4), but also the so-called weak solutions in the sense of distributions.
- in order to prove the existence of quasi-optimal and optimal solutions to the original optimization problem (1)–(6), we consider another family of approximation problems.

Before proceeding further, we recall the well known facts for nonlinear elliptic problems with variable exponent. Assuming that the temperature $\theta = \theta(x)$ is known for some admissible control $B(x)$, we introduce the Sobolev-Orlicz space

$$\begin{aligned} &W_0^{1,p(\cdot)}(\Omega;\Gamma_D) \\ &:= \left\{ u \in W^{1,1}(\Omega) \ : \ \int_\Omega |\nabla u(x)|^{p(x)}\,dx < +\infty,\ u = 0 \text{ on } \Gamma_D \right\} \end{aligned} \tag{7}$$

and equip it with the norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega;\Gamma_D)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)^N}$, where $p(x) = \sigma(\theta(x))$. Here, $|\cdot|$ denotes the Euclidean norm $|\cdot|_{\mathbb{R}^N}$ in $\mathbb{R}^N$, and $L^{p(\cdot)}(\Omega)^N$ stands for the set of all measurable functions $f : \Omega \to \mathbb{R}^N$ such that $\int_\Omega |f(x)|^{p(x)}\,dx < +\infty$. It is well known that (see, for instance, Diening, Harjulehto, Hästö and Rûžiĉka, 2011; Zhikov, 2011), unlike in classical Sobolev spaces, smooth functions are not necessarily dense in $W = W_0^{1,p(\cdot)}(\Omega;\Gamma_D)$. Let $C_0^\infty(\mathbb{R}^N;\Gamma_D) = \left\{\varphi \in C_0^\infty(\mathbb{R}^N) \ : \ \varphi = 0 \text{ on } \Gamma_D\right\}$. Hence, with variable exponent $p = p(x)$ $(1 < \alpha \le p(\cdot) \le \beta)$ another Sobolev space can be associated,

$$H = H_0^{1,p(\cdot)}(\Omega;\Gamma_D) \ \text{ as the closure of the set } \ C_0^\infty(\Omega;\Gamma_D) \\ \text{in } W_0^{1,p(\cdot)}(\Omega;\Gamma_D)\text{-norm.}$$

Defnition 1 *We say that a function $u \in W_0^{1,p(\cdot)}(\Omega;\Gamma_D)$ is a weak solution of the problem (2) if*

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u, \nabla\varphi\right)_{\mathbb{R}^N} \, dx = \int_\Omega (g, \nabla\varphi)_{\mathbb{R}^N} \, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega;\Gamma_D), \tag{8}$$

and we say that u is the H-solution of problem (2), if $u \in H_0^{1,p(\cdot)}(\Omega;\Gamma_D)$ and the integral identity (8) *holds for any test function $\varphi \in H_0^{1,p(\cdot)}(\Omega;\Gamma_D)$.*

Since we can fail with the density of the set $C_0^\infty(\Omega;\Gamma_D)$ in $W_0^{1,p(\cdot)}(\Omega;\Gamma_D)$ for some (irregular) variable exponents $p(x)$, it follows that a weak solution to the problem (2) is not unique, in general. To clarify this inference, let us associate with the Dirichlet problem (2) the Carathéodory vector-valued function $A_p : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, given by the rule $A_p(x,\xi) = |\xi|^{p(x)-2}\xi$. Then, it is easy to see that the following strict monotonicity, coercivity, and boundedness conditions hold:

$$(A_p(x,\xi) - A_p(x,\eta), \xi - \eta)_{\mathbb{R}^N} > 0, \quad \forall\, \xi \ne \eta, \tag{9}$$

$$(A_p(x,\xi), \xi)_{\mathbb{R}^N} \ge |\xi|^{p(x)} \quad \text{a.e. in } \Omega, \quad \forall\, \xi \in \mathbb{R}^N, \tag{10}$$

$$|A_p(x,\xi)|^{p'(x)} \le |\xi|^{p(x)} \quad \text{a.e. in } \Omega, \quad \forall\, \xi \in \mathbb{R}^N, \ \text{where } p'(x) = \frac{p(x)}{p(x)-1}. \tag{11}$$

Typically, such conditions are referred to as $p(x)$-monotonicity conditions. Let V be a closed subspace of $W_0^{1,p(\cdot)}(\Omega;\Gamma_D)$. Then, for a given $u \in V$ we can determine an element $\mathcal{A}_p u \in V^*$ as follows

$$\begin{aligned}(\mathcal{A}_p u, v) &= \int_\Omega (A_p(x, \nabla u(x)), \nabla v(x) \ dx \\ &= \int_\Omega \Big(|\nabla u(x)|^{p(x)-2}\nabla u(x), \nabla v(x)\Big) \ dx.\end{aligned} \tag{12}$$

Since

$$|(A_p(x,\nabla u), \nabla v)| \le \frac{1}{p'(x)} \left|A_p(x,\nabla u)\right|^{p'(x)} + \frac{1}{p(x)}|\nabla v|^{p(x)} \le |\nabla u|^{p(x)} + |\nabla v|^{p(x)}$$

by the Young inequality and condition (10), it follows that the given definition of $\mathcal{A}_p u \in V^*$ is valid and the operator $\mathcal{A}_p : V \to V^*$ is bounded. To verify that this operator is strictly monotone, coercive, and semicontinuous, we refer to Theorem 2.1 in Zhikov (2011). Hence, the existence and uniqueness of a weak solution to the problem (2) in V-subspace is a direct consequence of monotone operator theory (see, for instance, Roubíček, 2013). Thus, the H-solution exists, it is unique and satisfies the energy equality

$$\int_\Omega |\nabla u(x)|^{p(x)}\,dx = \int_\Omega (g(x), \nabla u(x))_{\mathbb{R}^N}\;dx \tag{13}$$

which immediately follows from (8) and density of $C_0^\infty(\Omega;\Gamma_D)$ in $H_0^{1,p(\cdot)}(\Omega;\Gamma_D)$.

As for the second equation (3), its right-hand side $|\nabla u|^p$ with $p(\cdot) = \sigma(\theta(\cdot))$, a priori belongs to the space $L^1(\Omega)$. So, $f = |\nabla u|^p$ is not an element of the dual space $H^{-1}(\Omega)$ and, hence, we cannot expect that the weak distributional solution of the Dirichlet problem (3) belongs to $H_0^1(\Omega)$. Moreover, the classical counterexample due to Serrin shows that the solution of (3) in the sense of distribution is not unique in general. Thus, in order to get both existence and uniqueness results it is necessary to make use of other notions of solution. Classical L^1-theory of the Dirichlet problem for the Laplace operator by Stampacchia and others says that the boundary value problem (3) admits a unique so-called duality solution.

Defnition 2 *A function $\theta \in L^1(\Omega)$ is a duality solution to problem* (3) *if*

$$\int_\Omega \theta\varphi\,dx = \int_\Omega |\nabla u|^p v\,dx, \quad \forall\,\varphi \in L^\infty(\Omega),$$

where $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ is the weak solution of

$$-\operatorname{div}(B\nabla v) = \varphi \quad \textit{in } \Omega \quad v = 0 \textit{ on } \partial\Omega.$$

It is, however, imperative to note that while every duality solution of (3) is also a distributional solution, i.e., satisfies the equation when tested with $C_0^\infty(\Omega)$ functions, the converse is in general false, because of the non-uniqueness of distributional solutions for equations with $L^1(\Omega)$ right hand sides (see Meyer, Panizzi and Schiela, 2011; Prignet, 1995). At the same time, every duality solution possesses the so-called SOLA-property (here, the abbreviation SOLA stands for the solution obtained as a limit of approximations): If $\theta \in L^1(\Omega)$ is a duality solution of (3), then for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ such that $f_n \to |\nabla u|^p$ strongly in $L^1(\Omega)$ and

$$\|f_n\|_{L^1(\Omega)} \le \||\nabla u|^p\|_{L^1(\Omega)} \ \text{ for all } \ n\in\mathbb{N},$$

we have $\theta_n \to \theta$ strongly in $L^1(\Omega)$ and weakly in $W_0^{1,\gamma}(\Omega)$ for all $\gamma \in [1, \frac{N}{N-1})$, where $\theta_n \in H_0^1(\Omega)\cap L^\infty(\Omega)$ is the weak solution of

$$-\operatorname{div}(B\nabla\theta_n) = f_n \quad \text{in } \Omega \quad \theta_n = 0 \text{ on } \partial\Omega.$$

The main result, concerning the existence of a duality solution to the problem (3), can be stated as follows (see, for instance, Theorems 3.3 and 4.1

Orsina, 2011): if Ω is a bounded domain with sufficiently smooth boundary and $|\nabla u|^{p(\cdot)} \in L^1(\Omega)$, then the Dirichlet problem (3) has the unique duality solution $\theta \in W_0^{1,\gamma}(\Omega)$ with $\gamma \in [1, \frac{N}{N-1})$; moreover, there exists a constant $C = C(\gamma)$, independent of $f = |\nabla u|^{p(\cdot)}$, such that

$$\|\theta\|_{W_0^{1,\gamma}(\Omega)} \leq C(\gamma)\|f\|_{L^1(\Omega)} = C(\gamma)\int_\Omega |\nabla u(x)|^{p(x)}\,dx. \tag{14}$$

Remark 1 *In fact, if the datum $f = |\nabla u|^p$ is more regular, say $f \in L^{1+\delta}(\Omega)$ for some $\delta > 0$, we have the following result (see Theorem 4.4 in Orsina, 2011): if $|\nabla u|^p \in L^{1+\delta}(\Omega)$, $0 < \delta < \frac{N-2}{N+2}$ then the unique duality solution of* (3) *belongs to $W_0^{1,q}(\Omega)$ with $q = \frac{N(1+\delta)}{N-1-\delta} = 1 + \delta + \frac{(1+\delta)^2}{N-1-\delta}$.*

The optimal control problem we consider in this paper is to minimize the discrepancy between a given distribution $\theta_d \in L^r(\Omega)$ and the temperature of thermistor $\theta \in W_0^{1,\gamma}(\Omega)$ by choosing an appropriate anisotropic heat conductivity matrix $B \in \mathfrak{B}_{ad}$. It is assumed here that $r \in \left(1, \frac{N}{N-2}\right)$, where the choice of such range is motivated by Sobolev Embedding Theorem. Namely, in view of the fact that the embedding $W_0^{1,\gamma}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, \frac{N}{N-2})$, the exponents γ and r can be related as follows: $\gamma = \frac{Nr}{N+r}$. As a result, for a given $r \in \left(1, \frac{N}{N-2}\right)$ we have $\gamma \in \left[1, \frac{N}{N-1}\right)$.

Since for a "typical" measurable or even continuous function $\sigma(\theta)$ with properties (6), the set $C_0^\infty(\Omega; \Gamma_D)$ is not dense in $W_0^{1,p(\cdot)}(\Omega; \Gamma_D)$, and, hence, no uniqueness of weak solutions to (2)–(4) can be expected, the mapping $B \mapsto (u, \theta)$, where (u, θ) is a weak solution to the boundary value problem (2)–(4), can be multi-valued in general. In view of this, we introduce the set of feasible solutions to the OCP (1)–(6) as follows:

$$\Xi_0 = \left\{ (B, u, \theta, p) \;\middle|\; \begin{array}{c} B \in \mathfrak{B}_{ad},\ u \in H_0^{1,p(\cdot)}(\Omega; \Gamma_D),\ \theta \in W_0^{1,\gamma}(\Omega), \\ p \in L^\infty(\Omega), \quad \gamma = \frac{Nr}{N+r}, \\ p(\cdot) = \sigma(\theta(\cdot)) \ \text{ a.e. in } \ \Omega, \\ u \text{ is the } H\text{-solution of (2)}, \\ \theta \text{ is a distributional solution to (3).} \end{array} \right\} \tag{15}$$

It is clear that $J(B, u, \theta, p) < +\infty$ for all $(B, u, \theta, p) \in \Xi_0$.

Remark 2 *The characteristic feature of the OCP* (1)–(6) *is the fact that a priori it is unknown whether the set Ξ_0 is nonempty. Using the assumption* (6) *and basing on a special technique of the weak convergence of fluxes to a flux, it was established in Zhikov (2011) that the thermistor problem (2)–*(4) *with Dirichlet condition and for $B = \xi I$, with $\xi \in [m_1, m_2]$, and for any measurable function $\sigma(\theta)$ admits a weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$. However, in this case the inclusion $u \in H_0^{1,p(\cdot)}(\Omega)$ is by no means obvious even for the diagonal constant matrix $B \in \mathfrak{B}_{ad}$ (see Section 7 in Zhikov, 2011). Hence, the OCP* (1)–(6) *requires* *some relaxation. The idea that we push forward in this paper is to consider the function $p(\cdot)$ as a fictitious control with some more regular properties and interpret the fulfilment of equality $p(\cdot) = \sigma(\theta(\cdot))$ with some accuracy.*

1.2. Relaxation of the original OCP

We consider the following extension of the set of feasible solutions to the original OCP. Let $k_0 > 0$ and $\tau \geq 0$ be given constants.

Defnition 3 *We say that a tuple (B, u, θ, p) is quasi-feasible to the OCP* (1)–(6) *if $(B, u, \theta, p) \in \widehat{\Xi}_0(\tau)$, where*

$$\widehat{\Xi}_0(\tau) = \left\{ (B, u, \theta, p) \;\middle|\; \begin{array}{c} B \in \mathfrak{B}_{ad},\ u \in H_0^{1,p(\cdot)}(\Omega; \Gamma_D),\ \theta \in W_0^{1,\gamma}(\Omega), p \in \mathfrak{S}_{ad}, \\ \|p - \sigma(\theta)\|_{L^2(\Omega)} \leq \tau, \quad \gamma = \frac{Nr}{N+r}, \\ u \text{ is the } H\text{-solution of (2)}, \\ \theta \text{ is a distributional solution to } (3). \end{array} \right\}, \tag{16}$$

$$\mathfrak{S}_{ad} = \left\{ q \in C(\overline{\Omega}) \;\middle|\; \begin{array}{c} |q(x) - q(y)| \leq \omega(|x - y|),\ \forall\, x, y, \in \Omega,\ |x - y| \leq 1/2, \\ \omega(t) = k_0 / \log(|t|^{-1}),\ 1 < \alpha \leq q(\cdot) \leq \beta \text{ in } \overline{\Omega}. \end{array} \right\}. \tag{17}$$

We also say that

$$(B^0, u^0, \theta^0, p^0) \in BV(\Omega)^{N\times N} \times H_0^{1,p(\cdot)}(\Omega; \Gamma_D) \times W_0^{1,\gamma}(\Omega) \times C(\overline{\Omega})$$

is a quasi-optimal solution to the problem (1)–(6) *if*

$$(B^0, u^0, \theta^0, p^0) \in \widehat{\Xi}_0(\tau) \quad \text{and} \quad J(B^0, u^0, \theta^0, p^0) = \inf_{(B,u,\theta,p) \in \widehat{\Xi}_0(\tau)} J(B, u, \theta, p),$$

and this tuple is called optimal if $p^0(\cdot) = \sigma(\theta^0(\cdot))$ a.e. in Ω.

