Control and Cybernetics

vol. **46** (2017) No. 1

Resource placement in the 4-dimensional cube-type processor networks with soft degradation*

by

Roman Kulesza, Jan Chudzikiewicz and Zbigniew Zieliński

Military University of Technology, Faculty of Cybernetics,
Institute of Teleinformatics and Automatics,
ul. S. Kaliskiego 2, 00-908 Warszawa, Poland
jan.chudzikiewicz@wat.edu.pl,
zbigniew.zielinski@wat.edu.pl

Abstract: The paper considers some properties of selected types of perfect resource placements in 4-dimensional cube-type networks with soft degradation. The conditions of existence of perfect placements and the ways of determining resource placement are presented. Examples of different types of placements are given. On the basis of the established forms of the network working structures along with network degradation the average number of working processors with specified order is determined. This value could be a measure of the network's computing capabilities loss along with the increasing degree of network degradation for a given type of resource placement.

Keywords: informatics, resource placement, soft degradation, hypercube networks, distributed database

1. Introduction

A processor network has the logical structure of k-dimensional cube-type if the graph nodes of its topology can be labeled by k-dimensional binary vectors such that the Hamming distance between the vectors of adjacent nodes is equal one. We are considering the case when the network is 4-dimensional.

We assume that the network processors can be divided into working processors and database processors (distributed database). Execution of some tasks by a working processor requires access to database, and also some results obtained by a working processor must be submitted to the database.

In the networks used in embedded systems (typically real-time systems) a processor, which has been permanently faulty, can not be repaired (or replaced), but is excluded from the network (the access to it is being blocked), and the new degraded network continues to operate under the condition that it meets certain requirements (Chudzikiewicz and Zieliński, 2010; Kulesza and

*Submitted: April 2013; Accepted: April 2017

Zieliński, 2010; Zieliński et al., 2011; Zieliński, 2012). Such type of a network is called network with soft (graceful) degradation. Diagnosing of a processor network aims to identify faulty processors and to eliminate them from the network. One of significant problems in the design and exploitation of such a network is skillful resource placement and modification of resource placement after each loss of the network processors.

The generalized cost of a network traffic with specified resource (database) placement and workload of the network is usually analyzed with experimental methods or by means of simulation. In the present paper we try to express this cost analytically for a given perfect placement. The definition of (m, d)-perfect placement (where m is the number of data resource processors attained by any working processor of a network at a distance not greater than d) provides a characteristic value of the generalized cost of information traffic in the network for the given workload. We will find the minimum number of database processors in such way that each working processor has access to at least m database processors attainable at a distance of not more than d. Such an approach is known in relation to the network of regular logical structure (AlBdaiwi and Bose, 2003, 2005; Bae and Bose, 1997; Imani et al., 2010; Moinzadeh et al., 2008). We use the definition perfect distance-d placement of data resources in the processor network (Definition 3) fairly commonly used in reference to torus-type networks (AlBdaiwi and Bose, 2003, 2005; Bae and Bose, 1997), to networks whose logical structures are described by coherent graphs of 4-dimensional cubes, which corresponds to the working structures of the 4-dimensional cube networks with soft degradation (Kulesza and Zieliński, 2011, 2012; Zieliński, 2012).

The main goal of this paper is to obtain a measure of loss of the network computing capabilities with increasing degree of network degradation for a given type of resource placement. We have achieved this goal by determining the average numbers of working processors for some types of perfect resource placements in the 4-dimensional cube type network.

The paper consists of three main sections and conclusions. The second section provides the basic properties of working structures which are induced by the network in the process of its soft degradation. This section contains the definition of the concept of a perfect placement (deployment) of computers with data resources in the working structure of a network G (Definition 3) and of the concept of access characteristics of working computers to a distributed database for a specific placement (Definition 4). Examples of different placement types are provided. The third section considers the conditions of existence of a specific type of perfect placements and the ways of determining them, with determination of the $(1, 1 \,|\, G)$, $(1, 2 \,|\, G)$ and $(2, 1 \,|\, G)$-perfect placements for working structures G (with the distinction of the cyclic and acyclic structures). On the basis of geometric forms of network working structures and their node groups, given in Kulesza and Zieliński (2011), (2012); Zieliński (2012), the average number of working processors in a network of order of at least six for selected resource placements is determined. This value could constitute a measure of loss of network computing capabilities with increasing degree of network degradation for a given type of resource placement.

In the summary, some conclusions from the results presented in the paper are formulated.

2. Basic definitions and properties

Definition 1 *The logical structure of computer network is called the structure of 4-dimensional cube if it is described by coherent ordinary graph $G = \langle E,\ U\rangle$ (E is the set of processors, U is the set of bidirectional data transmission lines), whose nodes can be labelled by 4-dimensional binary vectors without repetitions in such a way that*

$$[H\left(\varepsilon\left(e'\right),\ \varepsilon\left(e''\right)\right) = 1] \Leftrightarrow [(e',\ e'') \in U] \tag{1}$$

where $H\left(\varepsilon\left(e'\right), \varepsilon\left(e''\right)\right)$ is Hamming distance between the labels of nodes e' and e''.

If $|E| = 2^4$ and $|U| = 2\,|E|$, then the graph of type of 4-dimensional cube is denoted by H^4, and is called the non-labeled 4-dimensional cube.

In the process of network degradation some of the nodes are excluded from the network due to diagnosis of arising faults and the subsequent network reconfiguration. In the process of reconfiguration such a network structure that meets the diagnosability conditions of the network is determined. This new network structure will be called working structure if it possesses the ability to identify at least one faulty node. The detailed conditions of the hypercube network diagnosability were highlighted in Zieliński (2012). Let us note that the new working structure (subgraph) must be a coherent graph, which means that there exists a path from any node to all other graph nodes. We distinguish three classes of working structures, namely cyclic graphs, acyclic graphs, and trees, which will be denoted by the C, A, T symbols, respectively.

Let $\tilde{H}^4 = \left\langle H^4; \left\{\varepsilon(e) : e \in E(H^4)\right\}\right\rangle$ be the 4-dimensional cube with labeled nodes, where $E(H^4)$ is the set of nodes of the graph H^4.

We denote by $\& \in \{C,\ A,\ T\}$ the class of working structures and by p its order, i.e. the number of nodes (processors) in the working structure.

Denote by $\breve{G}_p^{\&}\left(H^4\right)$ and $\breve{G}_p^{\&}(\tilde{H}^4)$ the sets of coherent subgraphs of graphs H^4 and $\tilde{H}^4$ of order p, respectively, for a class $\& \in \{C,\ A,\ T\}$.

