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Uniqueness of the Riccati operator of the non-standard ARE of a third order dynamics with boundary control*

by

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Abstract: The Moore-Gibson-Thompson [MGT] dynamics is considered. This third order in time evolution arises within the context of acoustic wave propagation with applications in high frequency ultrasound technology. The optimal boundary feedback control is constructed in order to have on-line regulation. The above requires wellposedness of the associated Algebraic Riccati Equation. The paper by Lasiecka and Triggiani (2022) recently contributed a comprehensive study of the Optimal Control Problem for the MGT-third order dynamics with boundary control, over an infinite time-horizon. A critical missing point in such a study is the issue of uniqueness (within a specific class) of the corresponding highly non-standard Algebraic Riccati Equation. The present note resolves this problem in the positive, thus completing the study of Lasiecka and Triggiani (2022) with the final goal of having on line feedback control, which is also optimal.

Keywords: uniqueness, non-standard algebraic Riccati equation, Moore-Gibson-Thompson equation

1. Introduction

The Moore-Gibson-Thompson (MGT) equation is a prominent example of a Partial Differential Equation (PDE) model, which describes the acoustic velocity potential in wave propagation. The well known applications thereof are in the area of High Frequency Ultrasound, where acoustic waves propagate through a

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closed environment (detection of tumors, lithotripsy, welding). In addition, the MGT model removes the paradox of infinite speed of propagation of thermal signals by the use of the constitutive Cattaneo law for the heat flux, in place of the Fourier law. This then results in the presence of a third-order derivative in time, thus yielding a third order dynamics in time. Most important is the fact that the resulting dynamics changes character - from parabolic to hyperbolic. The latter is associated with much compromised regularity of solutions - making the quantitative analysis of the solutions challenging. While the model has been known and used in the context of applications for a long time, the mathematical analysis of the underlined dynamics is relatively new. It was the recent decade that has witnessed a quite intense set of activities in the area, mostly with respect to the quantitative theory of wellposedness of solutions (both linear and nonlinear), as well as their stability. New techniques have been developed in order to study the system, see Kaltenbacher (2015), Kaltenbacher, Lasiecka and Pospieszalska (2012), Kaltenbacher, Lasiecka and Marchand (2011), Marchand, Mc Devitt and Triggiani (2012).

A natural step forward is to consider the control problems, associated with such third order dynamics. Of particular interest are then boundary control problems, where the action of control is confined only to a portion of the boundary. From a physical point of view, this is the most interesting case, as only a small portion of the affected boundary may be accessible to external manipulations. One of the main challenges, faced at the start is the low regularity of solutions (due to hyperbolicity) when activated by non-smooth controls. Note that in the case of parabolic-diffusive dynamics, roughness of the data is immediately counteracted by the smoothing effect of analyticity, exhibited by parabolic models. This does not happen in hyperbolic-like models. New tools and a better understanding of the underlying dynamics are necessary.

In fact, such task has been initiated in Bucci and Lasiecka (2019), where optimal boundary control on a *finite time horizon only* has been considered. This has been followed by Lasiecka and Triggiani (2022), where the *infinite time horizon is studied* with L_2 only boundary controls. The construction developed in Lasiecka and Triggiani (2022) leads to a feedback control from the boundary, which also produces uniform stability of the the model. The feedback control constructed is based on the solvability of an appropriate - nonstandard - Riccati equation. The attribute 'nonstandard' comes in two flavors: first, from the algebraic structure due to the presence of the third time-derivative in the model; second, from the gain operators (feedback operators) that are heavily unbounded and need an appropriate interpretation via regularity theory. The latter is due to the boundary nature of the controls, which interact with the non-smooth dynamics. While the optimal solution is represented in a feedback form via an appropriate solution of the Algebraic Riccati Equation (ARE), in order to have full confidence in an on-line construction, one needs to assure that the computed Riccati solution is *unique* within a specified class. This aspect is challenging, due to the much compromised smoothness of boundary controls.

The importance of uniqueness of the ARE solution is further elaborated in more details, in Remark 1.1, at the end of Section 1. All this brings us to the main goal of the present paper: **to show that the solution to highly non-standard Algebraic Riccati equation, obtained in Lasiecka and Triggiani (2022) is unique (within a specified class) - so that the resulting on-line feedback construction, emanating from such solution, is the correct one, leading to the optimal actuation of the given Optimal Control Problem**. Only then one can be confident that solving only algebraic equation is sufficient to obtain on-line dynamic feedback, which is *optimal.*

To achieve this goal we shall start from the paper of Lasiecka and Triggiani (2022). After briefly recalling only the needed results of Lasiecka and Triggiani (2022), we shall proceed with the proof of uniqueness, which is inspired by our prior developments (Lasiecka, Lukes and Pandolfi, 1995; Lasiecka, Pandolfi and Triggiani, 1997; Triggiani, 1994a,b). The technical issue to contend with is due to the algebraic (rather than differential) nonstandard structure of the ARE with *unbounded coefficients*. The latter is the consequence of the hyperbolicity of the dynamics. The references mentioned just before deal only with parabolic-analytic structures. To overcome the difficulties one needs to exhibit propagation of some regularity through the functional cost and the related optimization. This is accomplished in Section 4.

2. The OCP for the MGTJ equation with boundary control: a selected review from Lasiecka and Triggiani (2022)

2.1. The developement

The present paper is a successor to Lasiecka and Triggiani (2022), which studies an Optimal Control Problem (OCP) for the third order (in time) MGT equation defined on a, say, 3-d bounded domain Ω, with boundary $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$, which is reflected in Fig. 1. With reference to a physically significant case that we wish to cover, the resulting 3-d domain is obtained by rotating the 2-d section, shown in Fig. 1, around the vertical axis of rotation, passing through the focus point. The resulting spherical part at the bottom of the domain is the part Γ_0 of the boundary, shown in Fig. 2. It is convex and satisfies the geometrical conditions (ii) of Theorem 2.2. The equation in the acoustic pressure $u(x,t)$ is subject to dissipation in the Neumann boundary control over the portion Γ_1 of the boundary, and control function g in the Robin boundary control of the complementary portion Γ_0 of the boundary:

$$
\begin{cases}
u_{ttt} + \alpha u_{tt} - c^2\Delta u - b\Delta u_t = 0 & \text{in } Q \equiv \Omega \times (0,\infty) \quad (2.1\text{a}) \\
\partial_\nu u + u_t = 0 & \text{on } \Sigma_1 \equiv \Gamma_1 \times (0,\infty) \quad (2.1\text{b}) \\
\partial_\nu u + \ell u = g & \text{on } \Sigma_0 \equiv \Gamma_0 \times (0,\infty) \quad (2.1\text{c}) \\
u(\cdot,0) = u_0; \quad u_t(\cdot,0) = u_1; \quad u_{tt}(\cdot,0) = u_2 & \text{in } \Omega. \quad (2.1\text{d})
\end{cases}
$$