Remark 3 *It is clear that $\widehat{\Xi}_0(\tau) \subset \Xi_0$ for $\tau = 0$ and, moreover, as we will see later on, the set $\widehat{\Xi}_0(\tau)$ is nonempty if only $\tau \geq \sqrt{|\Omega|}(\beta - \alpha)$. It is also worth emphasizing that the condition $p \in \mathfrak{S}_{ad}$ implies that $p(\cdot)$ has some additional regularity. Moreover, in view of the obvious relation $\lim_{t\to 0} |t|^\delta \log(|t|) = 0$ with $\delta \in (0, 1)$, it is clear that $p \in C^{0,\delta}(\Omega)$ implies $p \in \mathfrak{S}_{ad}$. Because of this, $p \in \mathfrak{S}_{ad}$ is often called a locally log-Hölder continuous exponent (see Definition 2.2 in Cruz-Uribe and Fiorenza, 2013). Another point, regarding the benefit from the choice of the subset $\mathfrak{S}_{ad}$ is related to the following properties: (i) $\mathfrak{S}_{ad}$ is a compact subset in $C(\overline{\Omega})$ and thus provides uniformly convergent subsequences; (ii) Every cluster point p of a sequence $\{p_k\}_{k\in\mathbb{N}} \subset \mathfrak{S}_{ad}$ is a regular exponent (i.e. in this case the set $C_0^\infty(\Omega; \Gamma_D)$ is dense in $W_0^{1,p(\cdot)}(\Omega; \Gamma_D)$), which plays* *a key role in the situation of Propositions 2 and 3; (iii) Because of the log-Hölder continuity of the exponent* $p \in \mathfrak{S}_{ad}$*, the corresponding weak solution* $u \in W_0^{1,p(\cdot)}(\Omega;\Gamma_D)$ *to the variational problem* (8) *is such that* $|\nabla u|^{(1+\delta)p(\cdot)} \in L^1(\Omega)$ *for some* $\delta > 0$ *and satisfies the estimate*

$$\int_\Omega |\nabla u(x)|^{(1+\delta)p(x)}\,dx \le C\int_\Omega |g(x)|^{(1+\delta)p'(x)}\,dx + C, \tag{18}$$

where $\delta > 0$ *and* $C > 0$ *depend only on* Ω*,* α*,* N*,* k_0*, and* $\int_\Omega |g|^{p'}\,dx$*. For the proof of the higher integrability of* $|\nabla u|^{p(\cdot)}$*, we refer to Theorem 16.4 in Zhikov (2011) and Lemma 3.3 in Zhikov and Pastukhova (2008). As it is shown in Section 4, the property* (18) *is crucial for the proof of existence of quasi-optimal solutions to the problem* (1)–(6)*.*

Remark 4 *It is easy to show that if* $(g,\nu)_{\mathbb{R}^N} = 0$ $\mathcal{H}^{N-1} = 0$*-a.e. on* Γ_N*, and if* $u \in W_0^{1,p(\cdot)}(\Omega;\Gamma_D)$ *is a solution to*

$$\begin{aligned}&\operatorname{div}(A(u)\nabla u) = \operatorname{div} g \quad \textit{in} \quad \Omega,\\&u = 0 \quad \textit{on} \quad \Gamma_D, \quad A(u)\partial_\nu u = 0 \quad \textit{on} \quad \Gamma_N\end{aligned}$$

in the sense of distributions, then

$$\big(A(u)\nabla u,\nabla u\big)_{\mathbb{R}^N} = \operatorname{div}\big((A(u)\nabla u - g)u\big) + (g,\nabla u)_{\mathbb{R}^N},$$

also in $[C_0^\infty(\Omega;\Gamma_D)]^*$*, where*

$$A(u) = |\nabla u(x)|^{p(x)-2} \quad \textit{or} \quad A(u) = |\nabla u|^{p(x)-2} + \varepsilon|\nabla u|^{\beta-2}$$

and $p(\cdot)$ *is a regular exponent. As a result, this allows for deducing the existence of the unique weak solution to the variational problem*

$$-\operatorname{div}(B\nabla\theta) = \operatorname{div}\big((A(u)\nabla u - g)u\big) + (g,\nabla u)_{\mathbb{R}^N} \quad \textit{in } \mathcal{D}'(\Omega)$$

which is also a distributional solution to the Dirichlet BVP

$$-\operatorname{div}(B\nabla\theta) = |\big(A(u)\nabla u,\nabla u\big)_{\mathbb{R}^N}| \quad \textit{in} \quad \Omega, \quad \theta|_{\partial\Omega} = 0.$$

Our main goal in this paper is to present the "approximation approach", based on the perturbation theory of extremal problems and the concept of fictitious controls. With that in mind, we make use of the following family of approximated problems

$$\begin{aligned}&\text{Minimize} \quad J_{\varepsilon,\tau}(B,u,\theta,p) =\\&\int_\Omega |\theta-\theta_d|^r\,dx + \varepsilon\int_\Omega |\nabla\theta|^\varrho\,dx + \frac{1}{\varepsilon}\mu_\tau\left(\int_\Omega |p-\sigma(\theta)|^2\,dx\right)\end{aligned} \tag{19}$$

subject to the constraints

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u\right) = \operatorname{div} g \quad \text{in} \quad \Omega, \tag{20}$$

$$u = 0 \ \text{ on } \ \Gamma_D, \quad \left[|\nabla u|^{p-2} + \varepsilon|\nabla u|^{\beta-2}\right]\partial_\nu u = 0 \ \text{ on } \ \Gamma_N, \tag{21}$$

$$\begin{aligned} -\operatorname{div}(B\nabla\theta) = \operatorname{div}&\left[\left(|\nabla u|^{p(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u - g\right)u\right] \\ &+ (g,\nabla u)_{\mathbb{R}^N} \quad \text{in} \quad \Omega, \quad \theta|_{\partial\Omega} = 0, \end{aligned} \tag{22}$$

$$B \in \mathfrak{B}_{ad}, \quad p \in \mathfrak{S}_{ad}, \quad \varrho = \max\left\{\frac{2N-1}{2(N-1)}, \frac{Nr}{N+r}\right\}. \tag{23}$$

Here, the function $\mu_\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as follows

$$\mu_\tau(s) = 0 \ \text{ if } \ 0 \le s \le \tau^2, \quad \text{and} \quad \mu_\tau(s) = s - \tau^2 \ \text{ if } \ s > \tau^2.$$

There are several principal points in the statement of approximated problem (19)–(23) that should be emphasized. The first one is related to $\varepsilon\Delta_\beta$-regularization of $p(\cdot)$-Laplacian. Though this is a standard trick, meant to establish the existence of H-solution to the Dirichlet problem (20) with a given exponent $p(\cdot)$ (see Theorems 3.1–3.3 in Zhikov, 2011), it does not allow for arriving at the existence of a weak solution $(u,\theta) \in H_0^{1,p(\cdot)}(\Omega;\Gamma_D) \times W_0^{1,\gamma}(\Omega)$ to the thermistor problem (2)–(4) (see Theorem 7.2 in Zhikov, 2011). This can be done only if the exponent $p(\cdot) = \sigma(\theta(\cdot))$ is regular, i.e. if the set $C_0^\infty(\Omega;\Gamma_D)$ is dense in $W_0^{1,p(\cdot)}(\Omega;\Gamma_D)$, and the energy density $|\nabla u(\cdot)|^{p(\cdot)}$ belongs to the space $L^{1+\delta}(\Omega)$ for some $\delta > 0$, so that the equation (22) holds in the sense of the distributions. With that in mind we consider the condition $p \in \mathfrak{S}_{ad}$ as an additional option for the regularization of the original OCP. Another point that should be indicated is related to some relaxation of the equation (3). Namely, it is easy to see that after the formal transformations, the equation (3) can be rewritten as follows

$$-\operatorname{div}(B\nabla\theta) = \operatorname{div}\left[\left(|\nabla u|^{\sigma(\theta)-2}\nabla u - g\right)u\right] + (g,\nabla u)_{\mathbb{R}^N} \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{24}$$

The benefit of such representation and condition $p \in \mathfrak{S}_{ad}$ is the fact that, due to the estimate (18), the expression $\left(|\nabla u|^{\sigma(\theta)-2}\nabla u - g\right)u$ under the divergence sign in (24) is integrable with degree greater than 1. As follows from our further analysis, this property plays an important role in the study of OCP (1)–(6) and we consider the representation (24) as some relaxation of the relation (3).

1.3. Main results

The main result of this paper is Theorem 1, in which we claim that if the OCP (1)–(6) has a sufficiently regular feasible point, then there exist optimal solutions to the OCP and some of them are the limit as $\varepsilon \searrow 0$ of optimal solutions to (19)–(23).

THEOREM 1 *Let Ω be an open bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary. Assume that $\widehat{\Xi}_0(\tau) \neq \emptyset$ for $\tau = 0$, i.e. there exist a matrix $\widehat{B} \in \mathfrak{B}_{ad}$, an exponent $\widehat{p} \in \mathfrak{S}_{ad}$, and a weak solution to the thermistor problem (2)–(4) $(\widehat{u}, \widehat{\theta}) \in W_0^{1,\sigma(\widehat{\theta}(\cdot))}(\Omega; \Gamma_D) \times W_0^{1,\gamma}(\Omega)$ with $\gamma = \frac{Nr}{N+r}$ and $B(\cdot) = \widehat{B}(\cdot)$ such that $\widehat{p} = \sigma(\widehat{\theta})$ almost everywhere in Ω. Then, OCP (1)–(6) has a non-empty set of optimal solutions and some of them can be attained in the following way*

$$B_\varepsilon^0 \overset{*}{\rightharpoonup} B^0 \quad in \quad BV(\Omega)^{N\times N}, \quad u_\varepsilon^0 \rightharpoonup u^0 \quad in \quad W_0^{1,\alpha}(\Omega), \tag{25}$$

$$\theta_\varepsilon^0 \rightharpoonup \theta^0 \quad in \quad W_0^{1,\gamma}(\Omega), \quad p_\varepsilon^0 \to p^0 \quad uniformly\ on \quad \overline{\Omega}, \tag{26}$$

as $\varepsilon \to 0$, where $(B_\varepsilon^0, u_\varepsilon^0, \theta_\varepsilon^0, p_\varepsilon^0)$ are the solutions to the approximated problems (19)–(23) with $\tau = \varepsilon$ in (19).

We prove this theorem further on at the end of Section 4.

REMARK 5 *It is clear that the condition $\widehat{p} = \sigma(\widehat{\theta})$ in the statement of Theorem 1, where $\widehat{p}$ has logarithmic modulus of continuity, imposes some additional and rather special constraint on the function $\sigma \in C(\mathbb{R})$. The principal point here is the fact that this relation has to be valid for a particular function $\widehat{\theta}$ and it is not required that the function $\sigma(\theta(\cdot))$ be at least continuous for every solution $\theta \in W_0^{1,\gamma}(\Omega)$ of (3). It is rather a delicate problem to guarantee the fulfilment of the equality $\widehat{p} = \sigma(\widehat{\theta})$ by the direct description of function $\sigma \in C(\mathbb{R})$ even if we make use of the "typical" assumption (see, for instance, Hömberg, Meyer, Rehberg and Ring, 2010; Hrynkiv, 2009; Howison, Rodrigues and Shillor, 1993): σ is a Lipschitz continuous function.*

Since it is unknown whether OCP (1)–(6) is solvable or the main assumptions of Theorem 1 are satisfied, it is reasonable to show that this problem admits the quasi-optimal solutions and they can be attained (in some sense) by optimal solutions to special approximated problems. We prove in Section 4 the following result.

THEOREM 2 *Let $\left\{(B_\varepsilon^0, u_\varepsilon^0, \theta_\varepsilon^0, p_\varepsilon^0)\right\}_{\varepsilon>0}$ be an arbitrary sequence of optimal solutions to the approximated problems (19)–(23). Assume that either there exists a constant $C^* > 0$ satisfying condition*

$$\limsup_{\varepsilon\to 0} \inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}(B,u,\theta,p) \le C^* < +\infty$$

or $\tau \geq \sqrt{|\Omega|}(\beta - \alpha)$, where $\widehat{\Xi}_\varepsilon$ stands for the set of feasible solutions to the problem (19)–(23). Then, any cluster tuple $\left(B^0, u^0, \theta^0, p^0\right)$ in the sense of convergence (25)–(26) is a quasi-optimal solution of the OCP (1)–(6). Moreover, in this case the following variational property holds

$$\lim_{\varepsilon\to 0} \inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}(B,u,\theta,p) = J\left(B^0, u^0, \theta^0, p^0\right) = \inf_{(B,u,\theta,p)\in\widehat{\Xi}_0(\tau)} J(B,u,\theta,p).$$

2. Preliminaries and some auxiliary results

2.1. On Orlicz spaces

Let $p(\cdot)$ be a measurable exponent function on Ω such that $1 < \alpha \le p(x) \le \beta < \infty$ a.e. in Ω, where α and β are given constants. Let $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ be the corresponding conjugate exponent. It is clear that $\beta' \le p'(\cdot) \le \alpha'$ a.e. in Ω, where β' and α' stand for the conjugates of constant exponents. Denote by $L^{p(\cdot)}(\Omega)^N$ the set of all measurable functions $f(x)$ on Ω such that $\int_\Omega |f(x)|^{p(x)}\,dx < \infty$. Then, $L^{p(\cdot)}(\Omega)^N$ is a reflexive separable Banach space with respect to the Luxemburg norm (see Cruz-Uribe and Fiorenza, 2013; Diening, Harjulehto, Hästö and Rûẑiĉka, 2011; Rădulescu and Repovš, 2015, for the details)

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} = \inf\left\{\lambda > 0 \ : \ \rho_p(\lambda^{-1} f) \le 1\right\},$$
$$\text{where } \ \rho_p(f) := \int_\Omega |f(x)|^{p(x)}\,dx. \tag{27}$$

As for the infimum in (27), it is obviously attained if $\rho_p(f) > 0$; moreover

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} = \lambda > 0 \ \Leftrightarrow \rho_p(\lambda^{-1} f) = 1. \tag{28}$$

The dual of $L^{p(\cdot)}(\Omega)^N$ with respect to $L^2(\Omega)$-inner product will be denoted by $L^{p'(\cdot)}(\Omega)^N$. The following estimates are well-known (see, for instance, Zhikov, 2008a; Cruz-Uribe and Fiorenza, 2013; Rădulescu and Repovš, 2015): if $f \in L^{p(\cdot)}(\Omega)^N$ then

$$\|f\|^{\alpha}_{L^{p(\cdot)}(\Omega)^N} \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^{\beta}_{L^{p(\cdot)}(\Omega)^N}, \ \text{if } \|f\|_{L^{p(\cdot)}(\Omega)^N} > 1, \tag{29}$$
$$\|f\|^{\beta}_{L^{p(\cdot)}(\Omega)^N} \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^{\alpha}_{L^{p(\cdot)}(\Omega)^N}, \ \text{if } \|f\|_{L^{p(\cdot)}(\Omega)^N} < 1,$$
$$\|f\|_{L^{p(\cdot)}(\Omega)^N} = \int_\Omega |f(x)|^{p(x)}\,dx, \ \text{if } \|f\|_{L^{p(\cdot)}(\Omega)^N} = 1,$$
$$\|f\|^{\alpha}_{L^{p(\cdot)}(\Omega)^N} - 1 \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^{\beta}_{L^{p(\cdot)}(\Omega)^N} + 1, \tag{30}$$
$$\|f\|_{L^{\alpha}(\Omega)^N} \le (1 + |\Omega|)^{1/\alpha} \|f\|_{L^{p(\cdot)}(\Omega)^N}. \tag{31}$$

Moreover, due to the duality method, it can be shown that

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} \le (1 + |\Omega|)^{1/\beta'} \|f\|_{L^{\beta}(\Omega)^N}, \quad \beta' = \frac{\beta}{\beta - 1}, \quad \forall\, f \in L^{\beta}(\Omega)^N. \tag{32}$$

We make use of the following results.

Lemma 1 (Lemma 13.3 in Zhikov, 2011) *If a sequence $\{f_k\}_{k\in\mathbb{N}}$ is bounded in $L^{p(\cdot)}(\Omega)$ and $f_k \rightharpoonup f$ in $L^{\alpha}(\Omega)$ as $k \to \infty$, then $f \in L^{p(\cdot)}(\Omega)$ and $f_k \rightharpoonup f$ in $L^{p(\cdot)}(\Omega)$, i.e.*

$$\lim_{k\to\infty} \int_\Omega f_k \varphi\,dx = \int_\Omega f_k \varphi\,dx, \quad \forall\, \varphi \in L^{p'(\cdot)}(\Omega).$$

Lemma 2 *Let $\{p_k\}_{k\in\mathbb{N}} \subset \mathfrak{S}_{ad}$ and $p \in \mathfrak{S}_{ad}$ be such that $p_k(\cdot) \to p(\cdot)$ uniformly in $\overline{\Omega}$ as $k \to \infty$. If a sequence $\left\{\|f_k\|_{L^{p_k(\cdot)}(\Omega)}\right\}_{k\in\mathbb{N}}$ is bounded and $f_k \rightharpoonup f$ in $L^\alpha(\Omega)$ as $k \to \infty$, then $f \in L^{p(\cdot)}(\Omega)$.*

Proof By analogy with the proof of Lemma 1 (see p. 536 in Zhikov, 2011), to deduce the inclusion $f \in L^{p(\cdot)}(\Omega)$, it is enough to note that

$$\int_\Omega |f(x)|^{p(x)}\,dx \le \liminf_{k\to\infty} \int_\Omega |f_k(x)|^{p_k(x)}\,dx \tag{33}$$

by the semicontinuity of convex functionals with respect to the weak convergence in $L^\alpha(\Omega)$. □

2.2. On the weak convergence of fluxes to flux

A typical situation, arising in the study of most of the optimization problems and which is of fundamental importance in many other areas of nonlinear analysis, can be stated as follows: we have the weak convergence $u_k \rightharpoonup u$ in some Sobolev space $W^{1,\alpha}(\Omega)$ with $\alpha > 1$ and we have the weak convergence of fluxes $A_k(\cdot, \nabla u_k) \rightharpoonup z$ in the Lebesgue space $L^\delta(\Omega)$, $\delta > 1$, where by flux we mean the vector under the divergence sign in an elliptic equation (in our case it is $A_k(\cdot, \nabla u_k) = |\nabla u_k|^{p_k(\cdot)-2}\nabla u_k$ or $A_k(\cdot, \nabla\theta_k) = B_k(\cdot)\nabla\theta_k$). Then, the problem is to show that $z = A(\cdot, \nabla u)$, although the validity of this equality is by no means obvious at this stage.

Assume that the fluxes $A_k(x, \xi)$ satisfy the following conditions:

$$A_k : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{are Carathéodory vector-valued functions,} \tag{34}$$

$$(A_k(x,\xi) - A_k(x,\eta), \xi - \eta)_{\mathbb{R}^N} \ge 0,\ A_k(x,0) = 0, \quad \text{for a.e. } x \in \Omega \text{ and } \forall \xi, \eta \in \mathbb{R}^N, \tag{35}$$

$$|A_k(x,\xi)|^{\beta'} \le C_1|\xi|^\beta + C_2,\ \lim_{k\to\infty} A_k(x,\xi) = A(x,\xi) \quad \text{for a.e. } x \in \Omega \text{ and} \forall\, \xi \in \mathbb{R}^N. \tag{36}$$

Let $\{v_k\}_{k\in\mathbb{N}}$ and $\{A_k(\cdot, v_k)\}_{k\in\mathbb{N}}$ be weakly convergent sequences in $L^1(\Omega)^N$, and let v and z be their weak L^1-limits, respectively. In order to clarify the conditions, under which the equality $z = A(x, v)$ holds and the fluxes $A_k(\cdot, v_k)$ weakly converge to the flux $A(\cdot, v)$, we cite the following result.