Let $\nu\left(G\right)$ $\left(G \in \breve{G}_p\left(H^4\right) \text{ for } p \geq 6\right)$ be the number of possible label assignments to the nodes of graph G, according to the formula (1). It can be easily seen that $\nu\left(G\right)$ is the number of subgraphs of graph $\tilde{H}^4$ which are isomorphic with the graph G after removal of labels of the nodes.

Example 1 *The structure G (Fig. 1) is the structure of type $H^4(G \in \breve{G}_6^A(H^4))$ because the nodes of this structure can be labeled in accordance with the formula (1). It is known (Kulesza and Zieliński, 2011, 2012) that $\nu\left(G\right) = 96$.*

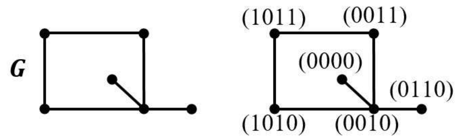

Figure 1. Illustration of verification that the structure of G is a structure of type H^4

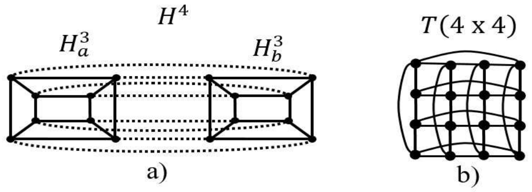

Figure 2. Geometric representations of the cube H^4

The graph G is a regular graph of degree 4 if the degree $\mu(e) = |A(e)|$ (where $A(e)$ is the set of nodes adjacent to a node $e \in E$) of each node of the graph G is equal to 4.

The graph H^4 will be presented as a composition (Kulesza and Zieliński, 2011) of graphs H_a^3 and H_b^3 (Fig. 2a) and as torus of size (4×4) (Fig. 2b, $T\,(4 \times 4)$).

Notice that any cube H^δ $(\delta \geq 2)$ can be represented in the form $T\,(\delta \times \delta)$, but not inversely.

Figure 3 presents the geometric forms of working structures $G \in \breve{G}_p\left(H^4\right)$ of order $p \geq 6$, induced by the network in the process of its degradation, that were presented in Kulesza and Zieliński (2011, 2012) and Zieliński (2012) in the form of a possibly minimum number of intersecting edge lines, which makes it easier to analyze their properties. The numbers $\nu\,(G)$ were determined by the method of structure composition (Zieliński, 2012).

The numbers of sets $\breve{G}_p(\tilde{H}^4)$ and $\breve{G}_p(H^4)$ for $p \geq 6$ with the distinction of the cyclic, acyclic and tree structures are specified in Table 1.

From among numerous properties of working structures of the network we will discuss only properties of a cycle, a tree, including also a simple chain of the highest order.

The cycle will be treated here as a subgraph of graph G and not as a cyclical chain (Korzan, 1978; Kulikowski, 1986).

Definition 2 *A cycle C in the graph G is called such a coherent subgraph of G that* $\forall_{e \in E(C)} : \mu\,(e\,|\,C) = 2$.

Lemma 1 $\forall_{C \in C(G)} : |E\,(C)| \in \{4, 6, 8\}$ $(C\,(G)$ - *the set of cycles in the graph* $G)$.

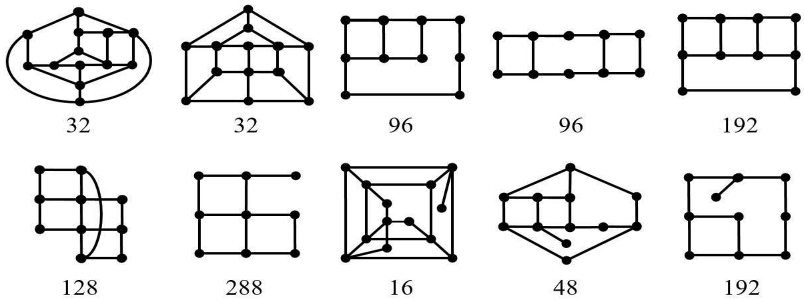

Figure 3. Examples of geometric forms of working structures of the network (the numbers $\nu\,(G)$ are specified)

p	16	15	14	13	12	11	10	9	8	7	6
$\lvert\breve{G}_p(\widetilde{H}^4)\rvert$	1	16	120	528	1412	2912	5144	7744	4920	3464	1568
$\lvert\breve{G}_p(H^4)\rvert$	1	1	5	9	19	37	40	36	31	16	9
$\lvert\breve{G}_p^C(\widetilde{H}^4)\rvert$	1	16	120	512	1136	1776	1824	1392	688	112	128
$\lvert\breve{G}_p^C(H^4)\rvert$	1	1	5	8	13	13	14	7	6	2	2
$\lvert\breve{G}_p^A(\widetilde{H}^4)\rvert$	0	0	0	16	276	1136	3320	6352	4232	3352	1440
$\lvert\breve{G}_p^A(H^4)\rvert$	0	0	0	1	6	24	26	29	25	14	7
$\lvert\breve{G}_p^D(\widetilde{H}^4)\rvert$	0	0	0	0	0	0	0	240	632	1384	720
$\lvert\breve{G}_p^D(H^4)\rvert$	0	0	0	0	0	0	0	2	4	5	3

Table 1. The numbers of sets of working structures $\breve{G}_p(\tilde{H}^4)$ and $\breve{G}_p(H^4)$

PROOF. The graph C is a graph of class H^4 and thus (in accordance with Definition 2)

$$\forall_{e\in E(C)} : [\{e',\ e''\} = A(e)] \Rightarrow [(H\,(\varepsilon\,(e)\,,\ \varepsilon\,(e')) = H\,(\varepsilon\,(e)\,,\ \varepsilon\,(e'')) = 1) \wedge$$

$$\wedge \left(\forall_{e'''\in\{E(C)\setminus\{A(e)\cup\{e\}\}\}} : H\,(\varepsilon(e),\ \varepsilon(e''')) > 1\right)\Big]$$

what is satisfied only if the chain lengths of the binary cycle labels are equal to 4, 6 or 8 (there is no need to justify that). ■

The examples of cycles existing in the H^4 are shown in Fig. 4.

The placement of data resources in the 4-dimensional cube type network with soft degradation will be regarded here as a labeled graph $\left\langle G; \dot{E}\right\rangle$ where $G \in \breve{G}_p\left(H^4\right)$ and $\dot{E} \subset E\,(G)$ ($\dot{E}$ is the set of database processors, $\left\{E\,(G)\setminus\dot{E}\right\}$ is the set of the working processors of the network G).

