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Continuous time non-smooth optimization through quasi efficiency*

by

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Abstract: The importance of quasi efficiency lies in its versatile nature as it permits a definite tolerable error that depend on the decision variables. This has been a motivating factor for us to introduce the notion of quasi efficient solution for the non-smooth multiobjective continuous time programming problem. Necessary optimality conditions are derived for this problem. To derive sufficient optimality conditions, the concept of approximate convexity has been extended to continuous case in this paper. A mixed dual is proposed for which weak and strong duality results are proved.

Keywords: continuous time problem, quasi efficiency, approximate convexity, optimality conditions, duality

1. Introduction

Continuous-time linear programming problem was introduced by Bellman (1953) and it was extended to the continuous-time nonlinear programming problem by Farr and Hanson (1974), Reiland and Hanson (1980) and Reiland (1980). Many researchers (see, in particular, Brandao, Rojas-Medar and Silva, 1998, 2001; Zalmai, Nobakhtian and Pouryayevali, 2008a,b) studied this problem through optimality conditions and duality results. While browsing through the literature one may come across several variants of solution concepts for multiobjective continuous-time programming problems such as weak efficiency / efficiency / proper efficiency. But in most of such problems the decision maker usually does not obtain an exact solution. It is evident that for the real world problems, generally the optimization algorithms terminate in finite number of steps and give

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only approximate solutions, hence it is theoretically, as well as computationally, useful to study approximate solutions. The concept of approximate solution can also be considered as an adequate compromise with a given prescribed error.

Loridan (1982) explored necessary conditions in mathematical programming with errors by introducing the concept of regular approximate solutions up to a certain degree of precision. These results were then extended to vector optimization in Loridan (1984). Golestani, Sadeghi and Tavan (2018) derived the characterization of quasi-efficient points in multiobjective problems with locally Lipschitz functions, using generalized Stampacchia vector variational inequalities. Upadhyay, Stancu-Minasian and Mishra (2013) utilized the characterization of an approximate convex function through its convexificator to elucidate the connections between solutions of Stampacchia-type vector variational inequality problems in relation to convexificator and quasi-efficient solutions of nonsmooth vector optimization problems involving locally Lipschitz functions. They also extend the results to internal valued programming problem in Upadhyay et al. (2019). Recent years have seen an increasing interest in quasi efficiency for the vector optimization problem (Bhatia, Gupta and Arora, 2013; Chuong and Kim, 2016; Gupta, Mehra and Bhatia, 2006; and Mishra and Upadhyay, 2013). The issues addressed in all these articles are static, but in the real world, a dynamic approach is more crucial. The essence of quasi efficiency lies in its versatile nature as it allows for the presence of some tolerable error that depends on the decision variable. Motivated by versatility of quasi efficiency we extend this notion to the class of multi-objective continuous time problems and utilize this notion to establish the necessary and sufficient optimality conditions. A mixed dual is proposed. Weak and strong duality theorems are proven under the notion of approximate convexity.

The paper is organized as follows. Section 2 is devoted to some basic definitions and preliminaries. In Section 3, necessary and sufficient optimality conditions are established for multi-objective continuous-time problem (P) using quasi efficiency. Mixed type dual of (P) is proposed in Section 4 and duality results are obtained under the assumptions of approximate convexity. Section 5 is devoted to conclusions and future developments of our work.

2. Definitions and preliminaries

Let r, p and n be three positive integers and X be a nonempty open convex subset of the Banach space $L^n_\infty[0,T]$ of all $n-$dimensional vector valued Lebesgue measurable essentially bounded functions, defined on the compact interval $[0,T] \subseteq \mathbb{R}$ with the norm

$$\|x\|_\infty = \max_{1\leq j\leq m} \text{ ess-sup}\,\{|x^j(t)| | 0 \leq t \leq T\},$$

where for each $t \in [0,T],\ x^j(t)$ denotes the j^{th} component of $x(t) \in \mathbb{R}^n$.

Let $t \to f^i(t,x(t)),\ i \in P = \{1,2,\ldots,p\}$ and $t \to g^j(t,x(t)),\ j \in M = \{1,2,\ldots,m\}$ be Lebesgue measurable and integrable functions for all $x \in X$. Also assume that $f^i(t,x(t)),\ i \in P$ and $g^j(t,x(t)),\ j \in M$ are locally Lipschitz on X throughout $[0,T]$. Define

$$g(t,x(t)) = (g^1(t,x(t)), g^2(t,x(t)), \ldots, g^m(t,x(t))) = G(x)(t)$$

and

$$f^i(t,x(t)) = \phi^i(x)(t), i \in P$$

where G is a map from X into the normed space $\Lambda_1^m[0,T]$ of all Lebesgue measurable essentially bounded $p-$dimensional vector functions defined on $[0,T]$ with the norm defined as

$$\|y\|_1 = \max_{1\le i\le p} \int_0^T |y^i(t)|dt,$$

and ϕ^i being a map from X into a normed space $\Lambda_1^1[0,T]$.