Theorem 3 (Theorem 4.6 in Zhikov, 2011) *Assume that $\{u_k\}_{k\in\mathbb{N}}$ and $\{A_k(\cdot, \nabla u_k)\}_{k\in\mathbb{N}}$ are the sequences such that conditions (35)–(36) hold true and*

(i) $u_k \rightharpoonup u$ in $W^{1,\alpha}(\Omega)$ and $u_k \in W^{1,\beta}(\Omega)$ for all $k \in \mathbb{N}$;

(ii) $\sup_{k\in\mathbb{N}} \|A_k(\cdot, \nabla u_k)\|_{L^{\beta'}(\Omega)^N} < +\infty$;

(iii) $\sup_{k\in\mathbb{N}} \|\,(A_k(\cdot, \nabla u_k), \nabla u_k)_{\mathbb{R}^N}\,\|_{L^1(\Omega)} < +\infty$;

(iv) the exponents α and β are related by the condition

$$1 < \alpha \le \beta < \begin{cases} +\infty, & \text{if } \alpha \ge N-1, \\ \frac{\alpha(N-1)}{N-1-\alpha}, & \text{if } \alpha < N-1. \end{cases} \tag{37}$$

Then, up to a subsequence, the fluxes weakly converge to the flux

$$A_k(\cdot, \nabla u_k) \rightharpoonup A(\cdot, \nabla u) \quad in \;\; L^{\beta'}(\Omega)^N.$$

It is worth to note that in the case of equality $\alpha = \beta$, Theorem 3 becomes the well-known result of Tartar and Murat (1978), also known as the div-curl Lemma.

3. On approximated optimal control problems in coefficients and their properties

This section deals with the description of the structure of the approximated optimal control problems with respect to the original one (1)–(6), and with the study of their main variational properties.

Let ε be a small parameter. Assume that the parameter ε varies within a strictly decreasing sequence of positive real numbers, which converges to 0. Let $\tau \geq 0$ be a given constant. We consider the collection of approximated optimal control problems in coefficients for nonlinear elliptic equations (19)–(23). For every $\varepsilon > 0$ we denote by $\widehat{\Xi}_\varepsilon$ the set of all feasible points to the problem (19)–(23).

Defnition 4 *We say that (B, u, θ, p) is a feasible solution to the problem (19)–(23) if $B \in \mathfrak{B}_{ad}$, $p \in \mathfrak{S}_{ad}$, and $u \in W_0^{1,\beta}(\Omega; \Gamma_D)$ and $\theta \in W_0^{1,\gamma}(\Omega)$ are the solutions to the following variational problems*

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u\right) = \operatorname{div} g \quad in \;\; [C_0^\infty(\Omega;\Gamma_D)]^*, \tag{38}$$

$$\left[|\nabla u|^{p-2} + \varepsilon|\nabla u|^{\beta-2}\right]\partial_\nu u = 0 \quad in \;\; \mathcal{D}'(\Gamma_N), \tag{39}$$

$$-\operatorname{div}(B\nabla\theta) = \operatorname{div}\left[\left(|\nabla u|^{p(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u - g\right)u\right] + (g, \nabla u)_{\mathbb{R}^N} \tag{40}$$

$$in \;\; \mathcal{D}'(\Omega).$$

We begin with the following lemma, reflecting the consistency of approximated optimal control problem (19)–(23).

Lemma 3 *Let $\theta_d \in L^r(\Omega)$ with $r \in \left(1, \frac{N}{N-2}\right)$ and $g \in L^\infty(\Omega)^N$ be given distributions such that $(g, \nu)_{\mathbb{R}^N} = 0$ $\mathcal{H}^{N-1} = 0$-a.e. on Γ_N. Let $\sigma \in C(\mathbb{R})$ be a function satisfying conditions* (6)*, and let τ be an arbitrary non-negative value. Then, the approximated optimal control problem (19)–*(23) *is consistent for each $\varepsilon > 0$, i.e. $\widehat{\Xi}_\varepsilon \neq \emptyset$.*

Proof Let us define the control functions $\widehat{B}$ and $\widehat{p}$ as follows: $\widehat{p}(\cdot) = \beta$ and $\widehat{B}(\cdot) = \xi I$, where $\xi \in [m_1, m_2]$. It is clear that $\widehat{B} \in \mathfrak{B}_{ad}$, $\widehat{p} \in \mathfrak{S}_{ad}$, and the variational problem (38) becomes a well-posed problem for the Δ_β-Laplacian and it admits a unique solution $\widehat{u}_\varepsilon \in W_0^{1,\beta}(\Omega; \Gamma_D)$. Let us show that there exists a unique element $\widehat{\theta} \in W_0^{1,\gamma}(\Omega)$ for $\gamma \in [1, \frac{N}{N-1})$, satisfying variational equality (40). Indeed, as follows from (40), we see that the identity

$$\begin{aligned}&\xi \int_\Omega \left(\nabla\widehat{\theta}, \nabla\varphi\right)_{\mathbb{R}^N} dx \\ &= \int_\Omega (g, \nabla\widehat{u})_{\mathbb{R}^N}\, \varphi\, dx - (1+\varepsilon)\int_\Omega \left(|\nabla\widehat{u}|^{\beta-2}\nabla\widehat{u} - g, \nabla\varphi\right)_{\mathbb{R}^N} \widehat{u} dx \end{aligned} \tag{41}$$

holds for all $\varphi \in C_0^\infty(\Omega)$ and $k \in \mathbb{N}$. Since $\widehat{u}$ is a solution to (38), it leads us to the following relations

$$(1+\varepsilon)\int_\Omega \left(|\nabla\widehat{u}|^{\beta-2}\nabla\widehat{u}, \nabla(\varphi\widehat{u})\right)_{\mathbb{R}^N} dx = \int_\Omega (g, \nabla(\varphi\widehat{u}))_{\mathbb{R}^N}\, dx, \quad \forall\,\varphi \in C_0^\infty(\Omega).$$

Hence,

$$\begin{aligned}(1+\varepsilon)\int_\Omega |\nabla\widehat{u}|^\beta \varphi\, dx &= (1+\varepsilon)\int_\Omega \left(\nabla\varphi, g - |\nabla\widehat{u}|^{\beta-2}\nabla\widehat{u}\right)_{\mathbb{R}^N} \widehat{u}\, dx \\ &\quad + \int_\Omega (g, \nabla\widehat{u})_{\mathbb{R}^N}\, \varphi\, dx, \quad \forall\,\varphi \in C_0^\infty(\Omega).\end{aligned}$$

Combining this equality with (41), we arrive at the integral identity

$$\xi \int_\Omega \left(\nabla\widehat{\theta}, \nabla\varphi\right)_{\mathbb{R}^N} dx = (1+\varepsilon)\int_\Omega |\nabla\widehat{u}|^\beta \varphi\, dx \quad \forall\,\varphi \in C_0^\infty(\Omega). \tag{42}$$

Thus, $\widehat{\theta}$ is the distributional solution to the Dirichlet problem

$$-\xi\, \mathrm{div}\,(\nabla\theta_{\varepsilon,k}) = (1+\varepsilon)|\nabla\widehat{u}|^\beta \ \text{ in } \ \Omega, \quad \widehat{\theta}\Big|_{\partial\Omega} = 0. \tag{43}$$

Since the domain Ω is smooth enough and the operator $-\xi\,\mathrm{div}\,(\nabla\theta) = -\xi\Delta\theta$ admits maximal elliptic regularity on $W^{-1,p_0}(\Omega)$ for some $p_0 > N$ (see Section 6 in Disser, Kaiser and Rehberg, 2015), it follows that distributional solution and duality solution to the problem (43) are the same (see Meyer, Panizzi and Schiela, 2011). Hence, by the Stampacchia theorem the duality solution exists, it is unique in $W_0^{1,\gamma}(\Omega)$ for every $\gamma \in [1, \frac{N}{N-1})$, and it can be found via approximation of $|\nabla\widehat{u}|^\beta \in L^1(\Omega)$ by $L^\infty(\Omega)$-functions. Moreover, there exists a constant $C = C(\gamma)$, independent of $|\nabla\widehat{u}|^\beta$, such that (see Theorem 4.1 in Orsina, 2011)

$$\int_\Omega |\nabla\widehat{\theta}|^\gamma\, dx \le C(\gamma)\||\nabla\widehat{u}|^\beta\|^\gamma_{L^1(\Omega)}, \quad \forall\,\gamma < \frac{N}{N-1}.$$

Thus, $\widehat{\theta} \in W_0^{1,\gamma}(\Omega)$ and the tuple $(\xi I, \widehat{u}, \widehat{\theta}, \beta)$ is a feasible solution to the problem (19)–(23). $\square$

The subsequent results are crucial for our further analysis.

Lemma 4 *The set of fictitious controls $\mathfrak{S}_{ad}$ is convex, bounded and compact with respect to the strong topology of $C(\overline{\Omega})$.*

Proof Let $\{p_k(\cdot)\}_{k\in\mathbb{N}} \subset \mathfrak{S}_{ad}$ be an arbitrary sequence of fictitious controls. Since $\max_{x\in\overline{\Omega}} |p_k(x)| \le \beta$ and each element of this sequence has the same modulus of continuity ω, it follows that the sequence $\{p_k(\cdot)\}_{k\in\mathbb{N}}$ is uniformly bounded and equi-continuous. Hence, by Arzelà–Ascoli Theorem the set $\{p_k\}_{k\in\mathbb{N}}$ is relatively compact with respect to the strong topology of $C(\overline{\Omega})$. Taking into account that the set $\mathfrak{S}_{ad}$ is closed with respect to the uniform convergence, we can deduce: there exists an element $p \in \mathfrak{S}_{ad}$ such that, up to subsequences, $p_k(\cdot) \to p(\cdot)$ uniformly in $\overline{\Omega}$ as $k \to \infty$. The convexity and boundedness of the set $\mathfrak{S}_{ad}$ obviously follows from its definition. $\square$

Lemma 5 *Let $\varepsilon > 0$ be a fixed value, and let $\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_\varepsilon$ be a minimising sequence for OCP (19)–(23). Then, the flux $A_{\varepsilon,k}(x,\xi) := |\xi|^{p_{\varepsilon,k}(x)-2}\xi + \varepsilon|\xi|^{\beta-2}\xi$ satisfies the properties (35)–(36).*

Proof The monotonicity property (35) is a direct consequence of the well-known inequalities:

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\right)_{\mathbb{R}^N} \ge 2^{2-p}|\xi - \eta|^p \quad \text{for} \ \ 2 \le p < +\infty,$$
$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\right)_{\mathbb{R}^N} \ge (|\xi| + |\eta|)^{p-2}\, |\xi - \eta|^2 \quad \text{for} \ \ 1 < p < 2,$$

whereas the boundedness condition $(35)_1$ is ensured by inclusion $p_{\varepsilon,k} \in \mathfrak{S}_{ad}$,

$$|A_{\varepsilon,k}(x,\xi)| \le |\xi|^{p_{\varepsilon,k}(x)-1} + \varepsilon|\xi|^{\beta-1} \le (1+\varepsilon)|\xi|^{\beta-1} + 1 \quad \text{a.e. in} \ \ \Omega.$$

It remains to establish the pointwise convergence $(36)_2$. Due to Lemma 4, we can suppose that there exists an element $p_\varepsilon \in \mathfrak{S}_{ad}$ such that, up to subsequences, $p_{\varepsilon,k}(\cdot) \to p_\varepsilon(\cdot)$ uniformly in $\overline{\Omega}$ as $k \to \infty$. As a result, we have

$$\begin{aligned} A_{\varepsilon,k}(x,\xi) &:= |\xi|^{p_{\varepsilon,k}(x)-2}\xi + \varepsilon|\xi|^{\beta-2}\xi \overset{k\to\infty}{\longrightarrow} |\xi|^{p_\varepsilon(x)-2}\xi + \varepsilon|\xi|^{\beta-2}\xi \\ &=: A_\varepsilon(x,\xi)\, \text{a.e. in}\ \Omega. \end{aligned}$$

$\square$

Lemma 6 *Let $\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_\varepsilon$ be an arbitrary sequence. Then there exist a distribution $u_\varepsilon \in W_0^{1,\beta}(\Omega;\Gamma_D)$, an exponent $p_\varepsilon \in \mathfrak{S}_{ad}$, and a subsequence of $\{u_{\varepsilon,k}\}_{k\in\mathbb{N}}$, still denoted by the suffix (ε,k), such that*

$$\varepsilon\|u_\varepsilon\|^\beta_{W_0^{1,\beta}(\Omega)} \le 2^{\alpha'+1}\left(\int_\Omega |g|^{\alpha'}\,dx + |\Omega|\right), \tag{44}$$

$$u_{\varepsilon,k} \rightharpoonup u_\varepsilon \quad in \ \ W_0^{1,\beta}(\Omega;\Gamma_D) \quad as \ \ k \to \infty, \tag{45}$$

$$u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega). \tag{46}$$

Proof To prove this result, we apply the line of reasoning coming from the paper of Zhikov (2009). Since $(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k}) \in \widehat{\Xi}_\varepsilon$ for each $k \in \mathbb{N}$, it follows that $u_{\varepsilon,k} \in W_0^{1,\beta}(\Omega;\Gamma_D)$ and the integral identity

$$\int_\Omega \left(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}-2}\nabla u_{\varepsilon,k} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta-2}\nabla u_{\varepsilon,k}, \nabla\varphi\right)_{\mathbb{R}^N}\,dx = \int_\Omega (g,\nabla\varphi)_{\mathbb{R}^N}\,dx$$

(47)

holds for all $\varphi \in C_0^\infty(\Omega)$. Hence, by density of $C_0^\infty(\Omega)$ in $W_0^{1,\beta}(\Omega;\Gamma_D)$, we obtain the energy equality

$$\int_\Omega \left(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}} + \varepsilon|\nabla u_{\varepsilon,k}|^\beta\right)\,dx = \int_\Omega (g, \nabla u_{\varepsilon,k})_{\mathbb{R}^N}\,dx. \tag{48}$$

By the Young inequality, we have

$$\begin{aligned}\int_\Omega (g, \nabla u_{\varepsilon,k})_{\mathbb{R}^N}\,dx &\le \frac{2^{\alpha'}}{\beta'}\int_\Omega |g|^{p'_{\varepsilon,k}}\,dx + \frac{1}{\alpha 2^\alpha}\int_\Omega |\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}}\,dx\\ &\le 2^{\alpha'}\int_\Omega |g|^{p'_{\varepsilon,k}}\,dx + \frac{1}{2}\int_\Omega |\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}}\,dx,\end{aligned}$$

where $\alpha' = \alpha/(\alpha-1)$. Then, it is easy to derive from (48) the following estimates

$$\begin{aligned}\int_\Omega \left(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}} + \varepsilon|\nabla u_{\varepsilon,k}|^\beta\right)\,dx &\le 2^{\alpha'+1}\int_\Omega |g|^{p'_{\varepsilon,k}}\,dx\\ &\le 2^{\alpha'+1}\left(\int_\Omega |g|^{\alpha'}\,dx + |\Omega|\right).\end{aligned} \tag{49}$$

Thus, the sequence $\{u_{\varepsilon,k}\}_{k\in\mathbb{N}}$ is relatively compact with respect to the weak topology of $W_0^{1,\beta}(\Omega;\Gamma_D)$. Without loss of generality, we assume that the weak convergence $u_{\varepsilon,k} \rightharpoonup u_\varepsilon$ takes place in $W_0^{1,\beta}(\Omega;\Gamma_D)$. Then it follows from (48) that

$$\begin{aligned}&\lim_{k\to\infty}\int_\Omega \left(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}} + \varepsilon|\nabla u_{\varepsilon,k}|^\beta\right)\,dx =\\ &\qquad = \lim_{k\to\infty}\int_\Omega (g, \nabla u_{\varepsilon,k})_{\mathbb{R}^N}\,dx = \int_\Omega (g, \nabla u_\varepsilon)_{\mathbb{R}^N}\,dx,\\ &\varepsilon\int_\Omega |\nabla u_\varepsilon|^\beta\,dx \le \int_\Omega (g, \nabla u_\varepsilon)_{\mathbb{R}^N}\,dx\end{aligned} \tag{50}$$

because, according to the property of lower semicontinuity of $\|\cdot\|_{W_0^{1,\beta}(\Omega;\Gamma_D)}$-norm with respect to the weak convergence, we have

$$\lim_{k\to\infty}\int_\Omega |\nabla u_{\varepsilon,k}|^\beta\,dx \ge \liminf_{k\to\infty}\int_\Omega |\nabla u_{\varepsilon,k}|^\beta\,dx \ge \int_\Omega |\nabla u_\varepsilon|^\beta\,dx. \tag{51}$$

Thus, the estimate (44) immediately follows from (49)–(51). □

LEMMA 7 *Let* $\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_\varepsilon$ *be an arbitrary sequence, and let* $p_\varepsilon \in \mathfrak{S}_{ad}$ *and* $u_\varepsilon \in W_0^{1,\beta}(\Omega;\Gamma_D)$ *be such that*

$$p_{\varepsilon,k}(\cdot) \to p_\varepsilon(\cdot) \text{ uniformly in } \overline{\Omega} \quad\text{and}\quad u_{\varepsilon,k} \rightharpoonup u_\varepsilon \text{ in } W_0^{1,\beta}(\Omega;\Gamma_D) \text{ as } k\to\infty. \tag{52}$$

Then, up to a subsequence, we have the weak convergence of fluxes to a flux:

$$|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}-2}\nabla u_{\varepsilon,k} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta-2}\nabla u_{\varepsilon,k} \rightharpoonup |\nabla u_\varepsilon|^{p_\varepsilon-2}\nabla u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon \quad in \quad L^{\beta'}(\Omega)^N. \quad (53)$$