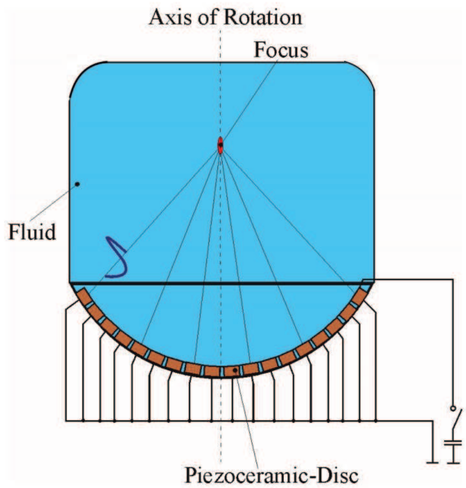

Figure 1. Illustration of the domain. The "red" convex portion of the boundary displays the control region Γ_0. The remaining part of the boundary - with no geometric constraints - represents the area Γ_1 of absorption. The picture is courtesy of B. Kaltenbacher

Here, in line with the literature, $c^2, b > 0$ are appropriate physical constants. We also take $\alpha > 0$, though this is not critical. Instead, taking the constant $\ell > 0$ (Robin boundary control) has the attractive implication of Theorem 2.2 below over the case $\ell = 0$ (Neumann boundary control). We refer to Bongarti, Lasiecka and Triggiani (2022), and Bongarti, Lasiecka and Rodrigues (2022) for recent work on problem (2.1a)–(2.1d). Here, we point out that the above mixed problem yields a non-standard (pathological) abstract model, see (2.6) below, which, in turn, is responsible for a non-standard (pathological) OCP, recently studied in Lasiecka and Triggiani (2022). Though pathological with respect to more traditional boundary control problems, as in Lasiecka and Triggiani (2000), the theory, achieved in Lasiecka and Triggiani (2022), is comprehensive. Because of space limitations, accorded to contributions of a special volume, the issue of uniqueness of the solution of the corresponding very non-standard (pathological) Algebraic Riccati Equation (ARE) – (2.21) below – was not treated in Lasiecka and Triggiani (2022). It is the object of the present note, see Theorem 2.4 below, following the announcement made in Lasiecka and Triggiani (2022). For purposes of brevity, we shall recall below only these results of Lasiecka and Triggiani (2022), which are needed for the proof of uniqueness, given in Section 4 of the present paper. The importance of the uniqueness issue is elaborated in Remark 2.1, at the end of Section 2.

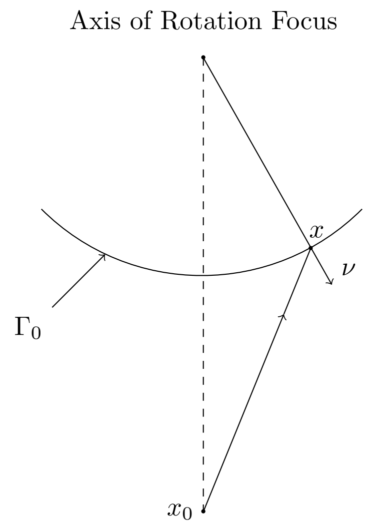

Figure 2. The boundary Γ_0

The optimal control problem. Our notation is the same as in Lasiecka and Triggiani (2022). With reference to problem (2.1a)–(2.1d), we let

$$y(t) = [u(t), u_t(t), u_{tt}(t)]; \quad y_0 = [u_0, u_1, u_2] \in Z = [\mathcal{D}(A^{*2})]';$$
$$g \in L^2(0, \infty; U); \quad U \equiv L^2(\Gamma_0) \tag{2.2}$$
$$Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L^2(\Omega); \quad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H^1(\Omega). \tag{2.3}$$

Here, $\mathcal{A}$ is the realization of $-\Delta$ in $L^2(\Omega)$ with appropriate boundary conditions, where ν is the outward normal vector to Γ:

$$\mathcal{A}u = -\Delta u,$$
$$\mathcal{D}(\mathcal{A}) = \left\{ u \in L^2(\Omega) : \Delta u \in\in L^2(\Omega), \ \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_1} = 0; \ \left[\frac{\partial u}{\partial \nu} + lu\right]_{\Gamma_0} = 0 \right\} \subset H^{2-\epsilon}(\Omega) \tag{2.4}$$

for $\epsilon > 0$ arbitrary, while A is the 3×3 operator on the space Y, that provides the abstract model of the problem (2.1a)–(2.1d) with $g \equiv 0$:

$$\frac{d}{dt}\begin{bmatrix} u \\ u_t \\ u_{tt} \end{bmatrix} = A \begin{bmatrix} u \\ u_t \\ u_{tt} \end{bmatrix}, \quad \text{or} \quad \frac{dy}{dt} = Ay, \ g \equiv 0. \tag{2.5}$$

It is given in Lasiecka and Triggiani (2022), Eq. (2.13). Its specific form is not strictly needed in the present paper. Only its properties, which are critical here, will be noted. More specifically, the abstract model of the mixed problem (2.1a)–(2.1d) is given by

$$y(t) = e^{At}y_0 + B_1 g(t) + (L_0 g)(t) \tag{2.6a}$$

$$(L_0 g)(t) \equiv \int_0^t e^{A(t-s)}B_0 g(s)ds + A\int_0^t e^{A(t-s)}B_1 g(s)ds. \tag{2.6b}$$