For any $x = (x^1,x^2,...,x^n)^T$, $y = (y^1,y^2,...,y^n)^T \in \mathbb{R}^n$, define

(i) $x = y \Leftrightarrow x^i = y^i$ for all $i = 1,2,...,n$.
(ii) $x < y \Leftrightarrow x^i < y^i$ for all $i = 1,2,...,n$.
(iii) $x \leqq y \Leftrightarrow x^i \le y^i$ for all $i = 1,2,...,n$.
(iv) $x \le y \Leftrightarrow x \leqq y$ and $x \ne y$.

$\mathbb{R}^n_+ = \{(x^1,x^2,...,x^n)^T \in \mathbb{R}^n | x^i \ge 0,\ i = 1,2,...,n\}$ and $\operatorname{int}\mathbb{R}^n_+$ denotes the interior of $\mathbb{R}^n_+$, that is, $\operatorname{int}\mathbb{R}^n_+ = \{(x^1,x^2,...,x^n)^T \in \mathbb{R}^n | x^i > 0, i = 1,2,...,n\}$.

Now, consider the Multiobjective Continuous time Problem (P) :

$$(P) \text{ Minimize } \Big(\int_0^T f^1(t,x(t))dt, \int_0^T f^2(t,x(t))dt, \ldots, \int_0^T f^p(t,x(t))dt\Big)$$

subject to

$$g^j(t,x(t)) \le 0,\ a.e.\ t \in [0,T], x \in X, j \in M. \tag{1}$$

Let X_0 be the set of all feasible solution of (P), that is

$$X_0 = \{x \in X | g^j(t,x(t) \le 0,\ a.e.\ t \in [0,T], x \in X, j \in M\}.$$

In the next part of this section we will define the efficient solution and the quasi efficient solution for (P).

Definition 2.1 *A feasible solution $\bar{x} \in X_0$ is said to be an efficient solution for (P) if there is no other $x \in X_0$ such that*

$$\int_0^T f^i(t, x(t))dt \leq \int_0^T f^i(t, \bar{x}(t))dt, \text{ for all } i \in P \text{ and}$$

$$\int_0^T f^j(t, x(t))dt < \int_0^T f^j(t, \bar{x}(t))dt, \text{ for at least one } j \in P.$$

Remark 2.1 *For $p = 1$, the concept of efficient solution for (P) reduces to the concept of optimal solution for single-objective continuous time problem.*

Definition 2.2 *A feasible solution $\bar{x} \in X_0$ is said to be a quasi efficient solution for (P) if there exists $c = (c^1, c^2, \ldots, c^p) \in int\mathbb{R}^p_+$ such that for no other $x \in X_0$*

$$\int_0^T f^i(t, x(t))dt \leq \int_0^T \{f^i(t, \bar{x}(t)) - c^i\|x(t) - \bar{x}(t))\|\}dt, \text{ for all } i \in P$$

and

$$\int_0^T f^j(t, x(t))dt < \int_0^T \{f^j(t, \bar{x}(t)) - c^j\|x(t) - \bar{x}(t))\|\}dt, \text{ for at least one } j \in P.$$

Remark 2.2 *For $p = 1$, the concept of quasi efficient solution for (P) reduces to the concept of quasi optimal solution for single-objective continuous time problem.*

Remark 2.3 *It can easily be proven that an efficient solution for (P) is a quasi efficient solution for (P), whereas the converse may not be true, as this is illustrated by the following example.*

Example 2.1

$$(P1) \text{ Minimize } \Big(\int_0^1 \{|x(t)| - x(t)\}dt, \int_0^1 \{x^2(t) - x(t)\}dt \Big)$$

subject to

$$x(t) \geqq 0, t \in [0, 1].$$

$\bar{x}(t) = 0$, $t \in [0, 1]$ is not an efficient solution for $(P1)$, as there exists a feasible solution $\hat{x}(t) = t(1 - t)$ for (P1) such that

$$\int_0^1 \{|\hat{x}(t)| - \hat{x}(t)\}dt = 0 \leq 0 = \int_0^1 \{|\bar{x}(t)| - \bar{x}(t)\}dt,$$

$$\int_0^1 \{\hat{x}^2(t) - \hat{x}(t)\}dt = \frac{-2}{15} < 0 = \int_0^1 \{\bar{x}^2(t) - \bar{x}(t)\}dt.$$

It is easy to check that $\bar{x}(t) = 0$, $t \in [0, 1]$ is a quasi efficient solution of (P1) for $c = (c^1, c^2) = (1, 1)$.

DEFINITION 2.3 *A feasible solution $\bar{x} \in X_0$ is said to be a locally quasi efficient solution for (P) if there exists $c = (c^1, c^2, \ldots, c^p) \in int\mathbb{R}^p_+$ and a neighborhood U of $\bar{x}$ such that for no other $x \in X_0 \cap U$*

$$\int_0^T f^i(t, x(t))dt \leq \int_0^T \{f^i(t, \bar{x}(t)) - c^i \|x(t) - \bar{x}(t)\|\}dt, \text{ for all } i \in P$$

and

$$\int_0^T f^j(t, x(t))dt < \int_0^T \{f^j(t, \bar{x}(t)) - c^j \|x(t) - \bar{x}(t)\|\}dt, \text{ for at least one } j \in P.$$

REMARK 2.4 *For $p = 1$, the concept of locally quasi efficient solution for (P) reduces to the concept of locally quasi optimal solution for the single-objective continuous time problem.*

The example given below verifies the existence of local quasi efficient solution, whereas the existence of quasi efficient solution has already been illustrated through Example 2.1.