Proof In order to prove the convergence (53), we show that all conditions of Theorem 3 are fulfilled. Taking into account Lemma 5 and the fact that $\frac{\alpha N}{N-\alpha} < \frac{\alpha(N-1)}{N-1-\alpha}$, we focus on the verification of conditions (ii)–(iii). Indeed, in view of the weak convergence $u_{\varepsilon,k} \rightharpoonup u_\varepsilon$ in $W_0^{1,\beta}(\Omega)$ and

$$\begin{aligned} &\|A_{1,k}(\cdot,\nabla u_{\varepsilon,k})\|_{L^{\beta'}(\Omega)^N} \\ &\overset{\text{by (31)}}{\leq} (1+|\Omega|)^{1/\beta'} \|A_{1,k}(\cdot,\nabla u_{\varepsilon,k})\|_{L^{p'_{\varepsilon,k}(\cdot)}(\Omega)^N} \\ &\overset{\text{by (30)}}{\leq} (1+|\Omega|)^{1/\beta'} \left(\int_\Omega |A_{1,k}(x,\nabla u_{\varepsilon,k}(x))|^{p'_{\varepsilon,k}(x)}\,dx + 1\right)^{1/\beta'} \\ &= (1+|\Omega|)^{1/\beta'} \left(\int_\Omega |\nabla u_{\varepsilon,k}(x)|^{p_{\varepsilon,k}(x)}\,dx + 1\right)^{1/\beta'} \\ &\leq (1+|\Omega|)^{1/\beta'} \left(\int_\Omega |\nabla u_{\varepsilon,k}(x)|^{\beta}\,dx + |\Omega| + 1\right)^{1/\beta'} < +\infty, \end{aligned} \quad (54)$$

$$\|A_{2,k}(\cdot,\nabla u_{\varepsilon,k})\|_{L^{\beta'}(\Omega)^N} = \left(\int_\Omega |\nabla u_{\varepsilon,k}(x)|^{\beta}\,dx\right)^{1/\beta'}, \quad (55)$$

where

$$\begin{aligned} A_{\varepsilon,k}(x,\nabla u_{\varepsilon,k}) &:= A_{1,k}(x,\nabla u_{\varepsilon,k}) + \varepsilon A_{2,k}(x,\nabla u_{\varepsilon,k}), \\ A_{1,k}(x,\nabla u_{\varepsilon,k}) &:= |\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}(x)-2}\nabla u_{\varepsilon,k}, \\ A_{2,k}(x,\nabla u_{\varepsilon,k}) &:= |\nabla u_{\varepsilon,k}|^{\beta-2}\nabla u_{\varepsilon,k}, \end{aligned}$$

we see that the fluxes $A_{\varepsilon,k}(\cdot,\nabla u_{\varepsilon,k})$ are bounded in $L^{\beta'}(\Omega)^N$. To check the condition (iii) of Theorem 3, it is enough to apply the estimate (44) and note that

$$\begin{aligned} \|\left(A_{\varepsilon,k}(\cdot,\nabla u_{\varepsilon,k}),\nabla u_{\varepsilon,k}\right)_{\mathbb{R}^N}\|_{L^1(\Omega)} &= \int_\Omega |\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}}\,dx + \varepsilon\int_\Omega |\nabla u_{\varepsilon,k}|^{\beta}\,dx \\ &\leq (1+\varepsilon)\sup_{k\in\mathbb{N}}\left(\int_\Omega |\nabla u_{\varepsilon,k}|^{\beta}\,dx\right) + |\Omega|. \end{aligned} \quad (56)$$

Thus, the weak convergence of fluxes to a flux (53) follows from Theorem 3. □

Lemma 8 *Let $p_\varepsilon \in \mathfrak{S}_{ad}$ and $u_\varepsilon \in W_0^{1,\beta}(\Omega;\Gamma_D)$ be as in Lemma 7. Then u_ε is the unique weak solution to the Dirichlet problem*

$$\operatorname{div}\left(|\nabla u|^{p_\varepsilon(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u\right) = \operatorname{div} g \quad in \quad \Omega,$$

$$u = 0 \quad on \quad \Gamma_D, \quad \left[|\nabla u|^{p_\varepsilon(x)-2} + \varepsilon|\nabla u|^{\beta-2}\right]\partial_\nu u = 0 \quad on \quad \Gamma_N. \quad (57)$$

Proof Since $\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}}$ are feasible points to OCP (19)–(23), it follows that the integral identity (47) holds true for all $\varphi \in C_0^\infty(\Omega; \Gamma_D)$ and for all $k \in \mathbb{N}$. Then, (53) implies that

$$\begin{aligned}\int_\Omega &\Big(|\nabla u_\varepsilon|^{p_\varepsilon(x)-2}\nabla u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon, \nabla\varphi\Big)_{\mathbb{R}^N}\, dx \\ &= \lim_{k\to\infty}\int_\Omega \Big(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}(x)-2}\nabla u_{\varepsilon,k} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta-2}\nabla u_{\varepsilon,k}, \nabla\varphi\Big)_{\mathbb{R}^N}\, dx \\ &= \int_\Omega (g, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega; \Gamma_D). \end{aligned} \tag{58}$$

Hence, u_ε is a weak solution to the boundary value problem (57). In view of the strict monotonicity of the operator $\mathcal{A}_\varepsilon : W_0^{1,\beta}(\Omega; \Gamma_D) \to W^{-1,\beta'}(\Omega; \Gamma_D)$, given by the equality

$$(\mathcal{A}_\varepsilon u, v) = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u, \nabla v\Big)_{\mathbb{R}^N}\, dx, \qquad \forall\, v \in W_0^{1,\beta}(\Omega; \Gamma_D), \tag{59}$$

this solution is unique. □

The next results are based on the compactness property of the set of matrices $\mathfrak{B}_{ad}$ with respect to the weak-$*$ topology of $BV(\Omega)^{N\times N}$. For motivation of BV-choice for the set of admissible controls, we refer to Buttazzo and Kogut (2011), Casas, Kogut and Leugering (2016), D'Apice, De Maio and Kogut (2010, 2012), Horsin and Kogut (2015), Kogut and Leugering (2013). We recall that a sequence $\{f_k\}_{k=1}^\infty$ converges weakly* to f in $BV(\Omega)$ if and only if the two following conditions hold (see Ambrosio, Fusco and Pallara, 2000): $f_k \to f$ strongly in $L^1(\Omega)$ and $Df_k \overset{*}{\rightharpoonup} Df$ weakly* in the space of Radon measures $\mathcal{M}(\Omega; \mathbb{R}^N)$. It is well known that each uniformly bounded set in BV-norm is relatively compact in $L^1(\Omega)$ with respect to the strong L^1-topology of this space. Moreover, if $\{f_k\}_{k=1}^\infty \subset BV(\Omega)$ converges strongly to some f in $L^1(\Omega)$ and satisfies $\sup_{k\in\mathbb{N}} \int_\Omega |Df_k| < +\infty$, then (see, for instance, Ambrosio, Fusco and Pallara, 2000; Giusti, 1984)

$$(i)\ f \in BV(\Omega) \ \text{ and } \int_\Omega |Df| \le \liminf_{k\to\infty}\int_\Omega |Df_k|; \quad (ii)\ f_k \overset{*}{\rightharpoonup} f \ \text{ in } BV(\Omega). \tag{60}$$

Since the set $\big\{B \in L^\infty(\Omega; \mathbb{R}^{N\times N}) \ : \ m_1 I \le B \le m_2 I, \text{ a.e. in } \Omega\big\}$ is closed with respect to the pointwise convergence almost everywhere, the following property of the class of admissible controls $\mathfrak{B}_{ad}$ holds true.

Proposition 1 *For any given $m_1 > 0$, $m_2 > m_1$, and $\mu > 0$, the set $\mathfrak{B}_{ad}$ is nonempty, uniformly bounded, convex, and sequentially compact with respect to the weak-$*$ topology of $BV(\Omega)^{N\times N}$.*

Lemma 9 *Let* $\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_\varepsilon$ *be a sequence such that* $B_{\varepsilon,k} \overset{*}{\rightharpoonup} B_\varepsilon$ *in* $BV(\Omega)^{N\times N}$ *and* $\theta_{\varepsilon,k} \rightharpoonup \theta_\varepsilon$ *in* $W_0^{1,\gamma}(\Omega)$ *for some* $\gamma \in [1, \frac{N}{N-1})$*. Then we have*

$$\lim_{k\to\infty} \int_\Omega (B_{\varepsilon,k}\nabla\theta_{\varepsilon,k}, \nabla\varphi)_{\mathbb{R}^N}\, dx = \int_\Omega (B_\varepsilon \nabla\theta_\varepsilon, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{61}$$

Proof Since $B_{\varepsilon,k} \to B_\varepsilon$ in $L^1(\Omega)^{N\times N}$ and $\{B_{\varepsilon,k}\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)^{N\times N}$, we infer $B_{\varepsilon,k} \to B_\varepsilon$ strongly in $L^r(\Omega)^{N\times N}$ for every $1 \le r < +\infty$. In particular, $B^t_{\varepsilon,k}\nabla\varphi \to B^t_\varepsilon\nabla\varphi$ in $L^{\gamma'}(\Omega)^N$ with $\gamma' = \frac{\gamma}{\gamma-1}$ and $\nabla\theta_{\varepsilon,k} \rightharpoonup \nabla\theta_\varepsilon$ in $L^\gamma(\Omega)^N$. Hence, it is immediate to pass to the limit and to deduce (61). □

The next lemma is crucial for our further analysis and it reveals some compactness properties of the set of feasible solutions $\widehat{\Xi}_\varepsilon$.

Lemma 10 *Let* $\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_\varepsilon$ *be a sequence of feasible points for OCP* (19)–(23)*. Then, there exist* $\delta > 0$ *and distributions* $p_\varepsilon \in \mathfrak{S}_{ad}$, $u_\varepsilon \in W_0^{1,\beta}(\Omega;\Gamma_D)$, $\widehat{\theta}_{\varepsilon,k} \in W_0^{1,\gamma}(\Omega)$, $\theta_\varepsilon \in W_0^{1,\gamma}(\Omega)$ *for* $\gamma \in [1, \frac{N}{N-1})$*, and a matrix* $B_\varepsilon \in \mathfrak{B}_{ad}$ *such that* $|\nabla u_\varepsilon|^\beta \in L^{1+\delta}(\Omega)$, $\left(B_{\varepsilon,k}, u_{\varepsilon,k}, \widehat{\theta}_{\varepsilon,k}, p_{\varepsilon,k}\right) \in \widehat{\Xi}_\varepsilon$ *for all* $k \in \mathbb{N}$*, and, up to subsequences,*

$$p_{\varepsilon,k}(\cdot) \to p_\varepsilon(\cdot) \text{ uniformly in } \overline{\Omega}, \quad u_{\varepsilon,k} \rightharpoonup u_\varepsilon \text{ in } W_0^{1,\beta}(\Omega;\Gamma_D), \tag{62}$$

$$B_{\varepsilon,k} \overset{*}{\rightharpoonup} B_\varepsilon \text{ in } BV(\Omega)^{N\times N}, \quad \widehat{\theta}_{\varepsilon,k} \rightharpoonup \theta_\varepsilon \text{ in } W_0^{1,\gamma}(\Omega), \tag{63}$$

where θ_ε *is a distributional solution to the Dirichlet problem*

$$\begin{aligned} &-\operatorname{div}(B_\varepsilon\nabla\theta)\\ &= \operatorname{div}\left[\left(|\nabla u_\varepsilon|^{p_\varepsilon(x)-2}\nabla u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon - g\right) u_\varepsilon\right] + (g, \nabla u_\varepsilon)_{\mathbb{R}^N}, \end{aligned} \tag{64}$$

$$\theta|_{\partial\Omega} = 0, \tag{65}$$

which satisfies the integral identity

$$\int_\Omega (B_\varepsilon\nabla\theta_\varepsilon, \nabla\varphi)_{\mathbb{R}^N}\, dx = \int_\Omega \left(|\nabla u_\varepsilon|^{p_\varepsilon(x)} + \varepsilon|\nabla u_\varepsilon|^\beta\right)\varphi\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{66}$$

Moreover, if $N > 2$*, then* $\theta_\varepsilon \in W_0^{1,q}(\Omega)$ *for* $q = \frac{N(1+\delta)}{N-1-\delta} = \frac{N}{N-1}\left(1 + \frac{N\delta}{N-1-\delta}\right)$ *provided* $\delta \in \left(0, \frac{N-2}{N+2}\right)$.

Proof To begin with, we note that the convergence $(63)_1$ and inclusion $B_\varepsilon \in \mathfrak{B}_{ad}$ is a direct consequence of Proposition 1, whereas (62) follows from Lemmas 4, 5, and 6. Moreover, by the higher integrability of the gradient (see Remark 3 for the details), we have

$$|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}(x)} + \varepsilon|\nabla u_{\varepsilon,k}|^\beta \in L^{1+\delta}(\Omega) \text{ for some } \delta > 0 \text{ independent of } k. \tag{67}$$

In view of the initial assumptions, for each $k \in \mathbb{N}$, the function $\theta_{\varepsilon,k}$ is the weak solution to (22) in the sense of distributions, i.e. the identity

$$\int_\Omega (B_{\varepsilon,k}\nabla\theta_{\varepsilon,k}, \nabla\varphi)_{\mathbb{R}^N}\, dx = \int_\Omega (g, \nabla u_{\varepsilon,k})_{\mathbb{R}^N}\, \varphi\, dx - \int_\Omega \Big(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}(x)-2}\nabla u_{\varepsilon,k} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta-2}\nabla u_{\varepsilon,k} - g, \nabla\varphi\Big)_{\mathbb{R}^N} u_{\varepsilon,k}\, dx \tag{68}$$

holds for all $\varphi \in C_0^\infty(\Omega)$ and $k \in \mathbb{N}$. Hence, arguing as in the proof of Lemma 3, we deduce that the integral identity

$$\int_\Omega (B_{\varepsilon,k}\nabla\theta_{\varepsilon,k}, \nabla\varphi)_{\mathbb{R}^N}\, dx = \int_\Omega \Big(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}(x)} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta}\Big)\varphi\, dx \tag{69}$$

holds for every $\varphi \in C_0^\infty(\Omega)$. Hence, $\theta_{\varepsilon,k}$ is the distributional solution to the Dirichlet problem

$$-\operatorname{div}(b_{\varepsilon,k}\nabla\theta_{\varepsilon,k}) = |\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}(x)} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta} \ \text{ in } \ \Omega, \quad \theta_{\varepsilon,k}|_{\partial\Omega} = 0. \tag{70}$$

In view of the condition (67), this Dirichlet boundary value problem admits a unique duality solution $\widehat{\theta}_{\varepsilon,k}$. Then, $\left\{\left(B_{\varepsilon,k}, u_{\varepsilon,k}, \widehat{\theta}_{\varepsilon,k}, p_{\varepsilon,k}\right)\right\}_{k\in\mathbb{N}}$ are feasible points to OCP (19)–(23). By the Stampacchia theorem, $\widehat{\theta}_{\varepsilon,k}$ is the duality solution to the problem (70) and it can be found via approximation of $f_{\varepsilon,k} \in L^1(\Omega)$ by $L^\infty(\Omega)$-functions, where $f_{\varepsilon,k} := \left(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}(x)} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta}\right) \in L^1(\Omega)$. Moreover, it can be shown that in this case $\widehat{\theta}_{\varepsilon,k}$ belongs to $W_0^{1,\gamma}(\Omega)$, for every $\gamma \in [1, \frac{N}{N-1})$, and there exists a constant C, independent of $f_{\varepsilon,k}$ and k, such that (for the details, we refer to Theorem 4.1 in Orsina, 2011) $\int_\Omega |\nabla\widehat{\theta}_{\varepsilon,k}|^\gamma\, dx \le C\|f_{\varepsilon,k}\|^\gamma_{L^1(\Omega)}$. In fact, the constant C depends only on N, γ, and m_1. Since

$$\int_\Omega |f_{\varepsilon,k}|\, dx = \int_\Omega \Big(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}(x)} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta}\Big)\, dx \overset{\text{by (56)}}{\le} \left((1+\varepsilon)\sup_{k\in\mathbb{N}}\left(\int_\Omega |\nabla u_{\varepsilon,k}|^\beta\, dx\right) + |\Omega|\right),$$

it follows that

$$\|\widehat{\theta}_{\varepsilon,k}\|_{W_0^{1,\gamma}(\Omega)} \le C^{\frac{1}{\gamma}}\left((1+\varepsilon)\sup_{k\in\mathbb{N}}\left(\int_\Omega |\nabla u_{\varepsilon,k}|^\beta\, dx\right) + |\Omega|\right), \quad \forall\, k \in \mathbb{N}. \tag{71}$$

Thus, the weak convergence $(63)_2$ immediately follows from the estimate (71). It remains to show that the limit function θ_ε is a weak solution to the problem (10) and that it satisfies the identity (66). With that in mind, we note that

$$\begin{aligned}|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}-2}\nabla u_{\varepsilon,k} + |\nabla u_{\varepsilon,k}|^{\beta-2}\nabla u_{\varepsilon,k} - g \rightharpoonup |\nabla u_\varepsilon|^{p_\varepsilon-2}\nabla u_\varepsilon \\ +\varepsilon|\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon - g \text{ in } L^{\beta'}(\Omega)^N, \\ u_{\varepsilon,k} \to u_\varepsilon \quad\text{in}\quad L^\beta(\Omega), \\ (g, \nabla u_{\varepsilon,k})_{\mathbb{R}^N} \to (g, \nabla u_\varepsilon)_{\mathbb{R}^N} \quad\text{in}\quad L^\beta(\Omega)\end{aligned}$$

by Lemma 7 and compactness of the embedding $W_0^{1,\beta}(\Omega) \hookrightarrow L^\beta(\Omega)$. Hence,

$$\begin{aligned}&\left(|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}-2}\nabla u_{\varepsilon,k} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta-2}\nabla u_{\varepsilon,k} - g\right) u_{\varepsilon,k} \\ &\qquad \rightharpoonup \left(|\nabla u_\varepsilon|^{p_\varepsilon-2}\nabla u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon - g\right) u_\varepsilon \quad\text{in}\quad L^1(\Omega).\end{aligned}$$

Taking these facts and Lemmas 4 and 9 into account, we can pass to the limit in (68) as $k \to \infty$. As a result, we obtain

$$\begin{aligned}\int_\Omega (B_\varepsilon \nabla\theta_\varepsilon, \nabla\varphi)_{\mathbb{R}^N}\,dx = \int_\Omega (g, \nabla u_\varepsilon)_{\mathbb{R}^N}\,\varphi\,dx \\ - \int_\Omega \left(|\nabla u_\varepsilon|^{p_\varepsilon-2}\nabla u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon - g, \nabla\varphi\right)_{\mathbb{R}^N} u_\varepsilon\,dx,\end{aligned} \tag{72}$$

i.e.

$$-\operatorname{div}(B_\varepsilon\nabla\theta_\varepsilon) = \operatorname{div}\left[\left(|\nabla u_\varepsilon|^{p_\varepsilon(x)-2}\nabla u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon - g\right) u_\varepsilon\right] + (g, \nabla u_\varepsilon)_{\mathbb{R}^N}$$

in the sense of distributions. In order to establish the integral identity (66), it is enough to observe that $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ is the unique weak solution to the Dirichlet problem (57) (see Lemma 8) and apply then the transformations that we used in the proof of Lemma 3, to the identity (72). As for the inclusions $|\nabla u_\varepsilon|^\beta \in L^{1+\delta}(\Omega)$ for $N \geq 2$ and $\theta_\varepsilon \in W_0^{1,q}(\Omega)$ for $N > 2$ and $q = \frac{N(1+\delta)}{N-1-\delta}$, we should apply the arguments of the higher integrability of the gradient (see Remark 3 for the details) and Theorem 4.4 from Orsina (2011). □

To conclude this section, we give the existence result for the approximated OCP (19)–(23).