Let $d\,(e',\ e''\,|\,G)$ be the distance between nodes e' and e'' in a coherent graph G, that is - the length of the shortest chain (in the graph G) connecting node e' with the node e''.

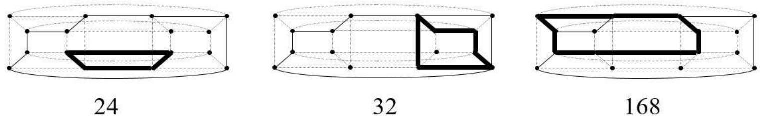

Figure 4. Examples of cycles in the graph H^4 (with given numbers $\nu(C)$ of cycles of specified order)

Definition 3 *We say (AlBdaiwi and Bose, 2003, 2005; Bae and Bose, 1997; Imani et al., 2010; Moinzadeh et al., 2008) that the labeled graph* $\left\langle G; \dot{E} \right\rangle$ $(\left|\dot{E}\right| \geq 1)$ *is the* $(m, d \,|\, G)$ ***-perfect placement*** $(m \in \{1, ..., \mu(G)\},\ d \in \{1, ..., D(G)\}$, *where* $D(G)$ *is the diameter of the graph* G), $\dot{E}$ *is the set of database processors in the network* G, *if there exists a set* $\dot{E}$ *of minimum cardinality such that*

$$\left[\forall_{e \in \left\{E(G) \setminus \dot{E}\right\}} : \left|\left\{e' \in \dot{E} : d\left(e,\ e'|G\right) \leq d\right\}\right| \geq m\right] \wedge$$

$$\wedge \left[\forall_{\{e^*,\ e^{**}\} \subset \dot{E}} : d\left(e^*,\ e^{**}|G\right) > d\right] \wedge \ \left[\left(\mu\left(e''|G\right) = 1\right) \Rightarrow \left(e'' \notin \dot{E}\right)\right]. \quad (2)$$

The $(m, d \,|\, G)$-perfect placement will be denoted by $(m, d \,|\, G)\, d$ in order to distinguish it from the placement of type $(m, d \,|\, G)$, in which the set $\dot{E}$ does not need to be of minimum cardinality.

Note that (according to Definition 3) a set $\dot{E}$ in placement $(m, d \,|\, G)$ is a particular kind of an externally stable set, i.e. a set of vertices no two of which are adjacent and the subgraph $\langle G(E, U) \rangle_{E(G) \setminus \dot{E}}$ is an empty graph.

Denote by $F_{(m,\ d)d}(G)$ the set of $(m, d \,|\, G)$-perfect placements in the network G.

Definition 4 *The vector* $\omega(f) = \left(\omega_1(f), \ldots, \omega_{D(f)}(f)\right)$ *for* $\left(f \in F_{(m,\ d)}(G)\right)$ *where* $\omega_i(f) = \left|\left\{e \in \left\{E(G) \setminus \dot{E}\right\} : \left(\exists_{e' \in \dot{E}} : d\left(e,\ e' \,|\, G\right) = i\right)\right\}\right|$ *is called **access characteristic** (numbers of working processors of network* G *with a given distance to distributed database) for placement* f.

Example 2 *Figure 5 presents the examples of* $\left(m, d \,|\, H^4\right)$*-perfect placements by giving their access characteristics. The network* H^4 *is presented in the form* $H_a^3 \otimes H_b^3$ *as well as in the form of torus* (4×4). *Note that the set* $\dot{E}$ *in* $\left(4, 1 \,|\, H^4\right)$*-perfect placement is a set of internally stable* $(U(\left\langle E\left(H^4\right) \setminus \dot{E} \right\rangle_{H^4}) = \emptyset)$.

Example 3 *Figure 6 presents six examples of* $(m, d \,|\, G)$*-perfect placement in five selected networks* $G \in \check{G}_p\left(H^4\right)$ *for* $p \in \{7,\ 8,\ 9\}$ *by giving their access characteristics. Placements of* f_4 *and* f_5 *are determined for the same network.*

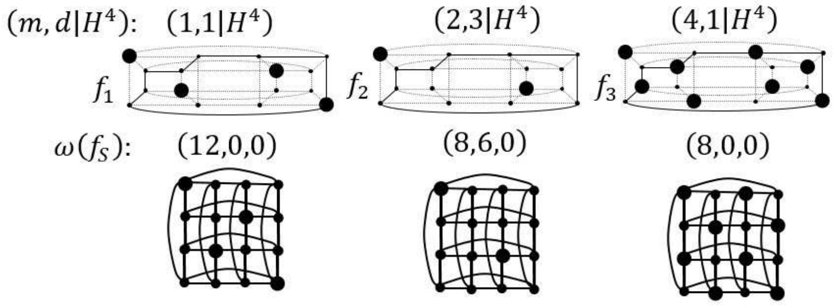

Figure 5. Examples of the $\left(m,d\,|\,H^4\right)$-perfect placement

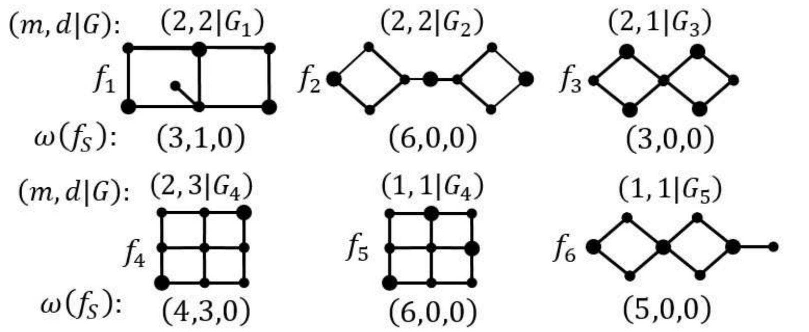

Figure 6. Examples of the $(m,d\,|\,G)$-perfect placements of database in five selected networks

We are interested in the conditions of existence of $(1,1\,|\,G)$, $(1,2\,|\,G)$ and $(2,1\,|\,G)$ perfect placements, the algorithms for their determination and the average number $R_{(m,d\,|\,G)d}\,(p)$ of working processors of the network of order p with $(m,d\,|\,G)$-*perfect placement* of data resources.