EXAMPLE 2.2

$$(P2) \; Minimize \left(\int_0^1 \{\psi(t, x(t))\}dt \right)$$

subject to

$$x(t) \geqq 0, \; t \in [0, 1],$$

where

$$\psi(t, x(t)) = \begin{cases} x(t) - 2|x(t)| & x(t) \leq 1, \\ -2x(t)(x(t) - 1) & x(t) > 1. \end{cases}$$

Then $\bar{x}(t) = 0$, $t \in [0, 1]$ is not an optimal solution for $(P2)$ as there exist $\hat{x}(t) = \frac{1}{2}, t \in [0, 1]$ such that

$$\int_0^1 \{\psi(t, \hat{x}(t))\}dt = \frac{-1}{2} < 0 = \int_0^1 \{\psi(t, \bar{x}(t))\}dt.$$

It is not a quasi optimal solution for (P2), as for $c > 0$, there exist $\tilde{x}(t) = c + 1$ such that

$$\int_0^1 \psi(t, \hat{x}(t))dt = -2(c+1)c < -c(c+1) = \int_0^1 \{\psi(t, \bar{x}(t)) - c|\hat{x}(t) - \bar{x}(t)|\}dt.$$

But it is a local quasi optimal solution for (P2) as there exist $c = 2$ and $U = (-\epsilon, \epsilon), 0 < \epsilon < 1$ such that for no other $x \in X_0 \cap U$

$$\int_0^1 \psi(t, x(t))dt < \int_0^1 \{\psi(t, \bar{x}(t)) - 2|x(t) - \bar{x}(t)|\}dt.$$

Since the functionals involved in the problem (P) are not differentiable, but rather they are locally Lipschitz, hence the concept of Clarke generalized directional derivative is used here.

Let us recall some basic concepts and tools from non-smooth analysis. Most of the material included here can be found in Clarke (1983).

Let $\phi : [0,T] \times X \to \mathbb{R}$ be Lebesgue measurable and integrable function and $\phi(t,\cdot)$ be locally Lipschitz on X throughout $[0,T]$.

We write $\phi(t,x(t)) = \Phi(x)(t)$, where Φ is map from X into a normed space $\Lambda_1^1[0,T]$.

Recall that for $\bar{x} \in X$ and $v \in L_\infty^n[0,T]$, the continuous Clarke generalized directional derivative of ϕ is defined by

$$\phi^o(t,\bar{x}(t);v(t)) = \Phi^o(\bar{x},v)(t) = \limsup_{y\to\bar{x}}\sup_{s\to 0} \frac{\Phi(y+sv)(t) - \Phi(y)(t)}{s} \ a.e.\ t \in [0,T],$$

$$\partial_c\phi(t,x(t)) = \{\xi : [0,T] \to \mathbb{R}^n | \xi(t)^T v(t) \le \phi^o(t,\bar{x}(t);v(t)), \forall v \in L_\infty^n[0,T]\}.$$

Let $\psi : [0,T] \times X \to \mathbb{R}$ be Lebesgue measurable and integrable function and $\psi(t,\cdot)$ be locally Lipschitz on X throughout $[0,T]$. Let $\lambda \in \mathbb{R}_+$. Note that

$$(\phi+\psi)^o(t,\bar{x}(t);v(t)) \le \phi^o(t,\bar{x}(t);v(t)) + \psi^o(t,\bar{x}(t);v(t)), v \in L_\infty^n[0,T], \quad (2)$$

$$(\lambda\phi)^o(t,\bar{x}(t);v(t)) = \lambda\phi^o(t,\bar{x}(t);v(t)). \quad (3)$$

3. Optimality conditions

Necessary optimality conditions are useful in the development of numerical algorithms for solving certain optimization problems. Further, these conditions are also responsible for the development of duality theory, on which there exists an extensive literature and a substantial use of which (i.e. of duality theory) has been made in theoretical as well as computational applications in many diverse fields.