Theorem 4 *Let $\theta_d \in L^r(\Omega)$ with $r \in \left[1, \frac{N}{N-2}\right)$ and $g \in L^\infty(\Omega)^N$ be given distributions, such that $(g, \nu)_{\mathbb{R}^N} = 0$ $\mathcal{H}^{N-1} = 0$-a.e. on Γ_N. Let $\sigma \in C(\mathbb{R})$ be a function satisfying the conditions (6), and let τ be an arbitrary non-negative value. Then the optimal control problem (19)–(23) admits at least one solution for each $\varepsilon > 0$.*

Proof Since the set $\widehat{\Xi}_\varepsilon$ is nonempty (see Lemma 3) and the cost functional $J_{\varepsilon,\tau}$ is bounded from below on $\widehat{\Xi}_\varepsilon$, it follows that there exists a minimizing sequence

$$\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_\varepsilon$$

to problem (19)–(23), i.e.

$$\inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}(B,u,\theta,p) = \lim_{k\to\infty}\left[\int_\Omega |\theta_{\varepsilon,k}(x)-\theta_d(x)|^r\,dx + \varepsilon\int_\Omega |\nabla\theta_{\varepsilon,k}|^\varrho\,dx + \frac{1}{\varepsilon}\mu_\tau\left(\int_\Omega |p_{\varepsilon,k}(x)-\sigma(\theta_{\varepsilon,k}(x))|^2\,dx\right)\right] < +\infty.$$

Hence, in view of Lemma 6 and definition of the sets $\mathfrak{B}_{ad}$ and $\mathfrak{S}_{ad}$, the minimizing sequence $\{(B_{\varepsilon,k},u_{\varepsilon,k},\theta_{\varepsilon,k},p_{\varepsilon,k})\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega)^{N\times N}\times W_0^{1,\beta}(\Omega;\Gamma_D)\times W_0^{1,\varrho}(\Omega)\times C(\overline{\Omega})$. From (45) and Lemmas 4, 8, 9, and 10, we deduce the existence of a subsequence, that we denote in the same way, and a tuple $\left(B^0_\varepsilon,u^0_\varepsilon,\theta^0_\varepsilon,p^0_\varepsilon\right)\in\widehat{\Xi}_\varepsilon$ such that $B_{\varepsilon,k}\overset{*}{\rightharpoonup}B^0_\varepsilon$ in $BV(\Omega)^{N\times N}$, $u_{\varepsilon,k}\rightharpoonup u^0_\varepsilon$ in $W_0^{1,\beta}(\Omega;\Gamma_D)$, $\theta_{\varepsilon,k}\rightharpoonup\theta^0_\varepsilon$ in $W_0^{1,\varrho}(\Omega)$, and $p_{\varepsilon,k}(\cdot)\to p^0_\varepsilon(\cdot)$ uniformly in $\overline{\Omega}$. Then, taking into account the compact embedding $W_0^{1,\varrho}(\Omega)\hookrightarrow L^q(\Omega)$ for all $q\in[1,\frac{N\varrho}{N-\varrho})$, where $\varrho=\max\left\{\frac{2N-1}{2(N-1)},\frac{Nr}{N+r}\right\}$, we have

$$q\in\begin{cases}[1,r), & \text{if } r>r^*:=\frac{2N-1}{2(N-1)},\\ \left[1,\frac{2N^2-N}{2N^2-4N+1}\right), & \text{if } 1\le r\le r^*.\end{cases}$$

Therefore,

$$\theta_{\varepsilon,k}(\cdot)\to\theta^0_\varepsilon(\cdot)\ \text{ in } L^{\widehat{r}}(\Omega) \text{ for } 1\le\widehat{r}<r, \text{ and }\ \theta_{\varepsilon,k}(\cdot)\rightharpoonup\theta^0_\varepsilon(\cdot)\ \text{ in } L^r(\Omega)$$

for the given $r\in(1,\frac{N}{N-2})$. So, we can suppose that $\theta_{\varepsilon,k}(\cdot)\to\theta^0_\varepsilon(\cdot)$ almost everywhere in Ω. Hence, in view of the fact that $\mu_\tau\in C_{loc}(\mathbb{R}_+)$, we have

$$\liminf_{k\to\infty}\int_\Omega |p_{\varepsilon,k}(x)-\sigma(\theta_{\varepsilon,k}(x))|^2\,dx \ge \int_\Omega |p^0_\varepsilon(x)-\sigma(\theta^0_\varepsilon(x))|^2\,dx,$$
$$\liminf_{k\to\infty}\int_\Omega |\theta_{\varepsilon,k}(x)-\theta_d(x)|^r\,dx \ge \int_\Omega |\theta^0_\varepsilon(x)-\theta_d(x)|^r\,dx,$$
$$\liminf_{k\to\infty}\int_\Omega |\nabla\theta_{\varepsilon,k}|^\varrho\,dx \ge \int_\Omega |\nabla\theta^0_\varepsilon|^\varrho\,dx.$$

As a result, we finally infer

$$\inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}(B,u,\theta,p) = \lim_{k\to\infty} J_{\varepsilon,\tau}\left(B_{\varepsilon,k},u_{\varepsilon,k},\theta_{\varepsilon,k},p_{\varepsilon,k}\right) \ge J_{\varepsilon,\tau}\left(B^0_\varepsilon,u^0_\varepsilon,\theta^0_\varepsilon,p^0_\varepsilon\right).$$

Thus, $\left(B^0_\varepsilon,u^0_\varepsilon,\theta^0_\varepsilon,p^0_\varepsilon\right)$ is a solution of the approximated OCP (19)–(23). □

Remark 6 *As follows from the proof of Lemma 7 and Theorem 3, the existence result for approximated OCPs (19)–*(23) *in the form of Theorem 4 remains valid even if we omit the condition* (23) *on the parameters α and β, because in this case, instead of Theorem 3, we can apply the celebrated div-curl Lemma of Tartar and Murat (1978).*

4. Asymptotic analysis of the approximated OCP (19)–(23) as $\varepsilon \to 0$

Our main intention in this section is to show that quasi-optimal solutions to the OCP (1)–(6) can be attained (in some sense) by optimal solutions to the approximated problems (19)–(23). In order to do it, we do not use the concept of variational convergence of constrained minimization problems (see Kogut and Leugering, 2011) but rather apply the direct analysis to the study of asymptotic behaviour of optimal solutions for OCPs (19)–(23) as $\varepsilon \to 0$.

Proposition 2 *Let $\left\{(B^0_\varepsilon, u^0_\varepsilon, \theta^0_\varepsilon, p^0_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of optimal solutions to the approximated problems (19)–(23) when the small parameter $\varepsilon > 0$ varies in a strictly decreasing sequence of positive numbers which converges to 0. Assume there exists a constant $C^* > 0$ such that*

$$\limsup_{\varepsilon\to 0} \inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}(B,u,\theta,p) \le \limsup_{\varepsilon\to 0} J_{\varepsilon,\tau}(B^0_\varepsilon, u^0_\varepsilon, \widehat{\theta}^0_\varepsilon, p^0_\varepsilon) \le C^* < +\infty, \tag{73}$$

where $\widehat{\theta}^0_\varepsilon \in W^{1,\gamma}_0(\Omega)$, $\gamma \in [1, \frac{N}{N-1})$, is the duality solution to the Dirichlet boundary value problem

$$-\operatorname{div}\left(B^0_\varepsilon \nabla\theta\right) = |\nabla u^0_\varepsilon|^{p^0_\varepsilon(x)} + \varepsilon |\nabla u^0_\varepsilon|^\beta \quad in\ \ \Omega, \quad \theta|_{\partial\Omega} = 0. \tag{74}$$

Then, there is a subsequence of $\left\{(B^0_\varepsilon, u^0_\varepsilon, \theta^0_\varepsilon, p^0_\varepsilon)\right\}_{\varepsilon>0}$, still denoted by the suffix ε, and distributions $B^0 \in \mathfrak{B}_{ad}$, $p^0 \in \mathfrak{S}_{ad}$, $u^0 \in W^{1,\alpha}_0(\Omega;\Gamma_D)$, and $\theta^0 \in W^{1,\varrho}_0(\Omega)$, such that

$$B^0_\varepsilon \overset{*}{\rightharpoonup} B^0 \quad in\ \ BV(\Omega)^{N\times N}, \quad u^0_\varepsilon \rightharpoonup u^0 \quad in\ \ W^{1,\alpha}_0(\Omega;\Gamma_D), \tag{75}$$

$$\theta^0_\varepsilon \rightharpoonup \theta^0 \quad in\ \ W^{1,\varrho}_0(\Omega), \quad p^0_\varepsilon(\cdot) \to p^0(\cdot) \quad uniformly\ on\ \ \overline{\Omega}, \tag{76}$$

$$\|p^0 - \sigma(\theta^0)\|_{L^2(\Omega)} \le \tau. \tag{77}$$

Proof Having applied the estimates (44) and (49) to elements of the sequence of optimal solutions, we obtain

$$\begin{aligned}
\int_\Omega |\nabla u^0_\varepsilon|^\alpha\,dx &\overset{\text{by (31)}}{\le} (1+|\Omega|)\,\|\nabla u^0_\varepsilon\|^\alpha_{L^{p^0_\varepsilon(\cdot)}(\Omega)^N} \overset{\text{by (30)}}{\le} (1+|\Omega|)\left(\int_\Omega |\nabla u^0_\varepsilon|^{p^0_\varepsilon}\,dx + 1\right)\\
&\overset{\text{by (3)}}{\le} (1+|\Omega|)\left(2^{\alpha'+1}\int_\Omega |g|^{(p^0_\varepsilon)'}\,dx + 1\right)\\
&\le (1+|\Omega|)\left(2^{\alpha'+1}\left(\int_\Omega |g|^{\alpha'}\,dx + |\Omega|\right) + 1\right),
\end{aligned} \tag{78}$$

$$\varepsilon\int_\Omega |\nabla u^0_\varepsilon|^\beta\,dx \le 2^{\alpha'+1}\left(\int_\Omega |g|^{\alpha'}\,dx + |\Omega|\right). \tag{79}$$

Hence, the sequence $\left\{u^0_\varepsilon\right\}_{\varepsilon>0}$ is relatively compact with respect to the weak topology of $W_0^{1,\alpha}(\Omega)$. Therefore, we can suppose that the convergence $(75)_2$ is valid. Moreover, in view of estimates (78)–(79), it is easy to conclude that the right-hand side of the equation (74) is L^1-bounded. In view of (71), the duality solution $\widehat{\theta}^0_\varepsilon$ to this equation is unique and obeying the estimate

$$\|\theta^0_\varepsilon\|_{W_0^{1,\gamma}(\Omega)} \le C(\gamma)\int_\Omega \left(|\nabla u^0_\varepsilon|^{p^0_\varepsilon(x)} + \varepsilon|\nabla u^0_\varepsilon|^\beta\right)\,dx \overset{\text{by (78)–(79)}}{\le} C(\gamma)2^{\alpha'+2}\left(\int_\Omega |g|^{\alpha'}\,dx + |\Omega|\right). \tag{80}$$

Since $(B^0_\varepsilon, u^0_\varepsilon, \widehat{\theta}^0_\varepsilon, p^0_\varepsilon) \in \widehat{\Xi}_\varepsilon$ for all $\varepsilon>0$, and

$$J_{\varepsilon,\tau}(B^0_\varepsilon, u^0_\varepsilon, \theta^0_\varepsilon, p^0_\varepsilon) \le J_{\varepsilon,\tau}(B^0_\varepsilon, u^0_\varepsilon, \widehat{\theta}^0_\varepsilon, p^0_\varepsilon), \tag{81}$$

it follows from (73) that the sequence $\left\{\theta^0_\varepsilon\right\}_{\varepsilon\to 0}$ is bounded in $W_0^{1,\varrho}(\Omega)$. Indeed, relations (80)–(81) and Hölder inequality imply that

$$\varepsilon\|\theta^0_\varepsilon\|^\varrho_{W_0^{1,\varrho}(\Omega)} \le \varepsilon\|\widehat{\theta}^0_\varepsilon\|^\varrho_{W_0^{1,\varrho}(\Omega)} \le C\varepsilon\|\widehat{\theta}^0_\varepsilon\|^\varrho_{W_0^{1,\gamma}(\Omega)} \overset{\text{by (80)}}{<} +\infty.$$

Thus, $\sup_{\varepsilon>0}\|\theta^0_\varepsilon\|_{W_0^{1,\varrho}(\Omega)} < +\infty$ and, hence, there exists a distribution $\theta^0 \in W_0^{1,\varrho}(\Omega)$ and a subsequence of $\left\{\theta^0_\varepsilon\right\}_{\varepsilon\to 0}$, still denoted by the same index, such that the property $(76)_1$ holds true. Therefore, up to a subsequence, we have the pointwise convergence

$$\sigma(\theta^0_\varepsilon) \to \sigma(\theta^0) \ \text{ a.e. in } \ \Omega \ \text{ as } \ \varepsilon \to 0. \tag{82}$$

By Lemma 4, the family $\{p_{\varepsilon,k}\}_{k\in\mathbb{N}} \subset \mathfrak{S}_{ad}$ is relatively compact in $C(\overline{\Omega})$. Hence, there exists an element $p^0 \in \mathfrak{S}_{ad}$ such that, up to a subsequence, $p^0_\varepsilon \to p^0$ uniformly in Ω and, in view of (82), the sequence $\left\{p^0_\varepsilon - \sigma(\theta^0_\varepsilon)\right\}_{\varepsilon>0}$ weakly converges in $L^2(\Omega)$ to $p^0 - \sigma(\theta^0)$ as $\varepsilon\to 0$. Taking this fact into account, we have

$$\liminf_{\varepsilon\to 0}\|p^0_\varepsilon - \sigma(\theta^0_\varepsilon)\|^2_{L^2(\Omega)} \ge \|p^0 - \sigma(\theta^0)\|^2_{L^2(\Omega)}. \tag{83}$$

At the same time, the condition (73) implies that

$$\mu_\tau\left(\int_\Omega |p^0_\varepsilon(x) - \sigma(\theta^0_\varepsilon(x))|^2\,dx\right) \le \varepsilon C^*. \tag{84}$$

Hence, $\|p^0_\varepsilon - \sigma(\theta^0_\varepsilon)\|_{L^2(\Omega)} \le \tau$ for ε small enough. Combining this fact with (83), we arrive at the desired property (77).

It remains to note that the existence of a matrix $B^0 \in \mathfrak{B}_{ad}$ with property $(75)_1$ is a direct consequence of Proposition 1. □

The next step of our analysis is to show that the tuple $\left(B^0, u^0, \theta^0, p^0\right)$ is a quasi-feasible point to the original OCP (1)–(6).