3. Perfect placements in the network working structures

Let $F_{(m,\ d)}\,(G)$ be the set of $(m,d\,|\,G)$-perfect placements in the network G and $\dot{E}\,(f)$, $\dot{E}_{(m,\ d)}\,(G)$ - the sets of database computers of the network G for the placement $f \in F_{(m,\ d)}\,(G)$.

Consider the algorithm for determining perfect placements $(1,1\,|\,G)$, $(1,2\,|\,G)$ and $(2,1\,|\,G)$ of $G \in \breve{G}_p\left(H^4\right)$ and we determine (for these placements) the average number

$$\Re_{(\,m,d|G)d}\,(p) = \left|\,\breve{G}_p(H^4)\,\right|^{-1} \sum_{G\in\breve{G}_p(H^4)} \nu\,(G)\,(p - \left|\dot{E}_{(m,d)d}\,(G)\right|) \qquad (3)$$

of working processors of the network of order p with the $(m,d\,|\,G)$-perfect placement of data resources (with distinction of cyclic and acyclic structures).

The average number $R_{(m,d\,|\,G)d}\,(p)$ characterizes the loss of the network computing potential (the number of working processors) along with the increase of its degradation degree for the specified $(m,d\,|\,G)$-perfect placement.

Denote by

$$E^{(d)}\,(e\,|\,G) = \{e' \in E\,(G) : d\,(e, e'\,|\,G) \leq d\}$$

for $d \in \{1, \ldots, D\,(G)\}$ and by

$$\hat{E}^1\,(G) = \left\{e \in E\,(G) : \left(\exists_{e' \in A(e)} : \mu\,(e') = 1\right)\right\}.$$

Note that

$$\left[f \in F_{(1,1)}\,(C)\right] \Longrightarrow \left[\left|\dot{E}\,(f)\right| = \left\lceil 3^{-1}\,|E\,(C)|\right\rceil\right]$$

for $(C \in C\,(G))$, and cyclic sequences $\lambda_{(1,1)}\,(|E\,(C)|)$ of numbers $d\,(e', e''\,|\,C)$ for $(\{e', e''\} \subseteq \dot{E}\,(f))$ have the form of $\lambda_{(1,1)}\,(4) = (2, 2)$, $\lambda_{(1,1)}\,(6) = (3, 3)$ and $\lambda_{(1,\,1)}\,(8) = (3,\ 3,\ 2)$. Here, $\lambda_{(1,\,1)}\,(|E\,(C)|)$ is the sequence of division of the number $|E\,(C)|$.

Lemma 2 $\left[F_{(1,\,1)}\,(G) = \emptyset\right] \Leftrightarrow \left[\exists_{\{e', e''\} \subset E(G) : \mu(e') = \mu(e'') = 1} : \left(A\,(e')\,, A\,(e'')\right) \in U\,(G)\right]$.

Proof. According to formula (2), the nodes adjacent to node e' and node e'' must belong to the set $\dot{E}$, which contradicts it being an externally stable set. ■

The rules of determining the $(1,\ 1|\,G)$-perfect placement are as follows:

1. If G is a cyclic structure $(G \in \breve{G}^C\left(H^4\right))$, then as the first node of the set $\dot{E}\,(f)$ for $\left(f \in F_{(1,\,1)d}\,(G)\right)$ we choose such a node $e_{i_1} \in E\,(G)$, whose degree is the biggest one and the subgraph $\bar{G}^{(1)}\,(G,\ e_{i_1}) =$ $\left\langle\left\{E\,(G) \setminus E^{(1)}\,(e_{i_1})\right\}\right\rangle_G$ has the smallest number of consistent components. Then, we determine a placement for every one of these consistent components, wherein if a consistent component is composed of a single node, then it belongs to the set $\dot{E}\,(f)$.

2. If $G \in \breve{G}^C\left(H^4\right)$, then we determine the subgraph of G, which is a cycle (in the sense of Definition (2)) C of the highest order and thus we choose the nodes of the set $\dot{E}\,(C)$ so that they form a cyclic sequence $\lambda_{\,(1,1)}\,(|E\,(C)|)$ and the expression $\sum\limits_{e \in \dot{E}(C)} \mu\,(e|\,G)$ reaches the maximum value. If $\exists_{e' \in \{E(G) \setminus E(C)\}} : \left\{E\,(e'|\,G) \cap \dot{E}\,(C)\right\} = \emptyset$ then $e' \in \dot{E}\,(G)$.

3. If $\left[G \in \breve{G}^A\left(H^4\right)\right] \wedge \left[\neg\exists_{\{e', e''\} \subseteq \hat{E}^1(G)} : (e', e'') \in U\,(G)\right]$, then we assume that $\hat{E}^1\,(G) \subset \dot{E}\,(f)$ and we determine $\dot{E}\,(f')$ for $\left(f' \in F_{(1,\,1)}\,(G'')\right)$,

where $G'' = \left\langle E(G) \setminus \left\{ \bigcup_{e' \in \hat{E}^1(G)} E^{(1)}(e') \right\} \right\rangle_G$ wherein each single-node component (which can be treated as a consistent component) of the graph G'' belongs to the set $\dot{E}(f)$.

Example 4 *In accordance with the above rules we determine the* $(1,\ 1|\,G)$-*perfect placements for the networks* $G_1 \in \breve{G}_{11}^A(H^4)$, $G_2 \in \breve{G}_9^C(H^4)$, *and* $G_3 \in \breve{G}_{12}^C(H^4)$ *(Fig. 7).*

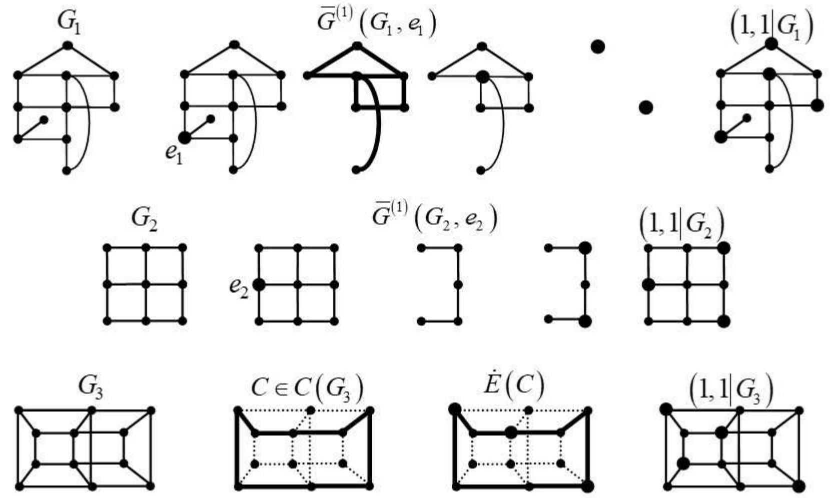

Figure 7. Illustration of the determination procedures of $(1,\ 1|\,G)$-perfect placement for networks $G_1 \in \breve{G}_{11}^A(H^4)$, $G_2 \in \breve{G}_9^C(H^4)$ and $G_3 \in \breve{G}_{12}^C(H^4)$

According to the adapted rules the $(1,\ 1|\,G)$-perfect placements were determined for 192 network working structures (these placements do not exist for 12 structures), ten of which are shown in Fig. 8 and in Table 2. Then, Fig. 9 shows the average numbers $\Re_{(1,1|G)d}(p)$ of working processors, depending on the order of structures (p) with the distinction of cyclic and acyclic structures.