Here, B_0, B_1 are suitable (unbounded) boundary operators, whose specific form is again strictly not needed in the present paper, see Lasiecka and Triggiani (2022), Eq. (2.17). Some critical properties are noted below:

$$A^{-1}B_0, A^{-2}B_1 : \text{compact } U \equiv L^2(\Gamma_0) \to Y \tag{2.7a}$$

$$\mathcal{B} \equiv B_0 + AB_1 : \text{compact } U \equiv L^2(\Gamma_0) \to Z \equiv [\mathcal{D}(A^{*2})]'. \tag{2.7b}$$

Theorem 2.1 (Well-posedness of the boundary homogeneous problem, $g \equiv 0$) *The (free dynamic) operator A in* (2.5), *modeling the problem* (2.1a)–(2.1d) *for $g \equiv 0$ generates s.c. semigroup e^{At} on the space Y, hence on $Z = [\mathcal{D}(A^{*2})]'$, see* (2.9) *below.*

The domain of $A(= A_{\text{ext}})$ as extended on Z is $\{x \in Z : Ax \in Z\}$, equivalently s.t. $A^{-2}Ax \in Y$, equivalently $A^{-1}x \in Y$ or $x \in [\mathcal{D}(A^*)]'$.

Theorem 2.2 (Uniform stability of e^{At} on Y and Z) *Let $g \equiv 0$ in* (2.1c). *Assume*

(i) $\gamma = \alpha - \frac{c^2}{b} \geq 0,\ \ell > 0$;

(ii) geometric condition on Γ_0:

(ii₁) Γ_0 *is either flat or convex;*

(ii₂) there exists some point $x_0 \in \mathbb{R}^n$ such that $(x - x_0)\cdot\nu \leq 0$ for all $x \in \Gamma_0$.

Then, there exist constants $C \geq 1$, $\omega > 0$ such that the semigroup solution, guaranteed by Theorem 2.1, satisfies

$$\|y(t)\|_Y = \|e^{At}y_0\|_Y \leq C\|y_0\|_Y e^{-\omega t}, \quad t > 0 \tag{2.8}$$

$$\|e^{At}z_0\|_Z = \|A^{-2}e^{At}z_0\|_Y = \|e^{At}A^{-2}z_0\|_Y \leq M\|z_0\|_Z e^{-\omega t}, \quad A^{-2}z_0 \in Y, \quad t \geq 0. \tag{2.9}$$

The Observation operator R is a (smoothing) operator, which satisfies the following assumptions

$$\begin{cases} R = R^* \text{ in } Y,\ RA^2 \in \mathcal{L}(Y), \text{ equivalently } R \in \mathcal{L}(Z;Y); \\ RB_i : \text{compact } U \to Y; i = 0,1, & (2.10\text{a}) \\ \text{so that } RR^* : [\mathcal{D}(A^*)]' \to \mathcal{D}(A) \text{ continuously,} \\ \text{or } ARR^*A^{-1} \in \mathcal{L}(Y) & (2.10\text{b}) \end{cases}$$

where $B_i, i = 0, 1$, are the (unbounded) boundary operators in (2.6), (2.7), see Lasiecka and Triggiani (2022), Eq. (2.17). Again, here we shall need only their properties. Next, we introduce the following cost functional

$$J(g, y) = \int_0^\infty \left[\|Ry(t)\|_Y^2 + \|g(t)\|_U^2 \right] dt \tag{2.11a}$$

and set up the corresponding OCP:

> **Minimize** $J(g, y)$ over all $g \in L^2(0, \infty; U)$,
> where y is the solution of the problem (2.1a)–(2.1d) or (2.6a)–(2.6b)
> due to g and $y_0 \in Z$. (2.11b)

Relative to the OCP (2.11), the main result of Lasiecka and Triggiani (2022) is

Theorem 2.3 *With reference to the control problem OCP, formulated above for the problem* (2.1a)–(2.1d) *or* (2.6a)–(2.6b), *the following results hold true:*

1. **Existence of a unique optimal pair and corresponding regularity.** *For any I.C.* $y_0 \in Z$, *there exists a unique optimal pair* $\{\widehat{g}(\cdot; y_0), \widehat{y}(\cdot; y_0)\}$, *with* $\widehat{g}(\cdot; y_0)$ *optimal control,* $\widehat{y}(\cdot; y_0)$ *the corresponding optimal solution, such that*

$$\widehat{g}(\cdot; y_0) \in C([0, \infty); U); \quad R\widehat{y}(\cdot; y_0) \in C([0, \infty); Y); \quad \text{and } \widehat{y}(\cdot; y_0) \in C([0, \infty); Z). \tag{2.12}$$

2. **Riccati operator.** *There exists a positive self-adjoint operator* P *on* $\mathcal{L}(Y)$, *defined by*

$$Py = \int_0^\infty e^{A^* \tau} R^* R\widehat{y}(\tau; y) d\tau, \quad y \in Y, \tag{2.13}$$

such that

$$A^{*2} P A^2 \in \mathcal{L}(Y), \textit{equivalently, } P \in \mathcal{L}(Z \equiv [\mathcal{D}(A^{*2})]'; \mathcal{D}(A^{*2})), \tag{2.14}$$

$$B_1^* A^* P \in \mathcal{L}(Y; U), \tag{2.15}$$

$$\widehat{J}(y_0) = (Py_0, y_0)_Y, y_0 \in Z, \textit{in the duality } \mathcal{D}(A^{*2}) \to [\mathcal{D}(A^{*2})]', \tag{2.16}$$

where

$$\widehat{J}(y_0) = J(\widehat{g}(\cdot; y_0), \widehat{y}(\cdot; y_0)) = \textit{optimal cost in } (2.11) \textit{ for } y_0 \in Z. \tag{2.17}$$

Further properties of P (regularity)

(i) Complementing (2.14)

$$P \in \mathcal{L}([\mathcal{D}(A^{*\gamma_1})]'; \mathcal{D}(A^{*\gamma_2})) \quad \textit{for any } \gamma_1, \gamma_2 \leq 2; \quad A^{*\gamma_2} P A^{\gamma_1} \in \mathcal{L}(Y). \tag{2.18}$$