Proposition 3 *Assume the condition* (73) *holds. Let* $\left(B^0, u^0, \theta^0, p^0\right) \in BV(\Omega)^{N\times N} \times W_0^{1,\beta}(\Omega;\Gamma_D) \times W_0^{1,\varrho}(\Omega) \times \mathfrak{S}_{ad}$ *be a cluster tuple (in the sense of convergence (75)–(76)) of a given sequence of optimal solutions* $\left\{(B^0_\varepsilon, u^0_\varepsilon, \theta^0_\varepsilon, p^0_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$. *Then,* $\left(B^0, u^0, \theta^0, p^0\right)$ *is an element of* $\widehat{\Xi}_0(\tau)$ *and*

$$\int_\Omega |\nabla u^0_\varepsilon|^{p^0_\varepsilon}\,dx + \varepsilon\int_\Omega |\nabla u^0_\varepsilon|^\beta\,dx \to \int_\Omega |\nabla u^0|^{p^0}\,dx, \tag{85}$$

$$\int_\Omega |\nabla u^0|^{p^0}\,dx = \int_\Omega \left(g, \nabla u^0\right)_{\mathbb{R}^N}\,dx, \tag{86}$$

$u^0 \in H_0^{1,p^0(\cdot)}(\Omega;\Gamma_D)$ *is the unique* H*-variational solution of the Dirichlet-Neumann problem*

$$\operatorname{div}\left(|\nabla u|^{p^0-2}\nabla u\right) = \operatorname{div} g \quad in\ \ \Omega, \quad u = 0 \ \ on\ \ \Gamma_D, \quad |\nabla u|^{p^0-2}\partial_\nu u = 0 \ \ on\ \ \Gamma_N, \tag{87}$$

and $\theta^0 \in W_0^{1,\varrho}(\Omega)$ *is a distributional solution to the Dirichlet problem with* L^1*-data*

$$-\operatorname{div}\left(B^0\nabla\theta\right) = |\nabla u^0|^{p^0} \quad in\ \ \Omega, \quad \theta|_{\partial\Omega} = 0. \tag{88}$$

Proof In order to conclude the energy equality (86), we apply the following reasoning. At the first step, let us show that distribution u^0 is a weak solution to the problem (87). With that in mind we have to pass to the limit as $\varepsilon \to 0$ in integral identity

$$\int_\Omega \left(|\nabla u^0_\varepsilon|^{p^0_\varepsilon(x)-2}\nabla u^0_\varepsilon + \varepsilon|\nabla u^0_\varepsilon|^{\beta-2}\nabla u^0_\varepsilon, \nabla\varphi\right)_{\mathbb{R}^N}\,dx = \int_\Omega (g, \nabla\varphi)_{\mathbb{R}^N}\,dx, \tag{89}$$

where $\varphi \in C_0^\infty(\Omega)$.

Taking into account the estimate (79), we see that the sequence $\left\{\varepsilon|\nabla u^0_\varepsilon|^{\beta-2}\nabla u^0_\varepsilon\right\}_{\varepsilon>0}$ is bounded in $L^{\beta'}(\Omega)^N$, and, in addition, for any vector-valued function $\psi \in C_0^\infty(\Omega)^N$, we have (see Lemma 3.8 in Zhikov, 2011)

$$\begin{aligned}
&\int_\Omega \left(\varepsilon|\nabla u^0_\varepsilon|^{\beta-2}\nabla u^0_\varepsilon, \psi\right)_{\mathbb{R}^N}\,dx \leq \\
&\quad \varepsilon\int_\Omega |\nabla u^0_\varepsilon|^{\beta-1}|\psi|\,dx \leq \varepsilon\left(\int_\Omega |\nabla u^0_\varepsilon|^\beta\,dx\right)^{1/\beta'}\|\psi\|_{L^\beta(\Omega)^N} \\
&\quad \leq \varepsilon^{1-\frac{1}{\beta'}}\left(\varepsilon\int_\Omega |\nabla u^0_\varepsilon|^\beta\,dx\right)^{1/\beta'}\|\psi\|_{L^\beta(\Omega)^N} \overset{\text{by (79)}}{\leq} C\varepsilon^{1-\frac{1}{\beta'}} \to 0.
\end{aligned}$$

Hence,

$$\varepsilon|\nabla u_\varepsilon^0|^{\beta-2}\nabla u_\varepsilon^0 \rightharpoonup 0 \quad \text{in} \quad L^{\beta'}(\Omega)^N. \tag{90}$$

It remains to show the weak convergence of fluxes to a flux:

$$|\nabla u_\varepsilon^0|^{p_\varepsilon^0-2}\nabla u_\varepsilon^0 \rightharpoonup |\nabla u^0|^{p^0-2}\nabla u^0 \quad \text{in} \quad L^{\beta'}(\Omega)^N. \tag{91}$$

Since $\frac{\alpha N}{N-\alpha} < \frac{\alpha(N-1)}{N-1-\alpha}$, $u_\varepsilon^0 \rightharpoonup u^0$ in $W_0^{1,\alpha}(\Omega;\Gamma_D)$,

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_\varepsilon^0|^{p_\varepsilon^0-1}\right)^{\beta'} dx \\
&\overset{\text{by (54)}}{\leq} (1+|\Omega|)\left(\int_\Omega |\nabla u_\varepsilon^0|^{p_\varepsilon^0}\, dx + 1\right) \\
&\overset{\text{by (3)}}{\leq} (1+|\Omega|)\left(2^{\alpha'+1}\left(\int_\Omega |g|^{\alpha'}\, dx + |\Omega|\right)+1\right) < +\infty,
\end{aligned}$$

and the sequence $\left\{|\nabla u_\varepsilon^0|^{p_\varepsilon^0}\right\}_{\varepsilon>0}$ is L^1-bounded (see estimate (49)), the convergence property of fluxes (91) follows from Theorem 3. Thus, in view of the properties (90)–(91), the limit passage in (4) as $\varepsilon \to 0$ immediately leads to the relation

$$\int_\Omega \left(|\nabla u^0|^{p^0-2}\nabla u^0, \nabla\varphi\right)_{\mathbb{R}^N} dx = \int_\Omega (g,\nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega;\Gamma_D). \tag{92}$$

Since the inclusion $u^0 \in W_0^{1,p^0(\cdot)}(\Omega;\Gamma_D)$ is guaranteed by Lemma 2 and convergence $(75)_2$, it follows that u^0 is a weak solution to the boundary value problem (87).

Taking into account the fact that $u^0 \in W_0^{1,p^0(\cdot)}(\Omega;\Gamma_D)$ and $\omega(t) = k_0/\log(|t|^{-1})$ is a modulus of continuity of the exponent $p^0 \in \mathfrak{S}_{ad}$, it follows that the set $C_0^\infty(\Omega)$ is dense in $W_0^{1,p^0(\cdot)}(\Omega;\Gamma_D)$ (see Theorem 13.10 in Zhikov, 2011). Hence, we can consider $\varphi = u^0$ in the identity (92) as a test function. As a result, we arrive at the energy equality (86). The fact that $u^0 \in W_0^{1,p^0(\cdot)}(\Omega)$ is the unique variational solution to the Dirichlet problem (87) follows from the strict monotonicity of the nonlinear operator $\mathcal{A}_{p^0} : W_0^{1,p^0(\cdot)}(\Omega;\Gamma_D) \to \left(W_0^{1,p^0(\cdot)}(\Omega;\Gamma_D)\right)^*$ given by the equality (12).

As for the property (85), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\left(\int_\Omega |\nabla u_\varepsilon^0|^{p_\varepsilon^0}\, dx + \varepsilon\int_\Omega |\nabla u_\varepsilon^0|^\beta\, dx\right) &\overset{\text{by (4)}}{=} \lim_{\varepsilon\to 0}\left(\int_\Omega \left(g,\nabla u_\varepsilon^0\right)_{\mathbb{R}^N} dx\right) \\
&\overset{\text{by }(75)_2}{=} \int_\Omega \left(g,\nabla u^0\right)_{\mathbb{R}^N} dx \overset{\text{by (86)}}{=} \int_\Omega |\nabla u^0|^{p^0}\, dx.
\end{aligned}$$

It remains to establish the relation (88). To this end, we have to pass to the limit in the integral identity (72) as $\varepsilon \to 0$. By Sobolev Embedding Theorem, the conditions (6) and the weak convergence $(75)_2$ imply the strong convergence $u^0_\varepsilon \to u^0$ in $L^\beta(\Omega)$. Combining this fact with the properties (90) and (91), we obtain

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u^0_\varepsilon|^{p^0_\varepsilon - 2} \nabla u^0_\varepsilon + \varepsilon |\nabla u^0_\varepsilon|^{\beta-2} \nabla u^0_\varepsilon, \nabla\varphi \Big)_{\mathbb{R}^N} \, dx \\
&\to \int_\Omega \Big(|\nabla u^0|^{p^0-2} \nabla u^0, \nabla\varphi \Big)_{\mathbb{R}^N} \, dx, \\
&\int_\Omega \big(g, \nabla u^0_\varepsilon\big)_{\mathbb{R}^N} \, dx \to \int_\Omega \big(g, \nabla u^0\big)_{\mathbb{R}^N} \, dx,
\end{aligned}$$

and $B^0_\varepsilon \nabla \theta^0_\varepsilon \rightharpoonup B^0 \nabla \theta^0$ in $L^1(\Omega)^N$ by Lemma 9. Thus, the limit passage in (72) leads to the equality

$$-\operatorname{div}\big(B^0 \nabla\theta^0\big) = \operatorname{div}\Big[\Big(|\nabla u^0|^{p^0-2}\nabla u^0 - g\Big) u^0\Big] + \big(g, \nabla u^0\big)_{\mathbb{R}^N} \quad \text{in } \mathcal{D}'(\Omega). \tag{93}$$

Since the set $C^\infty_0(\Omega; \Gamma_D)$ is dense in $W^{1,p^0(\cdot)}_0(\Omega;\Gamma_D)$, we can apply the transformations that we used in Lemma 3, to show that

$$\operatorname{div}\Big[\Big(|\nabla u^0|^{p^0-2}\nabla u^0 - g\Big) u^0\Big] + \big(g, \nabla u^0\big)_{\mathbb{R}^N} = |\nabla u^0|^{p^0}$$

in the sense of distributions. Thus, $\theta^0 \in W^{1,\varrho}_0(\Omega)$ is a duality solution to the boundary value problem (88). $\square$

Proposition 4 *Let $\tau \geq \sqrt{|\Omega|}(\beta - \alpha)$. Then, there exists a constant $C^* > 0$ such that estimate* (73) *holds true.*

Proof For an arbitrary $\xi \in [m_1, m_2]$, we set $\widehat{B}(\cdot) = \xi I$. Then, due to the fixed point principle, by analogy with Zhikov (2011) (see p.495), it can be shown that the system

$$\operatorname{div}\Big(|\nabla u|^{\sigma(\theta)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u\Big) = \operatorname{div} g \quad \text{in } \Omega, \tag{94}$$

$$u = 0 \text{ on } \Gamma_D, \quad \Big[|\nabla u|^{\sigma(\theta)-2} + \varepsilon|\nabla u|^{\beta-2}\Big]\partial_\nu u = 0 \text{ on } \Gamma_N, \tag{95}$$

$$-\xi \operatorname{div}(\nabla\theta) = |\nabla u|^{\sigma(\theta)} + \varepsilon|\nabla u|^\beta \quad \text{in } \Omega, \quad \theta|_{\partial\Omega} = 0 \tag{96}$$

has at least one solution $(u_\varepsilon, \theta_\varepsilon) \in W^{1,\beta}_0(\Omega;\Gamma_D) \times \Big(W^{2,1+\delta}(\Omega) \cap W^{1,1+\delta}_0(\Omega)\Big)$ for all $\varepsilon > 0$ with some positive $\delta > 0$. Moreover, there exists a constant $C > 0$ such that

$$\begin{gathered}
\sup_{\varepsilon>0} \|u_\varepsilon\|_{W^{1,\alpha}_0(\Omega;\Gamma_D)} \leq C, \quad \sup_{\varepsilon>0} \|\theta_\varepsilon\|_{W^{2,1+\delta}(\Omega)} \leq C \quad \text{for some } \delta > 0, \text{ and} \\
u_\varepsilon \rightharpoonup u \text{ in } W^{1,\alpha}_0(\Omega;\Gamma_D) \text{ and } \theta_\varepsilon \rightharpoonup \theta \text{ in } W^{2,1+\delta}(\Omega) \text{ as } \varepsilon \to 0,
\end{gathered} \tag{97}$$

where (u, θ) satisfies (in the sense of distributions) the following relations

$$\operatorname{div}\left(|\nabla u|^{\sigma(\theta)-2}\nabla u\right) = \operatorname{div} g,$$
$$u = 0 \text{ on } \Gamma_D, \quad |\nabla u|^{\sigma(\theta)-2}\partial_\nu u = 0 \text{ on } \Gamma_N,$$
$$-\xi \operatorname{div}(\nabla\theta) = \operatorname{div}\left[\left(|\nabla u|^{\sigma(\theta)-2}\nabla u - g\right) u\right] + (g, \nabla u)_{\mathbb{R}^N}.$$

We set $\widehat{p}_{\varepsilon,\lambda} = T_\lambda(\sigma(\theta_\varepsilon))$, where T_λ are smoothing operators, satisfying the properties

$$T_\lambda(\sigma(\theta_\varepsilon)) \to \sigma(\theta_\varepsilon) \quad \text{in } L^2(\Omega) \text{ as } \lambda \to 0, \tag{98}$$
$$\alpha \le T_\lambda(\sigma(\theta_\varepsilon)) \le \beta,\ T_\lambda(\sigma(\theta_\varepsilon)) \in C^1_{loc}(\mathbb{R}) \quad \forall\, \varepsilon \in (0, \lambda(\varepsilon)) \text{ with some } \lambda(\varepsilon) > 0. \tag{99}$$

As an example of such operators, we can take the following one

$$T_\lambda(\sigma(\theta_\varepsilon)) = \max\left\{\alpha, \min\left\{\sigma(\theta_\varepsilon) * \rho_\lambda, \beta\right\}\right\},$$

where $\{\rho_\lambda\}_{\lambda>0}$ is any rescaled family of smooth mollifiers such that $\operatorname{supp}\rho_\lambda \in B(0,\lambda)$.

Let $(\widehat{u}_{\varepsilon,\lambda(\varepsilon)}, \widehat{\theta}_{\varepsilon,\lambda(\varepsilon)}) \in W_0^{1,\beta}(\Omega;\Gamma_D) \times W_0^{1,1+\delta}(\Omega)$ be a unique solution to the system

$$\operatorname{div}\left(|\nabla u|^{p_{\varepsilon,\lambda(\varepsilon)}(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u\right) = \operatorname{div} g \quad \text{in } \Omega, \tag{100}$$
$$u = 0 \text{ on } \Gamma_D, \quad \left[|\nabla u|^{p_{\varepsilon,\lambda(\varepsilon)}(x)-2} + \varepsilon|\nabla u|^{\beta-2}\right]\partial_\nu u = 0 \text{ on } \Gamma_N, \tag{101}$$
$$-\,\xi \operatorname{div}(\nabla\theta) = |\nabla u|^{p_{\varepsilon,\lambda(\varepsilon)}(x)} \quad \text{in } \Omega, \quad \theta|_{\partial\Omega} = 0 \tag{102}$$

(here, we consider $\widehat{\theta}_{\varepsilon,\lambda(\varepsilon)}$ as the duality solution of (102), for the details we refer to the proof of Lemma 3).

We note that, in view of the estimate (14), the sequence $\left\{\widehat{\theta}_{\varepsilon,\lambda(\varepsilon)}\right\}_{\varepsilon>0}$ is bounded in $W_0^{1,\gamma}(\Omega)$ for all $\gamma \in [1, \frac{N}{N-1})$. Since

$$|\nabla\widehat{u}_{\varepsilon,\lambda(\varepsilon)}|^{\widehat{p}_{\varepsilon,\lambda(\varepsilon)}} = \operatorname{div}\left[\left(|\nabla\widehat{u}_{\varepsilon,\lambda(\varepsilon)}|^{\widehat{p}_{\varepsilon,\lambda(\varepsilon)}-2}\nabla\widehat{u}_{\varepsilon,\lambda(\varepsilon)} + \varepsilon|\nabla\widehat{u}_{\varepsilon,\lambda(\varepsilon)}|^{\beta-2}\nabla\widehat{u}_{\varepsilon,\lambda(\varepsilon)} - g\right)\widehat{u}_{\varepsilon,\lambda(\varepsilon)}\right] + \left(g, \nabla\widehat{u}_{\varepsilon,\lambda(\varepsilon)}\right)_{\mathbb{R}^N}$$

in the sense of distributions (see Lemma 10), it follows that $\left(\widehat{B}, \widehat{u}_{\varepsilon,\lambda(\varepsilon)}, \widehat{\theta}_{\varepsilon,\lambda(\varepsilon)}, \widehat{p}_{\varepsilon,\lambda(\varepsilon)}\right) \in \widehat{\Xi}_\varepsilon$ for all $\varepsilon > 0$. Then, by Lemmas 8, 9, and 10 and definition of the set $\mathfrak{B}_{ad}$, the sequence $\left\{\left(\widehat{B}, \widehat{u}_{\varepsilon,\lambda(\varepsilon)}, \widehat{\theta}_{\varepsilon,\lambda(\varepsilon)}, \widehat{p}_{\varepsilon,\lambda(\varepsilon)}\right)\right\}_{\varepsilon>0}$ is bounded in $BV(\Omega)^{N\times N} \times W_0^{1,\alpha}(\Omega;\Gamma_D) \times W_0^{1,\gamma}(\Omega) \times C(\overline{\Omega})$ and

$$\|\widehat{p}_{\varepsilon,\lambda(\varepsilon)} - \sigma(\widehat{\theta}_{\varepsilon,\lambda(\varepsilon)})\|_{L^2(\Omega)} \le (\beta - \alpha)\sqrt{|\Omega|} \quad \forall\, \varepsilon > 0.$$

Since $\tau \ge (\beta-\alpha)\sqrt{|\Omega|}$ and $\sup_{\varepsilon>0}\|\widehat{\theta}_{\varepsilon,\lambda(\varepsilon)}\|_{L^r(\Omega)} < +\infty$, it follows from definition of the function μ_τ that

$$\limsup_{\varepsilon\to 0} \inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}(B,u,\theta,p) \le \limsup_{\varepsilon\to 0} J_{\varepsilon,\tau}\left(\widehat{B},\widehat{u}_{\varepsilon,\lambda(\varepsilon)},\widehat{\theta}_{\varepsilon,\lambda(\varepsilon)},\widehat{p}_{\varepsilon,\lambda(\varepsilon)}\right)$$
$$= \limsup_{\varepsilon\to 0}\int_\Omega |\widehat{\theta}_{\varepsilon,\lambda(\varepsilon)}(x)-\theta_d(x)|^r\,dx < +\infty$$

provided the exponent $\gamma \in \left[1,\frac{N}{N-1}\right)$ was chosen as follows: $\gamma = \frac{Nr}{N+r}$. $\square$

Summing up Propositions 2 and 3, we are led to the following conclusion: the fulfilment of (73) or $\tau \ge (\beta-\alpha)\sqrt{|\Omega|}$ suffices to claim that any cluster tuple (in the context of convergence (75)–(76)) of the sequence of optimal solutions $\left\{(B^0_\varepsilon,u^0_\varepsilon,\theta^0_\varepsilon,p^0_\varepsilon)\in\widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ is a quasi-feasible point to the original OCP (1)–(6). We are now in a position to prove our main result.