Theorem 3.1 *(Necessary optimality conditions) Suppose $\bar{x}$ is a quasi efficient solution of (P). Then there exist $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^p) \in \mathbb{R}_+^p$, $c = (c^1, c^2, \ldots, c^p) \in int\mathbb{R}_+^p$ and $\mu^j \in L_\infty^1[0,T], j \in M$ such that*

$$\int_0^T \Big\{ \sum_{i\in P} \lambda^i (f^i)^o(t,\bar{x}(t);v(t)) + \sum_{i\in P} \lambda^i c^i \|v(t)\| + \sum_{j\in M} \mu^j(t)(g^j)^o(t,\bar{x}(t);v(t)) \Big\} dt \ge 0, \quad (4)$$

$$\forall\ v \in L_\infty^n[0,T].$$

$$\mu^j(t)g^j(t,\bar{x}(t)) = 0,\ \ a.e.\, t \in [0,T],\ j \in M. \tag{5}$$

$$\mu^j(t) \geq 0, j \in M,\ (\lambda^1, \dots, \lambda^p, \mu^1(t), \dots, \mu^m(t)) \neq 0,\ \ a.e.\ t \in [0,T]. \tag{6}$$

Proof Since $\bar{x}$ is a quasi efficient solution of (P), there exist $c = (c^1, c^2, \dots, c^p) \in \operatorname{int} \mathbb{R}^p_+$ such that for no other $x \in X_0$

$$\int_0^T f^i(t,x(t))dt \leq \int_0^T \{f^i(t,\bar{x}(t)) - c^i\|x(t) - \bar{x}(t))\|\}dt,\ \text{for all } i \in P$$

and

$$\int_0^T f^j(t,x(t))dt < \int_0^T \{f^j(t,\bar{x}(t)) - c^j\|x(t) - \bar{x}(t))\|\}dt,\ \text{for at least one } j \in P.$$

Fix $k \in P$, then $\bar{x}$ is an optimal solution of the following problem $(P(k))$:

$$(P(k))\ \text{Minimize} \left(\int_0^T \{f^k(t,x(t)) + c^k\|x(t) - \bar{x}(t))\|\}dt\right)$$

subject to

$$g^j(t,x(t)) \leq 0, j \in M,\ a.e.\ t \in [0,T], x \in X.$$

$$\int_0^T \{f^j(t,x(t)) + c^j\|x(t) - \bar{x}(t))\|\}dt \leq \int_0^T \{f^j(t,\bar{x}(t))\}dt, j \in P \setminus \{k\}.$$

Define $h(t,x(t)) = \|x(t) - \bar{x}(t)\|, t \in [0,T], x \in X$. Then it is easy to find

$$h^o(t,\bar{x}(t);v(t)) = \|v(t)\|, v \in L^n_\infty[0,T]. \tag{7}$$

Lemma 4.2 of Brandao, Rojas-Medar and Silva (2001) along with (2), (3) and (7) imply that there exist

$$\lambda = (\lambda^1, \lambda^2, \dots, \lambda^p) \in \mathbb{R}^p_+, \mu^j \in L^1_\infty[0,T], j \in M$$

$$\int_0^T \Big\{\sum_{i\in P} \lambda^i (f^i)^o(t,\bar{x}(t);v(t)) + \sum_{i\in P} \lambda^i c^i \|v(t)\| + \sum_{j\in M} \mu^j(t)(g^j)^o(t,\bar{x}(t);v(t))\Big\}dt \geq 0,$$

for all $v \in L^n_\infty[0,T]$.

$$\mu^j(t)g^j(t,\bar{x}(t)) = 0, a.e.\ t \in [0,T],\ j \in M.$$

$$\mu^j(t) \geq 0, j \in M,\ (\lambda^1, \dots, \lambda^p, \mu^1(t), \dots, \mu^m(t)) \neq 0,\ \ a.e.\, t \in [0,T].$$

Hence the results follow. ■

Definition 3.1 *A feasible solution $\bar{x} \in X_0$ is said to be a quasi normal efficient solution if it is a quasi efficient solution and $\lambda \neq 0$.*

Remark 3.1 *Through the following example one can see that these conditions are only necessary and not sufficient.*

Example 3.1

$$(P3)\ \textit{Minimize}\ \Big(\int_0^1 \{f(t,x(t))\}dt\Big)$$

subject to

$$x(t) \geqq 0, t \in [0,1],$$

where $f(t,x(t)) = -x^2(t)$. Then $\bar{x}(t) = 0, t \in [0,1]$. Note that $f^o(t,\bar{x}(t); v(t)) = 0$. For $\lambda = 1, c = 1$ and $\mu(t) = 0,\ t \in [0,1],\ \bar{x}(t)$ satisfies conditions (3.1), (5) and (6). But it is not quasi efficient for $(P3)$ as for each $c > 0$ there exists $\hat{x}(t) = 2c \in X_0$ such that

$$\int_0^1 f(t,\hat{x}(t))dt = -4c^2 < -2c^2 = \int_0^1 \{f(t,\bar{x}(t)) - c|\hat{x}(t) - \bar{x}(t)|\}dt.$$

To prove sufficient optimality conditions we extend the concept of approximate convexity to a continuous case.