(ii) *As a consequence of* (2.7a) *and* (2.18) *with* $\gamma_1 = \gamma_2 = 2$, *i.e. of* (2.14),

$$B_i^* P A^2 = (B_i^* A^{*-2})(A^{*2} P A^2) \in \mathcal{L}(Y;U),$$
$$\text{equivalently } B_i^* P \in \mathcal{L}(Z;U), i = 0, 1. \tag{2.19}$$

(iii) *Thus, the gain operator* $\mathcal{B}^* P$ *satisfies*

$$\mathcal{B}^* P A^2 \in \mathcal{L}(Y;U), \quad \text{equivalently} \quad \mathcal{B}^* P \in \mathcal{L}(Z;U). \tag{2.20}$$

3. **Riccati equation.** *The operator* P *in* (2.13) *satisfies the following non-standard Riccati equation: for all* $x_1, x_2 \in Y$ *(for whose well-posedness recall* (2.18), (2.10b) *and* (2.20)*)*

$$(Ax_1, Px_2)_Y + (Px_1, Ax_2)_Y + (Rx_1, Rx_2)_Y =$$
$$\left(B_1^* R^* R x_1 + \mathcal{B}^* P x_1, [I + B_1^* R^* R B_1]^{-1} [B_1^* R^* R x_2 + \mathcal{B}^* P x_2]\right)_U, \tag{2.21}$$

where (2.7b)

$$\mathcal{B} = B_0 + AB_1 : \text{compact } U \equiv L^2(\Gamma_0) \to Z = [\mathcal{D}(A^{*2})]';$$
$$\mathcal{B}^* \equiv B_0^* + AB_1^* : \text{compact } \mathcal{D}(A^{*2}) \to U. \tag{2.22}$$

4. **Feedback synthesis.** *The optimal control* $\widehat{g}(\cdot; y_0)$ *and the optimal solution* $\widehat{y}(\cdot; y_0)$ *are expressed by the following pointwise feedback synthesis for all* $t > 0$

$$\widehat{g}(\cdot; y_0) = -[I - \mathcal{B}^* P B_1]^{-1} [B_1^* R^* R + \mathcal{B}^* P] \widehat{y}(\cdot; y_0), \tag{2.23}$$

where the operator $[I - \mathcal{B}^* P B_1]$ *is boundedly invertible on* U, $(\mathcal{B}^* P A^2)(A^{-2} B_1) \in \mathcal{L}(U)$ *by* (2.20) *and* (2.7a).

Remark 2.1 *We note that while the optimal observed state* $R\widehat{y}$ *belongs to the state space* Y*- the optimal solution* $\widehat{y}(t)$ *is guaranteed only to reside in* $Z = [\mathcal{D}(A^{*2})]'$. *This is due to the rough behavior on the boundary. The additional regularity of the Riccati operator enables giving proper meaning to the operators in* (2.21).

The main result of the present paper is

Theorem 2.4 (Uniqueness) *Under the given assumptions of this section, the operator* P, *defined in* (2.12), *is the only positive self-adjoint solution of the ARE* (2.21) *within the class of operators, satisfying the regularity properties of Theorem 2.3.*

The proof is given in Section 4.

2.2. Literature

As noted in the abstract, the present paper is very focused: it completes the study of Lasiecka and Triggiani (2022) by showing uniqueness (within a specified class) of the Riccati operator, solution of the ARE. Thus, of the ever growing literature in third order SMGTJ-equation, we shall quote only some relevant papers. First, the name is justified by Stokes (1851), Moore and Gibson (1960), Thompson (1972), and Jordan (2004). Second, a mathematical study of these third order PDEs, both linear and non-linear, was initiated about a decade ago, see Kaltenbacher, Lasiecka and Marchand (2011) and Marchand, McDevitt and Triggiani (2012). Next, an optimal control problem over a finite time horizon, but for the Westervelt dynamics and related Kuznetsov-dynamics, was studied in and Clason and Kaltenbacher (2015), Clason, Kaltenbacher and Veljović (2009). The recent paper of Lasiecka and Triggiani (2022) provides a comprehensive study of the OCP, recalled above in Theorem 2.3. The present note completes that study by establishing the uniqueness of Theorem 2.4. In establishing (in Section 4) such uniqueness result, we are guided by the treatment of uniqueness in Lasiecka, Lukes and Pandolfi (1995). To this end, we point out that the papers by Lasiecka, Lukes and Pandolfi (1995), Lasiecka, Pandolfi and Triggiani (1997) and Triggiani (1994a,b) provide different approaches to study a sort of corresponding pathological OCP: however the one, where the basic standing assumption is that the original dynamics is *parabolic*; i.e. it generates a s.c. analytic semigroup. It intends to cover radically different classes from the hyperbolic third order SMGTJ-equation, namely, wave or plate equations with high Kelvin-Voigt damping, and boundary control. The study of *finite horizon* control problem for SMGTJ equation with a related Differential (rather than Algebraic) Riccati Equations has been carried out in Bucci and Lasiecka (2019).

REMARK 2.2 *This remark is inserted here at the request of a referee to clarify the impact of the paper. The importance of uniqueness of a positive self-adjoint solution of Eq* (2.21) *satisfying the regularity properties of Theorem 2.3 is at two levels: (i) a theoretical level; and (ii) a computational level.*

(i) Theoretical level. First, for any equation involving unknowns, one would typically like to claim the desirable property of uniqueness of a solution (within a specified class). We next make this concept more specific in the context of our present problem, as in Theorem 1.4. In line with the authors' approach over many years, see, e.g. Lasiecka and Triggiani (2000), the strategy followed in the study of the present OCP may be summarized as follows. In Step 1, one asserts the existence of a unique optimal pair in (2.12)*. In Step 2, one defines explicitly in* (2.13) *a positive self-adjoint operator P and proves for it various regularity properties: those in* (2.14)–(2.20)*; the feedback synthesis in* (2.23)*; and the semi-group property of Proposition 3.1, with the highly desirable exponential decay* ((ii))*. Thus far, there is no Riccati claim for such operator. Finally, in Step 3, one shows that such explicit operator P satisfies the ARE* (2.21)*. QUESTION:*

Is such operator P the unique solution of the ARE (2.21) *within the class of established properties? There may be another such solution and this will give no guarantee that, if used in the feedback synthesis* (2.23)*, it would provide the optimal control; nor that, if used in* (3.1)*, it would provide the optimal solution with the nice semigroup properties, established for the original operator P of Step 1. Thus, one needs that the ARE provides the "right" operator of the OCP, not a second spurious one, external to the OCP. This is so, also because the ARE is the de facto avenue for computing the "right" operator P, the one of Step 1. In short: if uniqueness is not guaranteed, the Algebraic Riccati solution operator selected may not yield the optimal feedback and the optimal on-line control.*

(ii) Computational level. Under the computational aspect, the desirability of uniqueness is even more pronounced. The lack of uniqueness can lead to a wrong quantity, which has nothing to do with the sought-after solution of the OCP.