Theorem 5 *Let* $\left\{(B^0_\varepsilon,u^0_\varepsilon,\theta^0_\varepsilon,p^0_\varepsilon)\in\widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ *be an arbitrary sequence of optimal solutions to the approximated problems* (19)–(23)*. If the condition* (73) *holds true, then any cluster tuple* $\left(B^0,u^0,\theta^0,p^0\right)$ *is a quasi-optimal solution of the OCP* (1)–(6)*. Moreover, in this case the following variational property holds*

$$\lim_{\varepsilon\to 0}\inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}(B,u,\theta,p) = J\left(B^0,u^0,\theta^0,p^0\right) = \inf_{(B,u,\theta,p)\in\widehat{\Xi}_0(\tau)} J(B,u,\theta,p). \tag{103}$$

Proof As follows from Propositions 2–4, the set of feasible points $\widehat{\Xi}_0(\tau)$ is nonempty. Let us assume that there exists a tuple $\left(\widehat{B},\widehat{u},\widehat{\theta},\widehat{p}\right)$ in $\widehat{\Xi}_0(\tau)$ such that

$$J\left(\widehat{B},\widehat{u},\widehat{\theta},\widehat{p}\right) < J\left(B^0,u^0,\theta^0,p^0\right). \tag{104}$$

By the definition of the set $\widehat{\Xi}_0(\tau)$, we have $\widehat{B}\in\mathfrak{B}_{ad}$, $\widehat{p}\in\mathfrak{S}_{ad}$, and $\|\widehat{p}-\sigma(\widehat{\theta})\|_{L^2(\Omega)}\le\tau$. We define the sequence $\left\{(\widehat{B}_\varepsilon,\widehat{u}_\varepsilon,\widehat{\theta}_\varepsilon,\widehat{p}_\varepsilon)\right\}_{\varepsilon>0}$ as follows

$$\begin{aligned}
&\widehat{p}_\varepsilon \to \widehat{p} \ \text{ in } \ C(\overline{\Omega}) \ \text{ as } \ \varepsilon\to 0,\\
&\widehat{B}_\varepsilon \equiv \widehat{B}, \quad \text{ and } \quad \widehat{p}_\varepsilon\in\mathfrak{S}_{ad}\ \forall\,\varepsilon>0,\\
&\|\widehat{p}_\varepsilon-\sigma(\widehat{\theta})\|_{L^2(\Omega)} \le \tau+\frac{\varepsilon^2}{2} \quad \text{for } \varepsilon\in(0,\tau) \text{ small enough,}
\end{aligned}\tag{105}$$

and each of the pairs $(\widehat{u}_\varepsilon,\widehat{\theta}_\varepsilon)$ is a weak solution to the boundary value problems (20)–(22) with $B=\widehat{B}_\varepsilon$ and $p=\widehat{p}_\varepsilon$, and such that $\widehat{\theta}_\varepsilon$ is the duality solution of (22). Since each of these problems admits the unique solution in

$W_0^{1,\beta}(\Omega;\Gamma_D)\times W_0^{1,\gamma}(\Omega)$, satisfying the a priori estimates (44) and (71), it follows that the tuples $(\widehat{B}_\varepsilon,\widehat{u}_\varepsilon,\widehat{\theta}_\varepsilon,\widehat{p}_\varepsilon)$ are feasible points to the corresponding approximated OCPs (19)–(23), i.e. $(\widehat{B}_\varepsilon,\widehat{u}_\varepsilon,\widehat{\theta}_\varepsilon,\widehat{p}_\varepsilon)\in\Xi_\varepsilon$ for all $\varepsilon>0$. Moreover, in view of Remark 3 and the estimates

$$
\begin{aligned}
\|\widehat{u}_\varepsilon\|^\alpha_{W_0^{1,\alpha}(\Omega)} &\overset{\text{by (31), (30)}}{\leq} (1+|\Omega|)\left(\int_\Omega |\nabla \widehat{u}_\varepsilon|^{\widehat{p}_\varepsilon}\,dx+1\right)\\
&\overset{\text{by (49)}}{\leq} (1+|\Omega|)\left(2^{\alpha'+1}\left(\int_\Omega |g|^{\alpha'}\,dx+|\Omega|\right)+1\right),\\
\|\widehat{\theta}_\varepsilon\|_{W_0^{1,\gamma}(\Omega)} &\overset{\text{by (44), (71)}}{\leq} C(\delta)2^{\alpha'+1}\left(\int_\Omega |g|^{\alpha'}\,dx+|\Omega|\right),\\
\|\widehat{\theta}_\varepsilon\|_{W_0^{1,\varrho}(\Omega)} &\overset{\text{by Höder ineq.}}{\leq} C\|\widehat{\theta}_\varepsilon\|_{W_0^{1,\gamma}(\Omega)}<+\infty,\quad \forall\,\gamma\geq\varrho,
\end{aligned}
\tag{106}
$$

the sequence $\left\{(\widehat{u}_\varepsilon,\widehat{\theta}_\varepsilon)\right\}_{\varepsilon>0}$ is bounded in $W_0^{1,\alpha}(\Omega;\Gamma_D)\times W_0^{1,\gamma}(\Omega)$ for some $\delta>0$. Hence, by analogy with Propositions 2 and 3, it can be shown that

$$
\widehat{u}_\varepsilon\rightharpoonup\widehat{u}\ \text{ in }\ W_0^{1,\alpha}(\Omega;\Gamma_D)\quad\text{and}\quad \widehat{\theta}_\varepsilon\rightharpoonup\widehat{\theta}\ \text{ in }\ W_0^{1,\gamma}(\Omega),
\tag{107}
$$

where $(\widehat{u},\widehat{\theta})$ is a weak solution to the boundary value problem (2)–(3) with $p=\widehat{p}$. Since the weak solution of this problem is unique (by the strict monotonicity and the log-Hölder continuity of the exponent $\widehat{p}$) it is not necessary to pass to a subsequence in (107). At the same time, since the embedding $W^{1,\gamma}(\Omega)\hookrightarrow L^\gamma(\Omega)$ is compact, we can suppose that there exists a subsequence $\left\{\widehat{\theta}_{\delta(\varepsilon)}\right\}_{\delta(\varepsilon)>0}$ of $\left\{\widehat{\theta}_\varepsilon\right\}_{\varepsilon>0}$ such that $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$, $\delta(\varepsilon)\leq\varepsilon$, and $\widehat{\theta}_{\delta(\varepsilon)}\to\widehat{\theta}$ almost everywhere in Ω. Hence, by the boundedness of the sequence $\left\{\sigma(\widehat{\theta}_\varepsilon)\right\}_{\varepsilon>0}$ and the Lebesgue theorem, we can suppose that $\sigma(\widehat{\theta}_{\delta(\varepsilon)})\to\sigma(\widehat{\theta})$ strongly in $L^2(\Omega)$ as $\varepsilon\to 0$ and

$$
\|\sigma(\widehat{\theta}_{\delta(\varepsilon)})-\sigma(\widehat{\theta})\|_{L^2(\Omega)}\leq\frac{\varepsilon^2}{2}\quad\text{for }\varepsilon\text{ small enough.}
\tag{108}
$$

As a result, (105) and (108) imply that

$$
\begin{aligned}
\|\widehat{p}_{\delta(\varepsilon)}-\sigma(\widehat{\theta}_{\delta(\varepsilon)})\|_{L^2(\Omega)}\leq\|\widehat{p}_{\delta(\varepsilon)}-\sigma(\widehat{\theta})\|_{L^2(\Omega)}&+\|\sigma(\widehat{\theta}_{\delta(\varepsilon)})-\sigma(\widehat{\theta})\|_{L^2(\Omega)}\\
&\leq\tau+\delta(\varepsilon)^2\leq\tau+\varepsilon^2
\end{aligned}
$$

for ε small enough. Hence, there exists $\varepsilon_0>0$ such that

$$
0\leq\mu_\tau\left(\int_\Omega|\widehat{p}_{\delta(\varepsilon)}(x)-\sigma(\widehat{\theta}_{\delta(\varepsilon)}(x))|^2\,dx\right)\leq\varepsilon^2,\quad\forall\,\varepsilon\in(0,\varepsilon_0).
\tag{109}
$$

Taking this fact into account, we get

$$\begin{aligned}
&J\left(B^0,u^0,\theta^0,p^0\right)\\
&\overset{\text{by (76)}}{=} \liminf_{\varepsilon\to 0}\int_\Omega |\theta^0_\varepsilon(x)-\theta_d(x)|^r\,dx \le \liminf_{\varepsilon\to 0} J_{\varepsilon,\tau}\left(B^0_\varepsilon,u^0_\varepsilon,\theta^0_\varepsilon,p^0_\varepsilon\right)\\
&= \liminf_{\varepsilon\to 0}\inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}\left(B,u,\theta,p\right) \le \liminf_{\varepsilon\to 0} J_{\varepsilon,\tau}\left(\widehat{B}_{\delta(\varepsilon)},\widehat{u}_{\delta(\varepsilon)},\widehat{\theta}_{\delta(\varepsilon)},\widehat{p}_{\delta(\varepsilon)}\right)\\
&= \liminf_{\varepsilon\to 0} J\left(\widehat{B}_{\delta(\varepsilon)},\widehat{u}_{\delta(\varepsilon)},\widehat{\theta}_{\delta(\varepsilon)},\widehat{p}_{\delta(\varepsilon)}\right) + \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mu_\tau\left(\int_\Omega |\widehat{p}_{\delta(\varepsilon)}-\sigma(\widehat{\theta}_{\delta(\varepsilon)})|^2 dx\right)\\
&\qquad\qquad + \lim_{\varepsilon\to 0}\varepsilon\int_\Omega |\nabla\widehat{\theta}_{\delta(\varepsilon)}|^\varrho\,dx \overset{\text{by (106)}}{=} J\left(\widehat{B},\widehat{u},\widehat{\theta},\widehat{p}\right).
\end{aligned}\tag{110}$$

Having come into conflict with (104), we conclude: relation (110) holds true only as an equality, which immediately yields (103). Thus, $\left(B^0,u^0,\theta^0,p^0\right)$ is a quasi-optimal solution of the OCP (1)–(6). □

It remains to discuss the question as to the optimal solutions to the OCP (1)–(6) in the sense of Definition 3. As follows from Theorem 5, it suffices to show that $\Xi_0\neq\emptyset$ and consider, instead of the approximated problems (19)–(23), the following ones (with $\tau=\varepsilon$ in the approximation cost functional)

$$\begin{aligned}
\text{Minimize }\ J_\varepsilon(B,u,\theta,p) = \int_\Omega |\theta(x)-\theta_d(x)|^r\,dx + \varepsilon\int_\Omega |\nabla\theta|^\varrho\,dx\\
+\frac{1}{\varepsilon}\mu_\varepsilon\left(\int_\Omega |p(x)-\sigma(\theta(x))|^2\,dx\right)
\end{aligned}\tag{111}$$

subject to the constraints (20)–(23).

In effect, the validity of the main Theorem 1 immediately results from the following result.

Theorem 6 *Let Ω be an open bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary. Assume that $\widehat{\Xi}_0(\tau)\neq\emptyset$ for $\tau=0$, i.e. there exist a matrix $\widehat{B}\in\mathfrak{B}_{ad}$, an exponent $\widehat{p}\in\mathfrak{S}_{ad}$, and a weak solution to the thermistor problem* (1)–(6) $(\widehat{u},\widehat{\theta})\in W_0^{1,\sigma(\widehat{\theta}(\cdot))}(\Omega;\Gamma_D)\times W_0^{1,\gamma}(\Omega)$ *with $B(\cdot)=\widehat{B}(\cdot)$ such that $\widehat{\theta}$ is the duality solution to* (3) *and $\widehat{p}=\sigma(\widehat{\theta})$ almost everywhere in Ω. Then, the OCP* (1)–(6) *has a non-empty set of optimal solutions and some of them can be attained (in the sense of convergence (75)–(76)) by solutions $(B^0_\varepsilon,u^0_\varepsilon,\theta^0_\varepsilon,p^0_\varepsilon)$ to the approximated problem* (111).

Proof To begin with, let us note that, in view of the initial assumptions and Remarks 3 and 4, the set Ξ_0, given by (15), is nonempty. To get the solvability of the original OCP (1)–(6), we pass to its perturbation in the form of the family of approximated problems (111). Due to Theorem 4, each of the problems (111) has a nonempty set of solutions. Let $(B^0_\varepsilon,u^0_\varepsilon,\theta^0_\varepsilon,p^0_\varepsilon)$ be optimal tuples to the approximated problems (111). As follows from Proposition 2, compactness of this sequence with respect to the convergence (75)–(76) can be guaranteed by the condition (73). In order to demonstrate that the estimate (73) holds true with some $C^* > 0$, we will closely follow the proof-line of Theorem 5. As a result, a sequence $\left\{(\widehat{B}_\varepsilon, \widehat{u}_\varepsilon, \widehat{\theta}_\varepsilon, \widehat{p}_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ can be constructed such that

$$
\begin{aligned}
&\widehat{p}_\varepsilon \to \widehat{p} \ \text{ in } \ C(\overline{\Omega}) \ \text{ as } \ \varepsilon \to 0, \quad \widehat{B}_\varepsilon \equiv \widehat{B}, \quad \text{ and } \quad \widehat{p}_\varepsilon \in \mathfrak{S}_{ad} \ \forall\, \varepsilon > 0,\\
&\|\widehat{p}_\varepsilon - \sigma(\widehat{\theta})\|_{L^2(\Omega)} \le \varepsilon/2 \quad \text{for } \varepsilon > 0 \text{ small enough},\\
&\widehat{u}_\varepsilon \rightharpoonup \widehat{u} \ \text{ in } \ W_0^{1,\alpha}(\Omega;\Gamma_D) \quad \text{and} \quad \widehat{\theta}_\varepsilon \rightharpoonup \widehat{\theta} \ \text{ in } \ W_0^{1,\gamma}(\Omega),\\
&\sigma(\widehat{\theta}_\varepsilon) \to \sigma(\widehat{\theta}) \quad \text{strongly in } \ L^2(\Omega) \ \text{ as } \ \varepsilon \to 0.
\end{aligned}
$$

Hence, by applying the arguments, having been used by us before in the proof of Theorem 5, we can conclude that there exists $\varepsilon_0 > 0$ and a subsequence $\{\delta(\varepsilon)\}$ of $\{\varepsilon\}$ such that

$$
\|\sigma(\widehat{\theta}_{\delta(\varepsilon)}) - \sigma(\widehat{\theta})\|_{L^2(\Omega)} \le \frac{\varepsilon}{2}
$$

for all $\varepsilon < \varepsilon_0$

and

$$
\|\widehat{p}_{\delta(\varepsilon)} - \sigma(\widehat{\theta}_{\delta(\varepsilon)})\|_{L^2(\Omega)} \le \|\widehat{p}_{\delta(\varepsilon)} - \sigma(\widehat{\theta})\|_{L^2(\Omega)} + \|\sigma(\widehat{\theta}_{\delta(\varepsilon)}) - \sigma(\widehat{\theta})\|_{L^2(\Omega)} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \quad \forall\, \varepsilon \in (0, \varepsilon_0).
$$

Therefore,

$$
\mu_{\delta(\varepsilon)} \left(\int_\Omega |\widehat{p}_{\delta(\varepsilon)}(x) - \sigma(\widehat{\theta}_{\delta(\varepsilon)}(x))|^2 \, dx \right) = 0, \quad \forall\, \varepsilon \in (0, \varepsilon_0). \tag{112}
$$

Thus,

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \inf_{(B,u,\theta,p)\in\widehat{\Xi}_{\delta(\varepsilon)}} J_{\delta(\varepsilon)}(B,u,\theta,p)\\
&\le \limsup_{\varepsilon\to 0} J_{\delta(\varepsilon)}(\widehat{B}_{\delta(\varepsilon)}, \widehat{u}_{\delta(\varepsilon)}, \widehat{\theta}_{\delta(\varepsilon)}, \widehat{p}_{\delta(\varepsilon)})\\
&= \limsup_{\varepsilon\to 0} \int_\Omega |\widehat{\theta}_{\delta(\varepsilon)} - \theta_d|^r \, dx = C^* < +\infty.
\end{aligned}
$$