Definition 3.2 *A functional $\bar{\Phi}(x) = \int_0^T \phi(t,x(t))dt$ is said to be approximately convex at $\bar{x} \in X$ if $\forall \alpha > 0$, there is a neighborhood U of $\bar{x}$ such that*

$$\bar{\Phi}(x) - \bar{\Phi}(\bar{x}) \geq \int_0^T \{(\phi)^o(t,\bar{x}(t); x(t) - \bar{x}(t)) - \alpha\|x(t) - \bar{x}(t)\|\}dt,$$
$$\textit{for all } x \in X \cap U,$$

Definition 3.3 *A functional $\bar{\Phi}(x) = \int_0^T \phi(t,x(t))dt$ is said to be strictly approximately convex at $\bar{x} \in X$ if $\forall \alpha > 0$, there is a neighborhood U of $\bar{x}$ such that*

$$\bar{\Phi}(x) - \bar{\Phi}(\bar{x}) > \int_0^T \{(\phi)^o(t,\bar{x}(t); x(t) - \bar{x}(t)) - \alpha\|x(t) - \bar{x}(t)\|\}dt,$$
$$\textit{for all } x \in (X \setminus \{\bar{x}\}) \cap U.$$

Remark 3.2 *A convex functional is an approximately convex functional, but converse need not be true.*

Let

$$\Phi(x) = \int_0^1 \phi(t, x(t))dt, \phi(t, x(t)) = x^3(t), x : [0, 1] \to \mathbb{R}, \bar{x}(t) = 0, t \in [0, 1],$$

$$(\phi)^o(t, x(t); v(t)) = 3x^2(t)v(t).$$

Note that $\Phi(x)$ *is not convex at* $\bar{x}$ *as there exists* $\hat{x}(t) = -1,\ t \in [0, 1]$ *such that*

$$\Phi(\hat{x}) - \Phi(\bar{x}) = -1 < 0 = \int_0^1 \{(\phi)^o(t, \bar{x}(t); \hat{x}(t) - \bar{x}(t))\}dt.$$

But it is approximate convex at $\bar{x}$ *for each* $\alpha > 0$, *since there exists a neighborhood*

$$U = \{y \in X | \|y\|_\infty < \sqrt{\alpha}\}$$

of $\bar{x}$ *such that*

$$\int_0^1 \{x^3(t) + \alpha|x(t)|\}dt = \int_{T_1} \{x^3(t) + \alpha|x(t)|\}dt + \int_{T_2} \{x^3(t) + \alpha|x(t)|\}dt \geq 0 \quad \text{for each } x \in X \cap U,$$

where

$$T_1 = \{t \in [0, 1] | x(t) \geq 0\},$$

$$T_2 = \{t \in [0, 1] | x(t) < 0\}.$$

Theorem 3.2 *(Sufficient optimality conditions) Let* $\bar{x} \in X_0$ *and let there exist* $\lambda = (\lambda^1, \dots, \lambda^p) \in int\mathbb{R}^p_+$, $c = (c^1, c^2, \dots, c^p) \in int\mathbb{R}^p_+$ *and* $\mu^j \in L^1_\infty[0, T], j \in M$ *such that* (3.1), (5) *and* (6) *hold. Let us write*

$$F^i(x) = \int_0^T f^i(t, x(t))dt, i \in P,$$

$$G(x) = \int_0^T \Big\{ \sum_{j \in M} \mu^j(t) g^j(t, x(t)) \Big\} dt.$$

Further, assume that the following conditions hold:

(a) Functional $F^i(x), i \in P$ *is approximately convex at* $\bar{x}$.

(b) Functional $G(x)$ *is approximately convex at* $\bar{x}$.

Then $\bar{x}$ *is a local quasi efficient solution of (P).*

Proof We shall prove the theorem by contradiction. Thus, suppose $\bar{x}$ is not a local quasi efficient solution of (P).

Then, for any $d = (d^1, d^2, \ldots, d^p) \in \operatorname{int} \mathbb{R}^p_+$ and for any neighborhood U of $\bar{x}$, there exist $\hat{x} \in X_0 \cap U$ such that

$$\int_0^T f^i(t, \hat{x}(t))dt \leq \int_0^T \{f^i(t, \bar{x}(t)) - d^i \|\hat{x}(t) - \bar{x}(t))\|\}dt, \quad \text{for all } i \in P$$

and

$$\int_0^T f^j(t, \hat{x}(t))dt < \int_0^T \{f^j(t, \bar{x}(t)) - d^j \|\hat{x}(t) - \bar{x}(t))\|\}dt,$$

for at least one $j \in P$.

By multiplying by λ^i each of the above inequalities and adding them, we get

$$\sum_{i \in P} \lambda^i (F^i(\hat{x}) - F^i(\bar{x})) < - \int_0^T \Big\{ \sum_{i \in P} \lambda^i d^i \|\hat{x}(t) - \bar{x}(t))\| \Big\} dt. \tag{8}$$

Using hypothesis (b), we infer that for any $\beta > 0$, there is a neighborhood U' of $\bar{x}$ such that

$$G(x) - G(\bar{x}) \geq \int_0^T \Big\{ \sum_{j \in M} \mu^j(t) (g^j)^o(t, \bar{x}(t); x(t) - \bar{x}(t)) - \beta \|x(t) - \bar{x}(t)\| \Big\} dt,$$
$$\text{for all } x \in X \cap U'. \tag{9}$$