3. Additional background from Lasiecka and Triggiani (2022) needed in the proof of Section 4

Evolution operator $\Phi(s)$ and feedback semigroup. In order to obtain an explicit feedback operator, Lasiecka and Triggiani (2022) consider an evolution operator, governing the dynamics of the controlled process. Unlike the standard Riccati theory, see Lasiecka and Triggiani (2000), such evolution does not coincide with the optimal trajectory. For $x \in Z \equiv [\mathcal{D}(A^{*2})]'$, one defines the operator $\Phi(t)$ by setting

$$\Phi(t)x = \widehat{y}(t;x) - B_1\widehat{g}(t;x) = e^{At}x + \{L_0\widehat{g}(\cdot;x)\}(t) \in C([0,T];Z) \cap L^2(0,\infty;Z) \tag{3.1}$$

$$(L_0\widehat{g})(t) \equiv \int_0^t e^{A(t-s)}B_0\widehat{g}(s;x)ds + A\int_0^t e^{A(t-s)}B_1\widehat{g}(s;x)ds \tag{3.2}$$

by (2.6a) where $\widehat{y}$ and $\widehat{g}$ are the optimal trajectory and the optimal control. We now collect several important properties.

PROPOSITION 3.1 *For the operator $\Phi(\cdot)$ defined in* (3.1)*, the following properties are valid:*

(i) $\Phi(t)$ is a strongly continuous semigroup on Z with infinitesimal generator

$$A_Fx = Ax + \mathcal{B}G_0x \quad \textit{for } x \in D(A_F), \qquad \mathcal{B} = B_0 + AB_1 \in \mathcal{L}(U;Z) \tag{3.3a}$$

$$[\mathcal{D}(A^*)]' \equiv D(A_F) \subset Z \equiv [\mathcal{D}(A^{*2})]' \to Z \tag{3.3b}$$

$$G_0 \equiv -T_0^{-1}[B_1^*R^*R + \mathcal{B}^*P] \in \mathcal{L}(Z;U); \qquad \mathcal{B}G_0 \in \mathcal{L}(Z) \tag{3.4a}$$

$$T_0 = [I + B_1^*R^*RB_1] = \textit{positive self-adjoint on } \mathcal{L}(U) \tag{3.4b}$$

$$\Phi(t) = e^{A_F t}; \qquad \frac{d\Phi(t)x}{dt} = A_F \Phi(t)x = \Phi(t)A_F x, \quad x \in D(A_F). \tag{3.5}$$

(ii) $\Phi(\cdot)$ *is exponentially stable in* $\mathcal{L}(Z)$*: there exist constants* $M \geq 1, \omega > 0$ *such that*

$$\|\Phi(t)\|_{\mathcal{L}(Z)} \leq Me^{-\omega t};$$
$$\text{hence } \|R\Phi(t)\|_{\mathcal{L}(Z;Y)} \leq \|RA^2\|_{\mathcal{L}(Y)} Me^{-\omega t}, \ t > 0. \tag{3.6}$$

4. Proof of uniqueness – Theorem 2.4

The proof consists of two main steps, discussed in Sections 4.1 and 4.2 below.

4.1. Uniform stability of the s.c. semigroup $\Phi(t)$ generated by any solution P of the ARE (1.20) through the procedure of Proposition 2.1

Let P be the operator, defined by (2.13) in terms of the optimal trajectory $\widehat{y}(\cdot; x)$. By Theorem 2.3, such P is a solution of the ARE (2.21) with regularity properties collected there. Moreover, through the procedure, described in Proposition 3.1, such operator P defines a s.c. semigroup $\Phi(t)x = \widehat{y}(t;x) - B_1\widehat{g}(t;x)$ in (3.1) that is uniformly stable on $Z = [\mathcal{D}(A^{*2})]'$, see ((ii)). The next result shows the uniform stability starting from *any* solution $\widehat{P}$ of the ARE (2.21).

Theorem 4.1 *Let* $\widehat{P} \in \mathcal{L}(Y)$ *be a positive self-adjoint solution of the ARE* (2.21)*, satisfying the same regularity properties as the operator* P *in* (2.13)*, listed in Theorem 2.3. Then, the operator*

$$\widehat{A}_F = A + \mathcal{B}\widehat{G}_0, \quad \widehat{G}_0 = -T_0^{-1}[B_1^* R^* R + \mathcal{B}^* \widehat{P}] \in \mathcal{L}(Z; U) \tag{4.1}$$

generates a s.c. semigroup $\widehat{\Phi}(t)x = e^{\widehat{A}_F t}x, x \in Z$*, that is uniformly stable in* Z*: there exist constants* $\widehat{C} \geq 1$*,* $\widehat{\omega} > 0$ *such that*

$$\|e^{\widehat{A}_F t}\|_{\mathcal{L}(Z)} = \|\widehat{\Phi}(t)\|_{\mathcal{L}(Z)} \leq \widehat{C}e^{-\widehat{\omega}t}, \ t \geq 0. \tag{4.2}$$

Proof Throughout this proof, to simplify the notation, we shall denote by $P, \Phi(t)$ – rather than $\widehat{P}, \widehat{\Phi}(t)$ – the operator in the statement of Theorem 4.1 and the corresponding semigroup. It should not be confused with the "optimality" operators P in (2.13), (2.17) and Φ in (3.1).

Step 1.