Now, let us denote by $d(e|\,G)$ the biggest distance from the node $e \in E(G)$ to another node of the set $E(G)$, while $r(G)$ and $D(G)$ (respectively) denote the radius and the diameter of the graph G, i.e., $r(G) = \min\{d(e|\,G) : e \in E(G)\}$ and $D(G) = \max\{d(e',\ e''|\,G) : \{e', e''\} \subset E(G)\}$.

It is known that $D(G) \leq 2\,r(G)$.

If $d(e|\,G) = r(G)$, then the node e is called the central node of the network G.

Let us denote $E^C(G) = \{e \in E(G) : d(e|\,G) = r(G)\}$.

Example 5 *The network* $G \in \breve{G}_9^A(H^4)$ *(see Fig. 10) has the radius* $r(G) = 3$ *and three central nodes.*

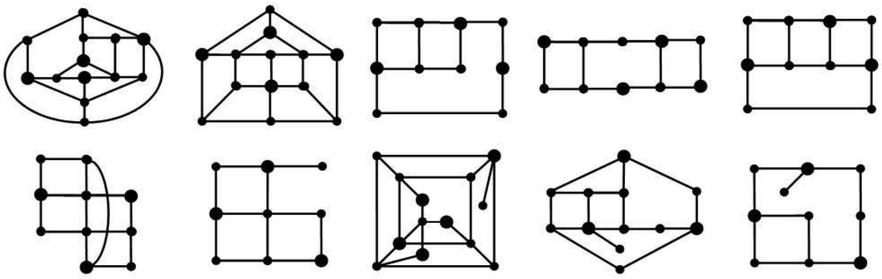

Figure 8. Examples of $(1,\ 1|\, G)$-perfect placements for a selected network $G \in \breve{G}_p\left(H^4\right)$

p	16	15	14	13	12	11	10	9	8	7	6
$\Re_{(1,1\vert G)d}(p)$	12	11	10	9.09	8.56	7.74	6.6	5.91	5.12	4.54	4
% of existing placements	100	100	100	100	99	100	97	94	93	95	90
$\Re^{C}_{(1,1\vert G)d}(p)$	12	11	10	9.19	8.46	8.17	6.92	6	5	4.57	4
$\Re^{A}_{(1,1\vert G)d}(p)$	-	-	-	9	8.65	7.3	6.27	5.82	5.23	4.5	4
% of existing placements	-	-	-	100	98	100	94	88	85	90	80

Table 2. The average numbers of working processors for $(1,\ 1|\, G)$-perfect placement in the network $G \in \breve{G}_p\left(H^4\right)$

Note that $D\left(H^4\right) = r\left(H^4\right) = 4$ and $\left|F_{(1,\,2)d}\left(\tilde{H}^4\right)\right| = 8$.

Obviously $[r\,(G) = 2] \Rightarrow \left[\,\left|\dot{E}_{(1,\,2)d}\,(G)\right| = 1\right]$.

One can ascertain that $\left|\left\{G \in \breve{G}_p\left(H^4\right) : r\,(G) = 2\right\}\right| = 23$, wherein a numerical series, defining the number of structures whose $(1,\ 2|\, G)$-perfect placement is a placement with a central database has the form $6x^6 + 7x^7 + 6x^8 + 4x^9$.

Note that $\left[f \in F_{(1,2)d}\,(C)\right] \Rightarrow \left[\left|\dot{E}\,(f)\right| = 1 + \nabla\left(|E\,(C)| - 4\right)\right]$ for $((a > 0) \Rightarrow (\nabla\,(a) = 1)$ and $(a \leq 0) \Rightarrow (\nabla\,(a) = 0))$, where $a = |E\,(C)| - 4$, and ∇a is bivalent function. The cyclic strings $\lambda_{(1,\,2)}\,(|E\,(C)|)$ of numbers $d\,(e',\ e''|\, C)$ for $\left(\{e', e''\} \subseteq \dot{E}\,(f)\right)$ have the form $\lambda_{(1,\,2)}\,(4) = (4)$, $\lambda_{(1,\,2)}\,(6) = (3,\ 3)$ and $\lambda_{(1,\,2)}\,(8) = \{(5,\ 3)\,, (4,\ 4)\}$.

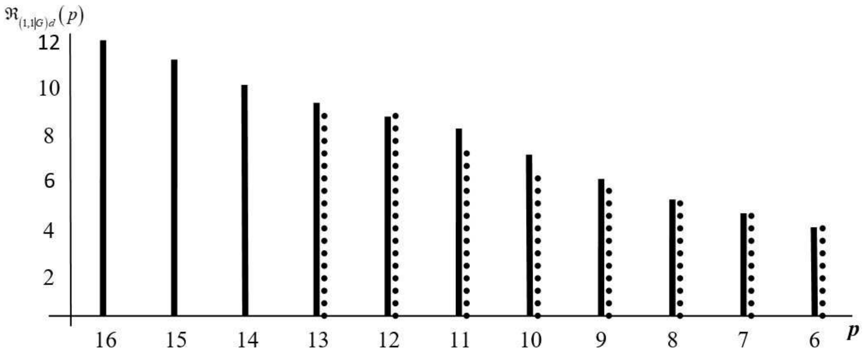

Figure 9. The average numbers of working processors in the network $G \in \breve{G}_p\left(H^4\right)$ with $(1,\ 1|\,G)$-perfect placement of database processors (indication: solid line for cyclic network; dotted line for acyclic network)