Similarly, upon using hypothesis (a), we deduce that for any $\alpha^i > 0$, $i \in P$ there is a neighborhood U_i of $\bar{x}$ such that

$$F^i(x) - F^i(\bar{x}) \geq \int_0^T \{(f^i)^o(t, \bar{x}(t); x(t) - \bar{x}(t)) - \alpha^i \|x(t) - \bar{x}(t)\|\} dt,$$
$$\text{for all } x \in X \cap U_i, i \in P.$$

$$\Rightarrow \sum_{i \in P} \lambda^i (F^i(x) - F^i(\bar{x}))$$
$$\geq \int_0^T \Big\{ \sum_{i \in P} \lambda^i (f^i)^o(t, \bar{x}(t); x(t) - \bar{x}(t)) - \sum_{i \in P} \lambda^i \alpha^i \|x(t) - \bar{x}(t)\| \Big\} dt,$$
$$\text{for all } x \in X \cap (\bigcap_{i \in P} U^i).$$

Set $U = (\bigcap_{i\in P} U^i) \cap U'$, as $\hat{x} \in X_0 \cap U$, then the above inequality yields

$$\begin{aligned}&\sum_{i\in P} \lambda^i (F^i(\hat{x}) - F^i(\bar{x})) \\ &\geq \int_0^T \Big\{ \sum_{i\in P} \lambda^i (f^i)^o(t, \bar{x}(t); \hat{x}(t) - \bar{x}(t)) - \sum_{i\in P} \lambda^i \alpha^i \|\hat{x}(t) - \bar{x}(t)\| \Big\} dt,\end{aligned}$$

The use of $\hat{x} \in X_0$, (5) and (6), along with (3), implies

$$0 \geq \int_0^T \Big\{ \sum_{j\in M} \mu^j(t)(g^j)^o(t, \bar{x}(t); \hat{x}(t) - \bar{x}(t)) - \beta \|\hat{x}(t) - \bar{x}(t)\| \Big\} dt.$$

By adding the above two inequalities and using (3.1), we get

$$\sum_{i\in P} \lambda^i (F^i(\hat{x}) - F^i(\bar{x})) \geq -\int_0^T \Big\{ \Big(\sum_{i\in P} \lambda^i \alpha^i + \sum_{i\in P} \lambda^i c^i + \beta) \|\hat{x}(t) - \bar{x}(t)\| \Big\} dt. \quad (10)$$

Choose $d^i,\ \alpha^i,\ i \in P$ and β such that

$$\sum_{i\in P} \lambda^i (d^i - \alpha^i) - \beta = \sum_{i\in P} \lambda^i c^i.$$

Then (10) contradicts (8). Hence the results follow. ∎

4. Duality

Duality results are significant as they lay down foundation for many computational techniques in optimization problems. Here, in this article we contemplate the mixed dual of (P).

Let J be a subset of $M = \{1, 2, \ldots, m\}$ and $K = M \setminus J$ such that $J \cup K = M$. Let

$$\mu^J(t)^T g^J(t, u(t)) = \sum_{i\in J} \mu^i(t) g^i(t, u(t))$$

and

$$\mu^K(t)^T g^K(t, u(t)) = \sum_{i\in K} \mu^i(t) g^i(t, u(t)).$$

Consider the following mixed dual of (P) :

$$(D)\ \text{Maximize}\ \Big(\int_0^T \{f^1(t,u(t)) + \mu^J(t)^T g^J(t,u(t))\}dt,$$
$$\ldots, \int_0^T \{f^p(t,u(t)) + \mu^J(t)^T g^J(t,u(t))\}dt\Big)$$

Subject to

$$\int_0^T \Big\{\sum_{i\in P}\lambda^i (f^i)^o(t,u(t);v(t)) + \sum_{i\in P}\lambda^i c^i \|v(t)\| + \sum_{j\in M}\mu^j(t)(g^j)^o(t,u(t);v(t))\Big\}dt \geq 0, \tag{11}$$

$$\forall v \in L_\infty^n[0,T].$$

$$\mu^j(t)g^j(t,u(t)) = 0,\ a.e.\ t\in[0,T],\ j\in K. \tag{12}$$

$$\sum_{i\in P}\lambda^i = 1, \lambda^i \geq 0, c^i > 0, i\in P, \mu^j(t)\geq 0, j\in M,\ a.e.\ t\in[0,T]. \tag{13}$$

Remark 4.1 *If $K=\emptyset$, then the dual (D) reduces to the well-known Wolf dual, and if $J=\emptyset$, then (D) reduces to the Mond-Weir type dual.*

Let Y_0 be the set of all feasible solutions of (D).

We shall prove various duality results connecting quasi efficient solutions of (P) and (D).

Theorem 4.1 *(Weak duality) Let $(u,\lambda^1,\ldots,\lambda^p,\mu^1(t),\ldots,\mu^m(t),c^1,\ldots,c^p)\in Y_0$.*

Let
(i) $\Psi^i(x) = \int_0^T\{f^i(t,x(t)) + \mu^J(t)^T g^J(t,x(t))\}dt, i\in P$, be strictly approximate convex at u.
(ii) $\bar{G}(x) = \int_0^T\{\mu^K(t)^T g^K(t,x(t))\}dt$ be approximately convex at u.
Then there exist $d=(d^1,d^2,\ldots,d^p)\in int\mathbb{R}_+^p$ and a neighborhood U of u such that for $x\in X_0\cap U$, the following inequalities do not hold:

$$\int_0^T f^i(t,x(t))dt \leq \int_0^T \{f^i(t,u(t)) + \mu^J(t)^T g^J(t,u(t)) - d^i\|x(t)-u(t)\|\}dt, \tag{14}$$

for all $i\in P$

and

$$\int_0^T f^j(t,x(t))dt < \int_0^T \{f^j(t,u(t)) + \mu^J(t)^T g^J(t,u(t)) - d^j\|x(t)-u(t)\|\}dt, \tag{15}$$

for at least one $j \in P$.