Proposition 4.2 *Let P be* any *positive self-adjoint solution of the ARE* (2.21), *satisfying the regularity properties of Theorem 2.3. Then, such P satisfies*

$$2Re\{(Ax, Px)_Y + (G_0x, \mathcal{B}^*Px)_U\} = -\|Rx + RB_1G_0x\|_Y^2 - \|G_0x\|_U^2, \quad \text{for any } x \in Y. \tag{4.3}$$

Proof (Proof of Proposition 4.2) We return to the ARE (2.21) with $x_1 = x_2 = x \in Y$, and T_0 defined in (3.4b):

$$(Ax, Px)_Y + (Px, Ax)_Y + (Rx, Rx)_Y = = \left(B_1^*R^*Rx + \mathcal{B}^*Px, T_0^{-1}[B_1^*R^*Rx + \mathcal{B}^*Px]\right)_U. \tag{4.4}$$

Notice that all terms are well-defined by the properties of P in Theorem 2.3: for $x \in Y$,

$$Ax \in [\mathcal{D}(A^*)]', \quad Px \in \mathcal{D}(A^*); \quad Rx \in Y, \quad B_1^*R^*Rx \in U; \quad \mathcal{B}^*Px \in U; \quad T_0^{-1}[B_1^*R^*R + \mathcal{B}^*P]x \in U. \tag{4.5}$$

Next, on the RHS of (4.1), add and subtract $(G_0x, \mathcal{B}^*Px)_U$ (G_0 having been defined in (3.4a)) to get

$$2\text{Re}\{(Ax, Px)_Y + (G_0x, \mathcal{B}^*Px)_U\} = -(Rx, Rx)_Y + (G_0x, \mathcal{B}^*Px)_U - (B_1^*R^*Rx, G_0x)_U. \tag{4.6}$$

Next, on the RHS of (4.1), we likewise add and subtract $(G_0x, B_1^*R^*Rx)_U$, thus obtaining

$$\text{RHS of (4.1)} = = -(Rx, Rx)_Y + (G_0x, B_1^*R^*Rx + \mathcal{B}^*Px)_U - 2\text{Re}\{(G_0x, B_1^*R^*Rx)_U\} \tag{4.7}$$

$$= -(Rx, Rx)_Y - 2\text{Re}\{(RB_1G_0x, Rx)_Y\} - (G_0x, T_0G_0x)_U \tag{4.8}$$

$$= -\text{Re}\{(Rx + 2RB_1G_0x, Rx)_Y\} - (T_0G_0x, G_0x)_U. \tag{4.9}$$

When going from (4.7) to (4.8) we have recalled $-T_0G_0 = B_1^*R^*R + \mathcal{B}^*P$ from (3.4a), while when going from (4.8) to (4.9) we have used the fact that T_0 in (3.4b) is self-adjoint. The new form (4.9) of the RHS of (4.1) induces to consider

$$\|Rx + RB_1G_0x\|_Y^2 = \text{Re}\{(Rx + 2RB_1G_0x, Rx)_Y\} + \|RB_1G_0x\|_Y^2. \tag{4.10}$$

Substituting (4.10) in (4.9) yields, upon recalling (4.1):

$$2\text{Re}\{(Ax, Px)_Y + (G_0x, \mathcal{B}^*Px)_U\} = \|RB_1G_0x\|_Y^2 - \|Rx + RB_1G_0x\|_Y^2 - (T_0G_0x, G_0x)_U = = ([B_1^*R^*RB_1 - T_0]G_0x, G_0x)_U - \|Rx + RB_1G_0x\|_Y^2, \tag{4.11}$$

or, recalling $B_1^* R^* R B_1 - T_0 = -I$ from (3.4b),

$$2\mathrm{Re}\left\{(Ax, Px)_Y + (G_0 x, \mathcal{B}^* P x)_U\right\} = -\|Rx + RB_1 G_0 x\|_Y^2 - \|G_0 x\|_U^2, \ x \in Y. \tag{4.12}$$

Thus, (4.12) proves Proposition 4.2. □

Step 2.

Lemma 4.3 *With $P(=\widehat{P})$ being any solution of the ARE* (2.21)*, as assumed in Theorem 4.1, we have that the operator*

$$A_F = A + \mathcal{B}G_0, \quad G_0 = -T_0^{-1}[B_1^* R^* R + \mathcal{B}^* P], \quad \mathcal{B}G_0 \in \mathcal{L}(Z) \tag{4.13}$$

generates a s.c. semigroup $\Phi(t) = e^{A_F t}$ on Z.

Proof 1 *By Theorem 2.2, A generates a s.c. semigroup e^{At} on Z while $\mathcal{B}$: bounded (in fact, compact) $U \to Z$ by* (2.7b)*, and $G_0 \in \mathcal{L}(Z;U)$, so that $\mathcal{B}G_0 \in \mathcal{L}(Z)$ and the Lemma follows.*

Step 3.

Proposition 4.4 *With $P(=\widehat{P})$ as in Theorem 4.1, consider $x(t) = e^{A_F t}x_0, x_0 \in Z$, as guaranteed by Lemma 4.3. Then, the following identity holds true, where $T > 0$ is arbitrary:*

$$(Px(T), x(T))_Y + \int_0^T \|Rx(t) + RB_1 G_0 x(t)\|_Y^2 dt + \int_0^T \|G_0 x(t)\|_U^2 dt = (Px_0, x_0)_Y. \tag{4.14}$$

Proof Consider $x(t) = e^{A_F t}x_0, x_0 \in \mathcal{D}(A_F) \subset Z$. Then, by (3.5)

$$\frac{d}{dt}(Px(t), x(t))_Y = (PA_F x(t), x(t))_Y + (Px(t), A_F x(t))_Y \tag{4.15}$$

$$= 2\mathrm{Re}\left\{([A + \mathcal{B}G_0]x(t), Px(t))_Y\right\} \tag{4.16}$$

$$= 2\mathrm{Re}\left\{(Ax(t), Px(t))_Y + (G_0 x(t), \mathcal{B}^* P x(t))_U\right\} \tag{4.17}$$

since $P = P^*$, recalling also (4.13).