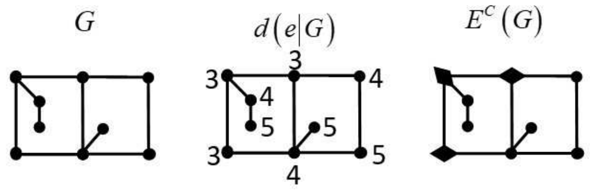

Figure 10. Illustration of determination of the radius $r(G)$ of the network $G \in \breve{G}_9^A\left(H^4\right)$

The rules of determining the $(1,\ 2|\,G)$-perfect placement are as follows:

1. If G is a cyclic structure $(G \in \breve{G}^C\left(H^4\right))$, as the first node of a set $\dot{E}(f)$ for $\left(f \in F_{(1,\ 2)}(G)\right)$ we choose such a node $e_{i_1} \in E(G)$, whose degree is the biggest one and the subgraph $\bar{G}^{(2)}(G,\ e_{i_1}) = \left\langle\left\{E(G) \setminus E^{(1)}(e_{i_1})\right\}\right\rangle_G$ has the smallest number of consistent components. Then, we determine a placement for every one of these consistent components, wherein if a consistent component is composed of a single node, then it belongs to the set $\dot{E}(f)$.

2. If $G \in \breve{G}^C\left(H^4\right)$, then we determine a partial subgraph of graph G, which is a cycle (in the sense of Definition 2) C of the highest order and we choose the nodes of the set $\dot{E}(C)$ in such a way that they form a cyclic sequence of $\lambda_{(1,2)}(|E(C)|)$ and the expression $\sum\limits_{e \in \dot{E}(C)} \mu(e|\,G)$ reaches the maximum value. If $\exists_{e' \in \{E(G) \setminus E(C)\}} : \left\{E(e'|\,G) \cap \dot{E}(C)\right\} = \emptyset$, then $e' \in \dot{E}(G)$.

3. If $\left[G \in \breve{G}^A\left(H^4\right)\right]$, then we assume that $\hat{E}^1(G) \subset \dot{E}(f)$ and we determine

$\dot{E}\,(f')$ for $\left(f' \in F_{(1,2)}\,(G'')\right)$, where $G'' = \left\langle E(G) \backslash \{ \bigcup\limits_{e' \in \hat{E}^1(G)} E^{(1)}\,(e')\} \right\rangle_G$,
wherein the single-node consistent component of the graph G'' belongs to the set $\dot{E}\,(f)$.

Example 6 *In accordance with the above rules we determine the* $(1,\ 2|\,G)$ *- perfect placement for the networks* $G_1 \in \breve{G}^C_{13}\left(H^4\right), G_2 \in \breve{G}^A_{12}\left(H^4\right)$*, and* $G_3 \in \breve{G}^C_{10}\left(H^4\right)$ *(see Fig. 11).*

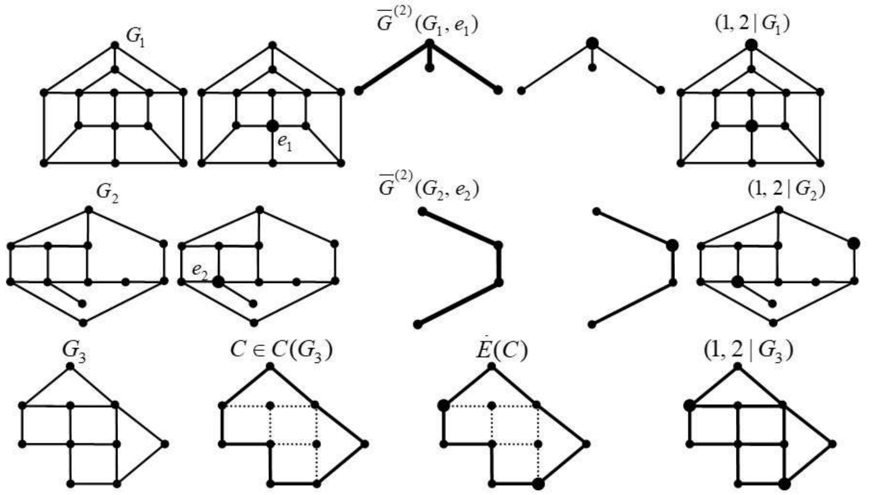

Figure 11. Illustration of the determination procedures of $(1,\ 2|\,G)$-perfect placement for networks $G_1 \in \breve{G}^C_{13}(H^4)$, $G_2 \in \breve{G}^A_{12}(H^4)$, and $G_3 \in \breve{G}^C_{10}(H^4)$

According to the established rules, the $(1,\ 2|\,G)$-perfect placements were determined for 204 network working structures, ten of which are shown in Fig. 12 and in Table 3. Then, Fig. 13 provides the illustration for the average numbers $\Re_{(\,1,\,2\,|G)d}\,(p)$ of working processors of the structure depending on its order p with distinction of cyclic and acyclic structures.

Lemma 3 $\left[G \in \breve{G}^A\left(H^4\right)\right] \Rightarrow \left[F_{(2,\,1)}\,(G) = \emptyset\right]$.

Proof. $\exists_{e \in E(G)}\ :\ \mu\,(e) = 1$ and according to Definition 1 $(\mu\,(e) = 1) \Rightarrow \left(e \notin \dot{E}_{(m,d)}\,(G)\right)$, and thus $|A\,(e)| < 2$, which contradicts the definition of $(2,\ 1|\,G)$ -perfect placement. ■

Note that $[f \in F_{(2,1)d}(C)] \Rightarrow [\,|\dot{E}(f)| = 2^{-1}|E(C)|\,]$ $(C \in C(G))$, and the cyclic sequences $\lambda_{(2,\,1)}(\,|E(C)|\,)$ of numbers $d(e', e''|C\,)$ $(\{e', e''\} \subseteq \dot{E}(f), f \in F_{(2,\,1)d}(C))$ have the form of $\lambda_{(2,\,1)}(\,4) = (2,\ 2)$, $\lambda_{(2,\,1)}(\,6) = (2,\ 2,\ 2)$, and $\lambda_{(2,\,1)}(\,8) = (2,\ 2,\ 2,\ 2)$.