As a result, Propositions 2 and 3 imply the existence of a cluster tuple

$$
\left(B^0, u^0, \theta^0, p^0\right) \in BV(\Omega)^{N\times N} \times W_0^{1,\beta}(\Omega;\Gamma_D) \times W_0^{1,\gamma}(\Omega) \times \mathfrak{S}_{ad},
$$

with $\gamma = \frac{Nr}{N+r}$, of the sequence $\left\{(\widehat{B}_\varepsilon, \widehat{u}_\varepsilon, \widehat{\theta}_\varepsilon, \widehat{p}_\varepsilon)\right\}_{\varepsilon>0}$ in the sense of convergence

(75)–(76), such that (see (104) and (110))

$$
\begin{aligned}
J\left(B^0,u^0,\theta^0,p^0\right) &= \liminf_{\varepsilon\to 0}\int_\Omega |\theta^0_\varepsilon-\theta_d|^r\,dx = \liminf_{\varepsilon\to 0}\int_\Omega |\theta^0_{\delta(\varepsilon)}-\theta_d|^r\,dx\\
&= \liminf_{\varepsilon\to 0} J_{\delta(\varepsilon)}\left(B^0_{\delta(\varepsilon)},u^0_{\delta(\varepsilon)},\theta^0_{\delta(\varepsilon)},p^0_{\delta(\varepsilon)}\right)\\
&= \liminf_{\varepsilon\to 0}\inf_{(B,u,\theta,p)\in\widehat{\Xi}_{\delta(\varepsilon)}} J_{\delta(\varepsilon)}\left(B,u,\theta,p\right)\\
&\ge \liminf_{\varepsilon\to 0}\inf_{(B,u,\theta,p)\in\widehat{\Xi}_{\varepsilon}} J_{\varepsilon}\left(B,u,\theta,p\right)\\
&\ge \liminf_{\varepsilon\to 0}\int_\Omega |\theta^0_\varepsilon-\theta_d|^r\,dx = J\left(B^0,u^0,\theta^0,p^0\right).
\end{aligned}
$$

Taking into account the strong convergence $\widehat{p}_{\delta(\varepsilon)}\to\widehat{p}=\sigma(\widehat{\theta})$ in $C(\overline{\Omega})$, we finally deduce that $\left(B^0,u^0,\theta^0,p^0\right)\in\Xi_0$. In order to show that $\left(B^0,u^0,\theta^0,p^0\right)$ is an optimal solution to the original problem, we can assume the converse statement and apply the arguments of the proof of Theorem 5. □

Remark 7 *As for the fulfilment of the conditions of Theorem 6, we make use of the following observation. Assume that $N=2$ or $N=3$, the constant m_2 in the definition of the class of admissible controls (5) is large enough, and the function σ is Lipschitz continuous on $[0,\infty)$, i.e. $|\sigma(s)-\sigma(s')|\le m|s-s'|$. Then, there exists a value $\xi\in[m_1,m_2]$ such that (see, for instance, Zhikov, 1997, 2008a) a unique solution $(u(B),\theta(B))\in H_0^{1,\sigma(\theta(B))}(\Omega;\Gamma_D)\times W_0^{1,\gamma}(\Omega)$ to the thermistor problem (2)–(4) is such that*

$$
\sigma(\theta(B))\in W^{1,\gamma}(\Omega),\quad \textit{for some } \gamma>N \textit{ and } \quad \int_\Omega |\nabla u(B)|^{\frac{N\sigma(\theta(B))}{N-2}}\,dx<+\infty
$$

for all matrices $B=\widehat{\xi}I\in\mathfrak{B}_{ad}$, where $\widehat{\xi}\in[\xi,m_2]$. Hence, by Sobolev embedding theorem, we can conclude that: $p:=\sigma(\theta(B))\in W^{1,\gamma}(\Omega)$ is Hölder continuous in Ω and, hence, p has a logarithmic modulus of continuity. So, $p\in\mathfrak{S}_{ad}$ and, therefore, $\Xi_0\ne\emptyset$. Thus, the OCP (1)–(6) *admits at least one solution by Theorem 6.*

We also note that without the assumption of large constant m_2, the existence of weak solutions to the thermistor boundary value problem (2)–(4) *with any matrix $B=\widehat{\xi}I\in\mathfrak{B}_{ad}$, $\widehat{\xi}\in[m_1,m_2]$, and any Lipschitz continuous function σ, has been proved only in the one-dimensional case (see Zhikov, 2008a). Hence, if $N=1$ then all conditions of Theorem 6 are satisfied and, therefore, the corresponding OCP* (1)–(6) *is solvable.*

There stands separately the special case of the thermistor problem with $\alpha=\beta$. Following the fixed point principle, it can be proven that for any admissible matrix-valued control $B\in\mathfrak{B}_{ad}$, the system (2)–(4) *has at last one solution $(u,\theta)\in W_0^{1,\alpha+\delta}(\Omega;\Gamma_D)\times\left[W_0^{1,1+\delta}(\Omega)\cap W^{2,1+\delta}(\Omega)\right]$, for some $\delta>0$, in domain Ω with a sufficiently smooth boundary (see p.494 in Zhikov, 2011). Moreover,* *in this case* $u \in H_0^{1,p(\cdot)}(\Omega;\Gamma_D)$ *and the equation* (3) *holds almost everywhere in* Ω*. As a result, we have: the tuple* (B,u,θ,σ)*, with* $\sigma(\theta)=\alpha$*, is a feasible solution to the OCP* (1)–(6) *and, therefore, Theorem 6 leads to the solvability of the OCP* (1)–(6).

In the general case, the question of non-emptiness of the set of feasible solutions Ξ_0 *to the OCP* (1)–(6) *remains open even for the smooth functions* σ *and homogeneous Dirichlet boundary conditions* ($\Gamma_N=\emptyset$)*. At the same time, if we consider, instead of equation* (3)*, its relaxation in the form* (24)*, Theorem 7.2 from Zhikov (2011) says that the relaxed version of the thermistor problem (2),* (4)*,* (24) *admits a solution* $(u,\theta)\in W_0^{1,\sigma(\theta(B))}(\Omega)\times W_0^{1,\gamma}(\Omega)$ *for any* $\gamma\in[1,\frac{N}{N-1})$*,* $B\in\mathfrak{B}_{ad}$*, and any continuous function* $\sigma(\theta)$*, satisfying the condition* (6) *with* $\beta<+\infty$ *if* $\alpha\geq N-1$*, and* $\beta<\frac{\alpha(N-1)}{N-1-\alpha}$ *if* $\alpha<N-1$*. Moreover, in this case we have* $u\in W_0^{1,\alpha}(\Omega)$*. However, as it was mentioned in Remark 2, in this case the inclusion* $u\in H_0^{1,p(\cdot)}(\Omega)$ *is by no means obvious. In order to circumvent this artefact, we can apply Theorem 7.2 from Zhikov (2011) to a function* $\sigma(\theta)$ *that has a logarithmic modulus of continuity. Then,* $W_0^{1,p(\cdot)}(\Omega)=H_0^{1,p(\cdot)}(\Omega)$ *with* $p:=\sigma(\theta(B))$*, i.e. the* H*-solution coincides with the* W*-solution, and, therefore, the tuple* (B,u,θ,σ) *is a feasible solution to the modified OCP* (1)*,* (2)*,* (24)*,* (4)*,* (6) *for any admissible control* $B\in\mathfrak{B}_{ad}$*. Thus, its solvability can be established by analogy with Theorem 6.*

5. Conclusion

Thermistor, as a generic name of a device that is made from materials whose electrical conductivity is highly dependent on temperature, is often used as temperature control element in a wide variety of military and industrial equipment ranging from space vehicles to the air conditioning controllers. It is well known that the large temperature gradients may cause thermistor to crack. Numerical experiences (see, for instance, Fowler, Frigaard and Howison, 1992; Zhou and Westbrook, 1997) indicate that the boundary value problem (2)–(4) is rather sensitive to the choice of a source function div g, the type of boundary conditions in (2), and magnitude of the heat conductivity, given by the matrix $B=B(x)$.

In spite of the fact that theoretical analysis of the thermistor equations with different types of boundary conditions has received a significant amount of attention, there are only optimal control papers on the thermistor problem (see Introduction) where either the source or the heat transfer coefficient in Robin boundary conditions are taken to be the control. This circumstance stimulated us to bring into consideration the optimization problem (1)–(6) with controls in coefficients of the elliptic operator div $(B(x)\nabla\theta(x))$. From this point of view, we deal with a material (or topology) optimization problem for an ill-posed nonlinear elliptic system, and there are two main sources of concern that make the analysis of optimization problem (1)–(6) nontrivial and rather complicate: the first one comes from nonlinearity of the state equations (2)–(3), and the second issue is related to the point-wise constraint (4).

The main idea that has been pushed forward in this paper (see Remark 2), is to free the original problem from having to deal with difficult constraints. In particular, we allow for a certain flexibility in dealing with the state restriction $p(\cdot) = \sigma(\theta(\cdot))$, in the sense that this equality between the function $p(\cdot)$ as a fictitious control and the temperature variable $\theta(\cdot)$ can be interpreted with some accuracy and the indicated pairs can run freely in their respective sets of feasibility. Properly speaking, a similar idea was recently promoted in a very interesting paper of P. Pedregal (2019), where the author not only allows for a bit of flexibility in the interpretation of state-equality constraints, but it is also proposed to estimate the corresponding 'defect' by introducing an additional variable in a collection of approximated problems. We sincerely hope that this idea can be leveraged for the deriving and substantiation of optimality condition for the thermistor problem (1)–(6).

References

Ambrosio, L., Fusco, N. and Pallara, D. (2000) *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, New York.

Antontsev, S. N. and Chipot, M. (1994) The thermistor problem: existence, smoothness, uniqueness, blowup. *SIAM J. Math. Anal.* **25**, 1128–1156.

Antontsev, S. N. and Rodrigues, J. F. (2006) On stationary thermorheological viscous flows. *Ann. Univ. Ferrara* **52** (1), 19–36.

Baranger, J. and Mikelić, A. (1995) Stationary solutions to a quasi-Newtonian flow with viscous heating. *Math. Models Methods Appl. Sci.* **5** (6), 725–738.

Buttazzo, G. and Kogut, P. I. (2011) Weak optimal controls in coefficients for linear elliptic problems. *Revista Matematica Complutense* **24**, 83–94.

Casas, E., Kogut, P. I. and Leugering, G. (2016) Approximation of optimal control problems in the coefficient for the p-Laplace equation. I. Convergence result. *SIAM J. Control Optim.* **54** (3), 1406–1422.

Cruz-Uribe, D. V. and Fiorenza, A. (2013) *Variable Lebesgue Spaces: Foundations and Harmonic Analysis.* Birkhäuser, New York.

D'Apice, C., De Maio, U. and Kogut, O. P. (2010) On shape stability of Dirichlet optimal control problems in coefficients for nonlinear elliptic equations. *Advances in Differential Equations* **15** (7–8), 689–720.

D'Apice, C., De Maio, U. and Kogut, O. P. (2012) Optimal control problems in coefficients for degenerate equations of monotone type: Shape stability and attainability problems. *SIAM Journal on Control and Optimization* **50** (3), 1174–1199.

D'Apice, C., De Maio, U. and Kogut, P. I. (2018) Thermistor problem: Multi-dimensional modelling, optimization, and approximation. *32nd European Conference on Modelling and Simulation*, May 22nd–May 25th, 2018, Wilhelmshaven, Germany.

D'Apice, C., De Maio, U. and Kogut, P. I. (2019) An indirect approach to the existence of quasi-optimal controls in coefficients for multi-dimensional thermistor problem. *Networks and Heterogeneous Media*, to appear.

Diening, L., Harjulehto, P., Hästö, P. and Růžiĉka, M. (2011) *Lebesgue and Sobolev Spaces with Variable Exponents.* Springer, New York.

Disser, K., Kaiser, H.-C. and Rehberg, J. (2015) Optimal Sobolev regularity for linear second-order divergence elliptic operators occurring in real-world problems. *SIAM J. Math. Anal.* **47** (3), 1719–1746.

Fowler, A.Ċ., Frigaard, I. and Howison, S. D. (1992) Temperature surges in current-limiting circuit devices. *SIAM J. on Applied Mathematics* **52**, 998–1011.

Giusti, E. (1984) *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser, Boston.

Hömberg, D., Meyer, C., Rehberg, J. and Ring, W. (2010) Optimal control for the thermistor problem. *SIAM Journal on Control and Optimization* **49** (5), 3449–3481.

Horsin, T. and Kogut, P.I. (2015) Optimal L^2-control problem in coefficients for a linear elliptic equation. *Mathematical Control and Related Fields* **5** (1), 73–96.

Horsin, T. and Kogut, P.I. (2016) On unbounded optimal controls in coefficients for ill-posed elliptic Dirichlet boundary value problems. *Asymptotic Analysis* **98**, 155–188.

Hrynkiv, V. (2009) Optimal boundary control for a time dependent thermistor problem. *Electr. Journal of Diff. Equations.* **2009** (83), 1–22.

Hrynkiv, V. and Koshkin, S. (2018) Optimal control of a thermistor problem with vanishing conductivity. *Applied Mathematics and Optimization* https://doi.org/10.1007/s00245-018-9511-z, 1–28.

Howison, S. D., Rodrigues, J. F. and Shillor, M. (1993) Stationary solutions to the thermistor problem. *J. Math. Anal. Appl.* **174** (2), 573–588.

Kogut, P.I. (2014) On approximation of an optimal boundary control problem for linear elliptic equation with unbounded coefficients. *Discrete and Continuous Dynamical Systems* Series A, **34** (5), 2105–2133.

Kogut, P. and Leugering, G. (2011) *Optimal Control Problems for Partial Differential Equations on Reticulated Domains. Approximation and Asymptotic Analysis.* Series: *Systems and Control*, Birkhäuser Verlag, Boston.

Kogut, P. and Leugering, G. (2013) Matrix-valued L^1-optimal control in the coefficients of linear elliptic problems, *Journal for Analysis and its Applications* **32** (4), 433–456.

Kogut, P.I., Manzo, R. and Putchenko, A. (2016) On approximate solutions to the Neumann elliptic boundary value problem with non-linearity of exponential type. *Boundary Value Problems* **2016** (1), 1–32.

Kuttler, K. L., Shillor, M. and Fernandez, J. R. (2008) Existence for the thermoviscoelastic thermistor problem. *Differential Equations and Dynamical Systems* **16** (4), 309–332.

Meyer, C., Panizzi, L. and Schiela, A. (2011) Uniqueness criteria for the adjoint equation in state-constrained elliptic optimal control. *Numer. Funct. Anal. Optim.* **32** (9), 983–1007.

Meinlschmidt, H., Meyer, C. and Rehberg, J. (2016) Optimal control of the thermistor problem in three spatial dimensions. *arXiv:1603.04305 [math.OC]*, 10.1137/16M1072656, 1–39.

Murat, F. (1978) Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa. Cl. Scic. Fis. Mat.* **5**, 481–507.

Orsina, L. (2011) *Elliptic Equations with Measure Data.* Preprint, Sapienza University of Rome.

Pedregal, P. (2019) On a remark of Tikhonov regularization in optimal control under PDEs. *European Journal of Applied Mathematics* **30** (to appear).

Prignet, A. (1995) Remarks on existence and uniqueness of solutions of elliptic problems with right-hand side measures. *Rend. Mat. Appl.* VII. **15** (3), 321–337.

Rădulescu, V. D. and Repovš, D. D. (2015) *Partial Differential Equations with Variable Exponents. Variational Methods and Qualitative Analysis.* CRC Press, New York.

Roubíček, T. (2013) *Nonlinear Partial Differential Equations with Applications.* Birkhäuser, Boston.

Růžička, M. (2000) *Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Math.*, **1748**, Springer-Verlag, Berlin.

Stampacchia, G. (1966) *Equations Elliptiques du Second Ordre a Coefficients Discontinus.* Les Presses de Université de Montréal.

Zhikov, V. V. (1997) On some variational problems. *Russian J. Math. Phys.* **5** (1), 105–116.

Zhikov, V. V. (2008a) Existence theorem for some pairs of coupled elliptic equations. *Russian Acad. Sci. Dokl. (Math.)* **77** (1), 80–84.

Zhikov, V. V. (2008b) Solvability of the three-dimensional thermistor problem. *Proceedings of the Steklov Institute of Mathematics* **281**, 98–111.

Zhikov, V. V. (2009) On the technique for passing to the limit in nonlinear elliptic problems. *Functional Analysis and its Appl.* **43** (2), 96–112.

Zhikov, V.V. (2010) On the weak convergence of fluxes to flux. *Russian Acad. Sci. Dokl. (Math.)* **81** (1), 58–62.

Zhikov, V. V. (2011) On variational problems and nonlinear elliptic equations with nonstandard growth conditions. *Journal of Mathematical Sciences* **173** (5), 463–570.

Zhikov, V.V. and Pastukhova, S. E. (2008) Improved integrability of the gradients of solutions of elliptic equations with variable nonlinearity exponent. *Mat. Sb.* **199** (12), 19–52.

Zhou, S. and Westbrook, D. R. (1997) Numerical solution of the thermistor equations. *J. of Computational and Applied Mathematics* **79**, 101–118.