Next, on the RHS of (4.17), we invoke identity (4.2) of Proposition 4.2. We obtain

$$\frac{d}{dt}(Px(t), x(t))_Y + \|Rx(t) + RB_1 G_0 x(t)\|_Y^2 + \|G_0 x(t)\|_U^2 \equiv 0. \tag{4.18}$$

Integrating (4.18) over $[0, T]$ yields (4.14), first for $x_0 \in \mathcal{D}(A_F)$, next for $x_0 \in Z$ by density. □

Step 4.

Corollary 4.5 *In the assumptions of Proposition 4.4 (i.e. Theorem 4.1) we have for* $x(t) = e^{A_F t} x_0, x_0 \in Z$

$$\int_0^\infty \|G_0 x(t)\|_U^2 dt \le (Px_0, x_0)_Y \le C\|x_0\|_Z^2, \quad x_0 \in Z, \tag{4.19}$$

where $C = \|P\|$, *the norm of* $P : Z \equiv [\mathcal{D}(A^{*2})]' \to \mathcal{D}(A^{*2})$ *as in* (2.14).

Proof In (4.14) we drop the first two positive terms on the LHS of (4.14), and let $T \to +\infty$ in the third integral term. On the RHS we estimate

$$(Px_0, x_0)_Y \le \|Px_0\|_{\mathcal{D}(A^{*2})} \|x_0\|_Z \le C\|x_0\|_Z^2. \tag{4.20}$$

□

Step 5.

This is the final step in the proof of Theorem 4.1. We return to equation (3.5):

$$\dot{x}(t) = A_F x(t) = (A + \mathcal{B}G_0)x(t) = Ax(t) + \mathcal{B}G_0 x(t) \tag{4.21}$$

or

$$x(t) = e^{A_F t} x_0 = e^{At} x_0 + \int_0^t e^{A(t-s)} \mathcal{B} G_0 x(s) ds. \tag{4.22}$$

We claim that

$$\int_0^\infty \|e^{A_F t} x_0\|_Z^2 dt \le C\|x_0\|_Z^2 \quad \text{for any } x_0 \in Z, \tag{4.23}$$

after which Datko's theorem (Datko, 1970) yields that the s.c. semigroup $e^{A_F t} (= e^{\widehat{A}_F t}, \text{i.e.} P = \widehat{P})$ is uniformly stable as asserted in (4.2). Claim (4.23) follows readily since e^{At} is uniformly stable on Z by (2.9) of Theorem 2.2; combined with $\mathcal{B} \in \mathcal{L}(U; Z)$ and (4.19) of Corollary 4.5:

$$\begin{aligned} &\left\| \int_0^t e^{A(t-s)} \mathcal{B} G_0 x(s) ds \right\| \\ &\le C \left\{ \int_0^t e^{-2\omega(t-s)} ds \right\}^{\frac{1}{2}} \|\mathcal{B}\|_{\mathcal{L}(U;Z)} \left\{ \int_0^\infty \|G_0 x(s)\|_U^2 ds \right\}^{\frac{1}{2}} \le \text{const} \|x_0\|_Z^2. \end{aligned} \tag{4.24}$$

Thus, Theorem 4.1 is established. □

4.2. Completion of the proof of uniqueness of Theorem 4.1

Step 1. Let P_1 and P_2 be two positive self-adjoint solutions of the ARE (2.21), satisfying the regularity properties of Theorem 2.3. Thus, for $x, y \in Y$, recalling (3.4b),

$$\begin{aligned}&(Ax, P_i y)_Y + (P_i x, Ay)_Y + (Rx, Ry)_Y = \\ &= \left(B_1^* R^* Rx + \mathcal{B}^* P_i x, T_0^{-1}[B_1^* R^* Ry + \mathcal{B}^* P_i y]\right)_U, \quad i = 1, 2. \end{aligned} \tag{4.25}$$

Define $Q = P_1 - P_2$, so that $Q = Q^*$ on Y satisfies the corresponding regularity properties of Theorem 2.3. Subtract (4.2) for $i = 2$ from (4.2) for $i = 1$. We obtain after cancellation of two terms

$$\begin{aligned}&(Ax, Qy)_Y + (Qx, Ay)_Y \\ &= \left(B_1^* R^* Rx, T_0^{-1}(\mathcal{B}^* Qy)\right)_U + \left(\mathcal{B}^* Qx, T_0^{-1}(B_1^* R^* Ry)\right)_U && (4.26a)\\ &+ \left(\mathcal{B}^* P_1 x, T_0^{-1}(\mathcal{B}^* P_1 y)\right)_U - \left(\mathcal{B}^* P_2 x, T_0^{-1}(\mathcal{B}^* P_2 y)\right)_U. && (4.26b)\end{aligned}$$

As to the last line, (4.26b), we compute, after adding and subtracting,

$$\begin{aligned}&(4.26b) = \\ &\left(\mathcal{B}^* P_1 x, T_0^{-1}(\mathcal{B}^* P_1 y) + T_0^{-1}(\mathcal{B}^* P_2 y) - T_0^{-1}(\mathcal{B}^* P_2 y)\right)_U - \left(\mathcal{B}^* P_2 x, T_0^{-1}(\mathcal{B}^* P_2 y)\right)_U && (4.27)\\ &= \left(\mathcal{B}^* Qx, T_0^{-1}(\mathcal{B}^* P_2 y)\right)_U + \left(\mathcal{B}^* P_1 x, T_0^{-1}(\mathcal{B}^* Qy)\right)_U. && (4.28)\end{aligned}$$

Using (4.28) in (4.26) yields

$$\begin{aligned}(Ax, Qy)_Y + (Qx, Ay)_Y &= \left(B_1^* R^* Rx, T_0^{-1}(\mathcal{B}^* Qy)\right)_U + \left(\mathcal{B}^* Qx, T_0^{-1}(B_1^* R^* Ry)\right)_U \\ &+ \left(\mathcal{B}^* Qx, T_0^{-1}(\mathcal{B}^* P_2 y)\right)_U + \left(\mathcal{B}^* P_1 x, T_0^{-1}(\mathcal{B}^* Qy)\right)_U. && (4.29)\\ &= \left(B_1^* R^* Rx + \mathcal{B}^* P_1 x, T_0^{-1}(\mathcal{B}^* Qy)\right)_U \\ &+ \left(\mathcal{B}^* Qx, T_0^{-1}(B_1^* R^* Ry) + T_0^{-1}(\mathcal{B}^* P_2 y)\right)_U. && (4.30)\end{aligned}$$