Proof Upon using the hypothesis (ii), there is a neighborhood U' of u such that

$$\bar{G}(x) - \bar{G}(u) \geq \int_0^T \Big\{ \sum_{j\in K} \mu^j(t)(g^j)^o(t,x(t);x(t)-u(t)) - \beta\|x(t)-u(t)\| \Big\} dt,$$

for all $x \in X \cap U'$, for any $\beta > 0$.

Hypothesis (i) implies that there is a neighborhood U_i of u such that

$$\Psi^i(x) - \Psi^i(u) > \int_0^T \{(f^i)^o(t,x(t);x(t)-u(t)) + \sum_{j\in J} \mu^j(t)(g^j)^o(t,x(t);x(t)-u(t)) - \alpha^i\|x(t)-u(t)\|\}dt,$$

for all $x \in X \cap U_i, i \in P,$ for any $\alpha^i > 0, i \in P$.

Set $U = (\bigcap_{i\in P} U^i) \cap U'$. Let $x \in X_0 \cap U$. We will prove the result by contradiction.

Suppose that for any $d = (d^1, d^2, \ldots, d^p) \in \operatorname{int}\mathbb{R}^p_+$, (14) and (15) hold.

Using $x \in X_0$ along with (13), (14) and (15), we arrive at

$$\Psi^i(x) \leq \Psi^i(u) - d^i \int_0^T \|x(t)-u(t)\|dt, \text{ for all } i \in P$$

and

$$\Psi^j(x) < \Psi^j(u) - d^j \int_0^T \|x(t)-u(t)\|dt, \text{ for at least one } j \in P.$$

Proceeding along the similar lines as in the proof of Theorem 3.2, we arrive at contradiction. Hence the result follows. ■

The following example illustrates the weak duality theorem:

EXAMPLE 4.1 *In a marketing campaign for a toy, the team wants to maintain a consistent level of sales over time. The objective is to minimize deviations from a target sales level, while optimizing the sales with restrictions set on the advertising budget. Let $x(t)$ denote the sales of the toy at time t. Target sale level is 200 toys.*

Mathematical formulation of the above problem is

$$(P4)\ \text{Minimize} \Big(\int_0^{100} \{|x(t) - 200|\}dt, \int_0^{100} \{-x(t)\}dt \Big)$$

subject to

$$x(t) \geqq 0,\ t \in [0, 100],$$
$$x(t) \leqq 400,\ t \in [0, 100],$$
$$x^2(t) - x(t) \leqq 0,\ t \in [0, 100].$$

Here, $\bar{x}(t) = 200$, $t \in [0, 1]$ *is a feasible solution of* $(P4)$. *We have* $n = 1$, $M = \{1, 2, 3\}$, $J = \{1\}$ *and* $K = \{2, 3\}$. *Consider the mixed dual of* $(P4)$*:*

$$(D4)\ \text{Maximize} \Big(\int_0^{100} \{|u(t) - 200| - \mu^1(t)u(t)\}dt, \int_0^{100} \{-u(t) - \mu^1(t)u(t)\}dt \Big)$$

subject to

$$\int_0^{100} \Big\{ \lambda^1(h)^o(t, u(t); v(t)) - \lambda^2 v(t) + \lambda^1 c^1 |v(t)| + \lambda^2 c^2 |v(t)| - \mu^1(t)v(t) + \mu^2(t)v(t) + \mu^3(t)\{2u(t) - 1\}v(t) \Big\} dt \geq 0 \quad \text{for all } v \in L^1_\infty[0, 100]$$

$$\mu^2(t)(u(t) - 400) = 0,\ \text{a.e. } t \in [0, 100]$$
$$\mu^3(t)(u^2(t) - u(t)) = 0,\ \text{a.e. } t \in [0, 100]$$
$$\lambda^1 + \lambda^2 = 1,\ \lambda^i \geq 0,\ c^i > 0,\ i \in P, \mu^j(t) \geq 0,\ j \in M,\ \text{a.e. } t \in [0, 100],$$

where $h(t, u(t), v(t)) = |u(t) - 200|$. *Let* $\bar{u}(t) = 0, t \in [0, 100]$. *Then,* $(\bar{u}, \lambda^1 = \frac{1}{2}, \lambda^2 = \frac{1}{2}, \mu^1(t) = 1, \mu^2(t) = 0, \mu^3(t) = 0, c^1 = 1, c^2 = 2)$ *is a feasible solution for* $(D4)$. *It is easy to verify that* $d = (d^1 = 1, d^2 = 1) \in int\mathbb{R}^2_+$ *and in the neighborhood* $U = (-\epsilon, \epsilon)$ *of* $\bar{u}$ *the inequalities* (14) *and* (15) *do not hold.*