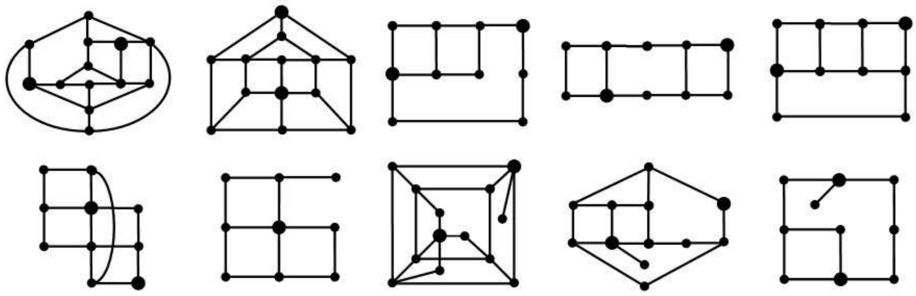

Figure 12. Examples of $(1,\ 2|\,G)$-perfect placements for a selected network $G \in \breve{G}_p\left(H^4\right)$

p	16	15	14	13	12	11	10	9	8	7	6
$\Re_{(1,2\|G)d}(p)$	14	13	12	11	10	8.98	8	7.11	6.27	5.28	4.74
$\Re^{C}_{(1,2\|G)d}(p)$	14	13	12	11	10	9	8	7.17	6.36	5.43	4.75
$\Re^{A}_{(1,2\|G)d}(p)$	-	-	-	11	10	8.96	8	7.05	6.17	5.12	4.73

Table 3. The average numbers of working processors for $(1,\ 2|\,G)$-perfect placement in the network $G \in \breve{G}_p\left(H^4\right)$

Property 1 *The coherent cyclic graph has a kernel if each cycle in this graph is of evennumbered order (Richardson, 1953).*

Property 2 *If the cyclic network G has a kernel then $F_{(2,1)}(G) \neq \emptyset$ because $\forall_{e\in\{E(G)\setminus\dot{E}\}} : |A(e)| \geq 2$.*

From Properties 1 and 2, the following property results directly:

Property 3 $[G \in \breve{G}^C(H^4)] \Rightarrow [(F_{(2,\,1)}(G) \neq \emptyset)]$ *because each cycle in the graph G is of the even-numbered order (Lemma 1).*

Determining the placement $f \in F_{(2,1)d}(G)$ for $\left(G \in \breve{G}^C\left(H^4\right)\right)$ is therefore relatively simple and involves the determination of the set $E^* \subset E(G)$ that is internally and externally stable and assuming that the set $\dot{E}(f)$ is of lesser cardinality than the sets E^* and $E(G)\setminus E^*$.

The set E^* can be determined in many different ways. One of them (applied in this paper) is coloring the nodes of the graph G using two colors in such a way that the nodes of the same color are not adjacent.

Figure 14 shows $(2,\ 1|\,G)$-perfect placements for five selected (out of 204) networks of the set $\breve{G}^C_p\left(H^4\right)$ for $p \geq 6$, and Table 4 and Fig. 15 present the

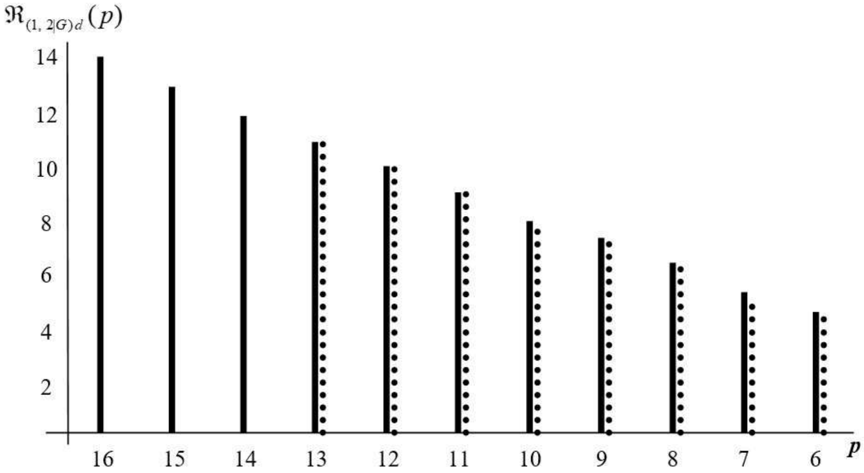

Figure 13. The average numbers of working processors in the network $G \in \breve{G}_p\left(H^4\right)$ with $(1,\ 2|\,G)$-perfect placement of database processors (indication: solid line for cyclic network; dotted line for acyclic network)

determined average numbers $\Re^C_{(2,1|G)d}\,(p)$ of working computers of the network of order p with $(2,\ 1|\,G)$-perfect placement of data resources.

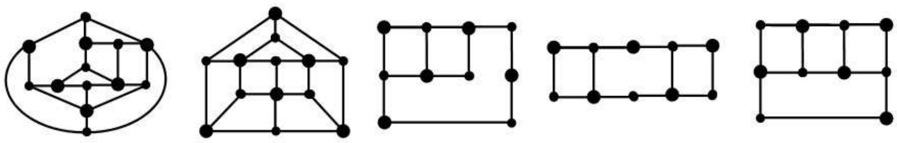

Figure 14. Examples of $(2,\ 1|\,G)$-perfect placements for selected networks $G \in \breve{G}_p\left(H^4\right)$

4. Conclusions

We considered the conditions of existence of certain types of perfect placements and the ways to determine them, and determined $(1,\ 1|\,G), (1,\ 2|\,G)$ and $(2,\ 1|\,G)$-perfect placements for the working structures of the 4-dimensional cube type network with soft degradation for orders equal at least 6. This allowed us (with the knowledge of geometrical forms of the working structures) to determine the average number $\Re_{(\,m,\ d|G)d}\,(p)$ $\big((m,d|G) \in \{(1,\,1|G)\,d, (1,\,2|G)\,d, (2,\,1|G)\,d\}\big)$ of the working processors of the network with the order p for a specific data resource placement (see Fig. 16).