In the first inner product term in (4.30), we use the fact that the operator T_0^{-1} (see (3.4b)) is self-adjoint on U, while the last two inner product terms [????], we invoke the definition $G_i = -T_0^{-1}[B_1^* R^* R + \mathcal{B}^* P_i], i = 1, 2$, as in (4.1). We obtain

$$(Ax, Qy)_Y + (Qx, Ay)_Y = -(G_1 x, \mathcal{B}^* Qy)_U - (\mathcal{B}^* Qx, G_2 y)_U, \quad x, y \in Y \tag{4.31}$$

where we recall that then: $Qx, Qy \in \mathcal{D}(A^*)$, $G_1 x, G_2 y \in U$, $\mathcal{B}^* Qx, \mathcal{B}^* Qy \in U$; or

$$((A + \mathcal{B}G_1)x, Qy)_Y + (Qx, (A + \mathcal{B}G_2)y)_Y = 0; \tag{4.32}$$

ultimately

$$\begin{aligned}&(A_{F_1} x, Qy)_Y + (Qx, A_{F_2} y)_Y = 0, \ x \in \mathcal{D}(A_{F_1}), \ y \in \mathcal{D}(A_{F_2}), \\ &\mathcal{D}(A_{F_i}) = [\mathcal{D}(A^*)]' \end{aligned} \tag{4.33}$$

with $A_{F_i} = A + \mathcal{B}G_i, G_i = -T_0^{-1}[B_1^* R^* R + \mathcal{B}^* P_i], i = 1, 2$, the infinitesimal generators (Lemma 4.3) of s.c. semigroups $e^{A_{F_1} t}$ and $e^{A_{F_2} t}$, both uniformly stable by Theorem 4.1. Equation (4.2), where $Q = Q^*$, resembles (but is not) a Lyapunov equation, where a critical feature is that its RHS is zero.

Equation (4.2) is well-defined for $x \in \mathcal{D}(A_{F_1})$ and $y \in \mathcal{D}(A_{F_2})$. Then we re-write (4.2) for $x = e^{A_{F_1} t} x_0, x_0 \in \mathcal{D}(A_{F_1})$ and $y = e^{A_{F_2} t} y_0, y_0 \in \mathcal{D}(A_{F_2})$. Thus, we obtain

$$0 \equiv \left(A_{F_1} e^{A_{F_1} t} x_0, Q e^{A_{F_2} t} y_0\right)_Y + \left(Q e^{A_{F_1} t} x_0, A_{F_2} e^{A_{F_2} t} y_0\right)_Y \tag{4.34}$$

$$= \frac{d}{dt} \left(e^{A_{F_1} t} x_0, Q e^{A_{F_2} t} y_0\right)_Y, \quad \text{for any } t \geq 0. \tag{4.35}$$

Integrating (4.35) over $[0, T]$ yields

$$\left(e^{A_{F_1} T} x_0, Q e^{A_{F_2} T} y_0\right)_Y - (x_0, Q y_0)_Y = 0, \quad \text{for any } T > 0. \tag{4.36}$$

Letting $T \to \infty$ and recalling the uniform stability of $e^{A_{F_1} t}$ and $e^{A_{F_2} t}$ (Theorem 4.1) yields

$$\begin{aligned} &(x_0, Q y_0)_Y = 0, x_0 \in \mathcal{D}(A_{F_1}), y_0 \in \mathcal{D}(A_{F_2}) \\ &\qquad \Longrightarrow (x_0, Q y_0)_Y = 0, \text{ for any } x_0, y_0 \in Y \Longrightarrow Q = 0, \text{as desired.} \end{aligned} \tag{4.37}$$

5. Final remark

In the present paper we take throughout the constant $\ell > 0$ (Robin control), in which case the homogeneous problem (2.1a)-(2.1d) with $g \equiv 0$ is uniformly stable (Theorem 2.2). In a subsequent paper we intend to extend to results of Lasiecka and Triggiani (2022) and of the present paper to the case $\ell = 0$ (Neumann control). To this end, we intend to pursue the approach, proposed in Barbu, Lasiecka and Triggiani (2006), Appendix C, for an abstract system modeling more traditional boundary control problems as in Lasiecka and Triggiani (2000). In the context of the present paper, this means taking $B_1 = 0$ in model (2.6). Though such Appendix is introduced with reference to the linearized, 3D-Navier-Stokes equations, the assumption of analyticity (parabolicity) of the s.c. semigroup, generated by the free dynamic operator (A in (2.6)), is *not* needed. Thus, within the context of the abstract model $\dot{y} = Ay + Bg$, such Appendix provides a useful *equivalence* in the study of optimal control problems with quadratic cost functionals: between an original *unstable* dynamics, assumed to satisfy, of course, the Finite Cost Condition (FCC) and a suitable *stable* version, which then a-fortiori satisfies the FCC. Thus, without loss of generality, the former *unstable* problem may be reduced to the latter *stable* one, where then a more streamlined and, above all, explicit treatment is possible.

In the setting of the present paper, the original unstable dynamics is now

$$\text{problem (2.1a)–(2.1d), with } \ell = 0 \text{ and } g = 0. \tag{5.1}$$

For such a problem as in (5.1), we need to verify that it satisfies the FCC. In fact, to this end, we replace (5.1) with

$$\begin{array}{l} \text{problem (2.1a), (2.1b), (2.1d), with (2.1c) replaced by} \\ \qquad \partial_\nu u = g = -u \qquad \text{on } \Sigma_0 \equiv \Gamma_0 \times (0,\infty). \end{array} \tag{5.2}$$

Such control g as in (5.2) produces the required FCC for the original problem (5.1). Thus, the key task to be addressed is to extend the proof(s) of Barbu, Lasiecka and Triggiani (2006), Appendix C, to the present more pathological model (2.6) with $B_1 \neq 0$; more precisely, with the triple $\{A, B_0, B_1\}$ satisfying (2.7a)-(2.7b).

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