THEOREM 4.2 *(Strong duality) Let* $\bar{x}$ *be quasi normal efficient solution for* (P). *Then there exist* $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^p) \in \mathbb{R}^p_+$, $c = (c^1, c^2, \dots, c^p) \in int\mathbb{R}^p_+$ *and* $\mu^j \in L^1_\infty[0, T], j \in M$ *such that* $(\bar{x}, \lambda^1, \dots, \lambda^p,\ \mu^1(t), \dots, \mu^m(t), c^1, \dots, c^p) \in Y_0$. *Further, if the weak duality theorem holds, then* $(\bar{x}, \lambda^1, \dots, \lambda^p, \mu^1(t), \dots, \mu^m(t), c^1, \dots, c^p)$ *is a locally quasi efficient solution for* (D).

PROOF Let $\bar{x}$ be a quasi normal efficient solution for (P), then, by Theorem 3.1, there exist $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^p) \in \mathbb{R}^p_+,\ c = (c^1, c^2, \ldots, c^p) \in \operatorname{int} \mathbb{R}^p_+$ and $\mu^j \in L^1_\infty[0,T],\ j \in M$ such that $(\bar{x}, \lambda^1, \ldots, \lambda^p, \mu^1(t), \ldots, \mu^m(t),\ c^1, \ldots, c^p) \in Y_0$.

Suppose that the weak duality theorem holds. Let, for the purposes of this proof, suppose that $(\bar{x}, \lambda^1, \ldots, \lambda^p, \mu^1(t), \ldots, \mu^m(t), c^1, \ldots, c^p)$ is not a locally quasi efficient solution for (D), then for any $d = (d^1, d^2, \ldots, d^p) \in \operatorname{int} \mathbb{R}^p_+$ and for any neighborhood U of $\bar{x}$, there exist $(\hat{x}, \hat{\lambda}^1, \ldots, \hat{\lambda}^p, \hat{\mu}^1(t), \ldots, \hat{\mu}^m(t), \hat{c}^1, \ldots, \hat{c}^p) \in Y_0$ with $\hat{x} \in U$ such that

$$\int_0^T \{f^i(t, \bar{x}(t)) + \bar{\mu}^J(t)^T g^J(t, \bar{x}(t)) + d^i \|\bar{x}(t) - \hat{x}(t)\|\} dt$$

$$\leq \int_0^T \{f^i(t, \hat{x}(t)) + \hat{\mu}^J(t)^T g^J(t, \hat{x}(t))\} dt, \text{ for all } i \in P$$

and

$$\int_0^T \{f^j(t, \bar{x}(t)) + \bar{\mu}^J(t)^T g^J(t, \bar{x}(t)) + d^j \|\bar{x}(t) - \hat{x}(t)\|\} dt$$

$$< \int_0^T \{f^j(t, \hat{x}(t)) + \hat{\mu}^J(t)^T g^J(t, \hat{x}(t))\} dt, \text{for at least one } j \in P.$$

On using $\bar{\mu}^j(t) g^j(t, \bar{x}(t)) = 0,\ a.e.\ t \in [0,T],\ j \in J$ we obtain

$$\int_0^T \{f^i(t, \bar{x}(t)) + d^i \|\bar{x}(t) - \hat{x}(t)\|\} dt \leq \int_0^T \{f^i(t, \hat{x}(t)) + \hat{\mu}^J(t)^T g^J(t, \hat{x}(t))\} dt,$$

$$\text{for all } i \in P$$

and

$$\int_0^T \{f^j(t, \bar{x}(t)) + d^j \|\bar{x}(t) - \hat{x}(t)\|\} dt < \int_0^T \{f^j(t, \hat{x}(t)) + \hat{\mu}^J(t)^T g^J(t, \hat{x}(t))\} dt,$$

$$\text{for at least one } j \in P.$$

Which is a contradiction to the weak duality theorem. ■

5. Conclusions and future developments

In this paper, we have introduced the concept of quasi-efficient solutions for non-smooth multi-objective continuous-time programming problems, highlighting its versatility, due to allowing for some level of error, based on the decision variables. This notion accommodates real-world constraints, where exact solutions may not always be feasible, and an approximation is often acceptable. The study derives necessary optimality conditions for multi-objective continuous-time problems and extends the concept of approximate convexity to address sufficient optimality conditions. By proposing a mixed dual, we demonstrate both weak and strong duality results, providing a more comprehensive framework for analyzing these types of optimization problems.

Examples are provided to demonstrate the significance of the concepts introduced in this paper. Results of this paper can be extended to continuous time fractional programming problem using approximate pseudo-convex and approximate quasi-convex functions. As suggested by one of the reviewers, we will explore multiobjective continuous time problem with infinite constraints. Overall, we hope that our study inspires further research into more flexible and practical optimization methods, contributing to the ongoing development of this field.

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