The generalized cost of information traffic in a network for a given placement of database computers depends on the nature of the tasks performed by the

p	16	15	14	13	12	11	10	9	8	7	6
$\Re^{C}_{(2,1\|G)d}(p)$	8	8	7.4	7.1	6.36	6.06	5.3	5	4	4	3

Table 4. The average numbers of working computers for $(2,\ 1|\,G)$-perfect placement in the network $G \in \breve{G}_p\left(H^4\right)$

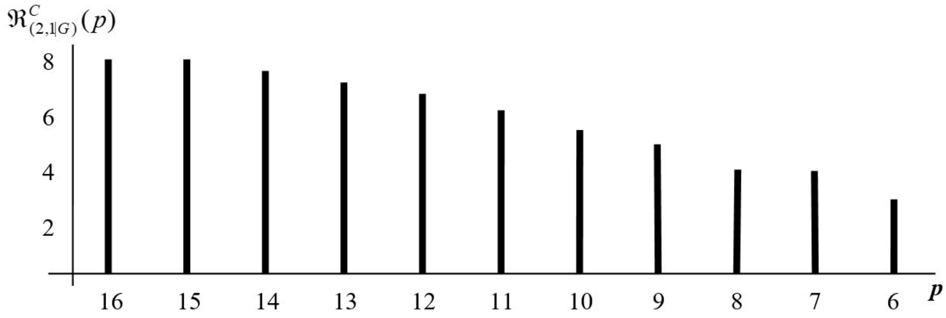

Figure 15. The average numbers of working computers in the network $G \in \breve{G}^C_p\left(H^4\right)$ with $(2,\ 1|\,G)$-perfect placement of database computers (for acyclic networks such placement does not exist)

network. Such an analysis is not the subject of this paper. This is a separate problem, which can be examined with the use of simulation methods for a specified (m, d)-perfect placement and a particular type of task load of the network.

References

Albdaiwi, B. F. and Bose, B. (2003) On resource placements in 3D tori. *Journal of Parallel and Distribiuted Computing*, 63, 838–845.

Albdaiwi, B. F. and Bose, B. (2005) Quasi-perfect resource placements for two-dimensional toroidal networks. *Journal of Parallel Distributed Computing*, 65, 815–831.

Bae, M. M. and Bose, B. (1997) Resource Placement in Torus-Based Networks. *IEEE Transactions on Computers*, **46** (10), 1083–1092.

Chudzikiewicz, J. (2002) *Sieci komputerowe o strukturze logicznej typu hipersześcianu (Computer networks having logical structure of a hypercube*; in Polish. Edition I, Instytut Automatyki i Robotyki, Wydział Cybernetyki, WAT, Warszawa.

Chudzikiewicz, J. and Zieliński Z. (2010) Reconfiguration of a processor cube-type network. *Przegląd Elektrotechniczny (Electrical Review)*, **86** (9), 149–153.

Imani, N., Sarbazi-Azad, H. and Zomaya A.Y. (2010) Resource placement

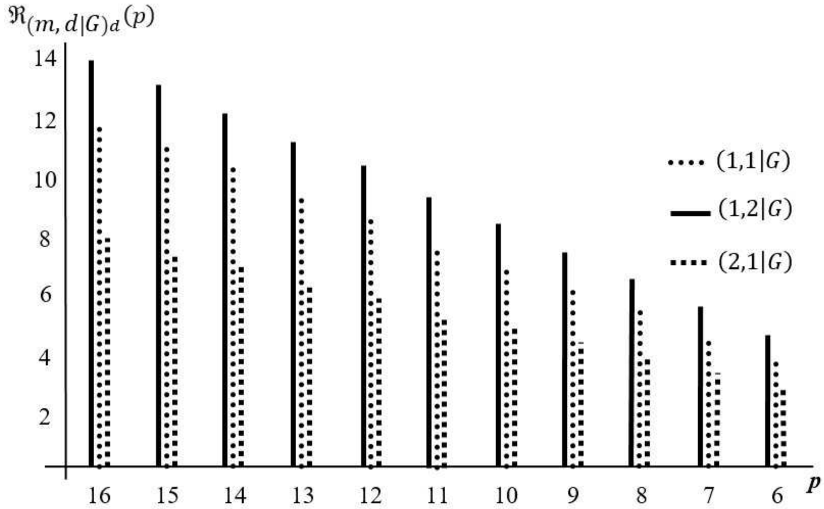

Figure 16. The average number of working processors in the network $G \in \breve{G}_p\left(H^4\right)$ for the selected $(m, d \,|\, G)$-perfect placements of database processors

in Cartesian product of networks. *Journal of Parallel and Distribiuted Computing* 70, 481–495.

Moinzadeh, P., Sarbazi-Azad, H. and Yazdani, N. (2008) Resource Placement in Cube-Connected Cycles. *The International Symposium on Parallel Architectures, Algorithms, and Networks.* IEEE Computer Society, 83–89.

Korzan, B. (1978) *Elementy teorii grafów i sieci. Metody i zastosowania (Elements of the theory of graphs and networks. Methods and Applications; in Polish).* Edition I. Wydawnictwo Naukowo-Techniczne, Warszawa.

Kulesza, R. (2000) *Podstawy diagnostyki sieci logicznych i komputerowych (Foundations of diagnostics of the logical and computer networks; in Polish)*, Edition II. Instytut Automatyki i Robotyki, Wydział Cybernetyki, WAT, Warszawa.

Kulesza, R. and Zieliński, Z. (2010) The life period of the hypercube processors' network diagnosed with the use of the comparison method. *Monographs of System Dependability - Technical Approach to Dependability.* Oficyna Wydawnicza Politechniki Wrocławskiej, 65–78.

Kulesza, R. and Zieliński, Z. (2011) Metoda generowania struktur logicznych sieci procesorów o łagodnej degradacji typu 4-wymiarowego sześcianu. (The method of generating the logical structures of the processor networks of the 4-dimensional cube type with soft degradation; in Polish) *Biuletyn WAT,* **60** (4), 345–358.

Kulesza, R. and Zieliński, Z. (2012) Wyznaczanie struktur logicznych sieci procesorów o łagodnej degradacji i strukturze 4-wymiarowego hipersześcianu (Determination of the logical structure of the processor networks with soft degradation and 4-dimensional hypercube structure; in Polish). *Biuletyn WAT,* **61** (4), 293–306.

KULIKOWSKI, J. (1986) *Zarys teorii grafów (An outline of graph theory*; in Polish). Wydawnictwo Naukowo Techniczne, Warszawa.

RICHARDSON, M. (1953) Solutions of Irreflexive Relations. *Annals of Mathematics Second Series,* **58** (2), November, 573–590.

ZIELIŃSKI, Z., STRZELECKI, Ł. and KULESZA, R. (2011) Diagnosability characterization of the 4dimensional cube type soft degradable processors' network. *Monographs On System Dependability - Problems of Dependability and Modeling.* Oficyna Wydawnicza Politechniki, 283–296.

ZIELIŃSKI, Z. (2012) *Podstawy diagnostyki systemowej sieci procesorów o łagodnej degradacji i strukturze hiperszęścianu (Foundations for the systemic diagnostics of the processor networks with soft degradation and hypercube structure*; in Polish). Edition I, Wojskowa Akademia Techniczna, Warszawa.