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The problem on the shape of a pile of granular substance

by

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Abstract: The problem of the increase of an ideal granular substance pile is considered. By using the variational formulation of the problem the existence of the solution in weak sense is proved. For the case of the small angle of gradient of the initial surface two numerical methods of solving are proposed. First one is based on the classical lagrangian technique, in the second the Herskovits' interior point algorithm is used. Numerical examples for 1-D and 2-D cases are presented.

Keywords: granual substance, evolutionary variational inequality, Lagrangian, interior point algorithm.

1. Formulation of the problem

Let us assume that a substance spreads over a surface of the pile $u(x, t)$, $x \in \Omega \subset R^2$ with a thin layer, the lower layers remain immovable and the particles have no mass inertia. We denote by $(x) = u(x, 0)$ the shape of the initial surface, a is the substance angle of repose. Formation of the pile in $[0, T]$ is described by the following relations (Prigozhin, 1986, 1993):

- state equation:

$$u' - \nabla(m\nabla u) = f \quad \text{in} \quad (0, T) \times 0, \tag{1}$$

  where $f(x, t)$ is the intensity of the external source, $f(x, t) \quad 0$, $m(x, t) \quad 0$ is an unknown scalar function, $u' = \frac{\partial u}{\partial t}$;
- boundary conditions:

$$u(x, 0) = , \qquad u(x, t)\Big|_{\partial n} = 0; \tag{2}$$

e surface conditions:

$$\begin{cases} v.(x, t) \quad -, \\ u(x, t) > \text{--+} \quad \text{IVv.(x, t)I::::; } 1 = \text{ltgal}, \\ \text{IVu(.'E, t)I} < 1 \quad \text{-+} \; m(x, t) = 0. \end{cases} \tag{3}$$

The problem is to determine the shape of the pile $v.(x, t)$ E (0, T) x O with given a, (x) and $f(x, t)$.

2. Variational formulation

Let O be a bounded subset of R^2. We define

$$V = HJ(O),\ V = L^2(0,T;V),\ H = L^{00}(0),\ H = L^{00}(0,T;H),$$

where $HJ(0)$ is the Sobolev space of distributions having a zero trace at the boundary 80. We denote by V' the dual space of V.

Assume that there exists u being a solution of (1)-(3) such that v. E V, u' E V', m E 'H. Then, problem (1) - (3) cari be written in the following form:

$$u' - V(mVu) = f \text{ in } 0, \quad u(x, 0) = \text{ft}, \quad u(x, t)I_{an} = 0, \tag{4}$$

$$(r^2(v.) - \text{IVv,l}^2, m) = 0, \tag{5}$$

$$r^2(u) - 1Vul^2 \quad 0, \quad m \quad 0, \quad u(E, t) \quad -, \tag{6}$$

where

$$\Gamma(u) = \begin{cases} |\nabla \overset{\circ}{u}| & \text{for} \quad u(x, t) = ., \\ \gamma & \text{for} \quad v.(x, t) > .. \end{cases}$$

Let us define the following function sets:

$$K_{*} = \{ v \text{ E } V, vJ_{80} = 0 \ \text{I} v(x) \quad (x),\ 1Vv(x)J^2 \ \text{=;} \ \text{r2} (u) \},$$

$$Ku = \{ V \text{ E } VI \quad v(t) \text{ E } Ku,\ t \text{ E } [O, T], \},$$

$$V = \{ v' \text{ E } V',\ v(0) = \}$$

Multiplying equation (4) by $(u - v.)$, $\forall u$ $EK.$ and using relation (5) and the following estimation:

$$-(\ V(mVu), u - u) = (mvu, Vu - Vu) \ \text{=;}$$

$$\text{=;} \ (m, \text{JVulJVul} - 1Vul^2) \ \text{=;} \ (m, r^2(u) - \text{JVuJ}^2) = 0,$$

we obtain the quasi-variational formulation of our problem:

$$\left(u', \bar{u} - u\right) \geq \left(f, \bar{u} - u\right), \quad u \in \mathcal{K}_u \cap \mathcal{D}, \ \forall \bar{u} \in \mathcal{K}_u. \tag{7}$$

In other hand, the solution of quasi-variational inequality (7) is a solution of boundary problem (1) - (3) (Prigozhin, 1986).

3. Existence of the solution

Let us assume that IV t, I ≤ Y, when problem (7) can by written as a variational inequality. We shall consider this inequality in the weak sense:

$$\left(\bar{u}', \bar{u} - u\right) \geq \left(f, \bar{u} - u\right), \quad u \in \mathcal{K}_o, \forall \bar{u} \in \mathcal{K}_o \cap \mathcal{D}, \tag{8}$$

where

$$K_o = \{ v \in V, v|_{\partial O} = 0 \mid v(x) \geq \psi(x), \ |v'(x)|^2 \leq \gamma^2 \},$$

$$\mathcal{K}_o = \{ v \in V \mid v(t) \in K_o, \ t \in [0, T], \},$$

$$V = \{ v' \in V', \ v(0) = \varphi \}.$$

We shall approximate the solution $u(x, t)$ of (8) by

$$v_{(k)}(x, t) = v_n(x) \quad \text{for } t \in [nk, (n+1)k],$$

where $v_n(x)$ is defined by the inequality:

$$\left(\frac{u^n - u^{n-1}}{k}, \bar{v} - v^n\right) \geq \left(f^n, \bar{u} - u^n\right), \quad \forall \bar{v} \in K_o, \tag{9}$$

with

$$k = \Delta t = T/M, \quad u^0 = \varphi, \quad f^n = \int_{nk}^{(n+1)k} f(\theta)\, d\theta, \quad n = 1, \ldots, M-1,$$

or, equivalently, by the solution of the following minimization problem:

$$\inf_{u^n \in K_o} J^k(u^n), \tag{10}$$

where

$$J^k(u^n) = \left(\frac{1}{2k} u^n - f^n - \frac{1}{k} u^{n-1}, u^n\right).$$

In turn, problem (10) has a unique solution and the sequence $u^{(k)}(x, t)$ converges to a solution of evolution inequality (8) when k tends to 0 (Glowinski, Lions, Tremolieres, 1976).

We propose here two numerical methods of solving this problem. The first one is based on the classical Lagrangian algorithm (Ekeland, Temam, 1976). The second one use the Herskovits' interior point techniques (Herskovits, 1986, 1993).

4. The Lagrangian method

Let us define the Lagrangian:

$$\mathcal{L}^k(u,p) = J^k(u) + (p, |\nabla u|^2 - \gamma^2)$$

on $V \times H^+$, $H^+ = \{p \in H | p \geq 0 \text{ a.e. in } D\}$.

There exists a saddle point of $\mathcal{L}^k$, the first component of this point is uniquely determined and it is the only solution of problem (10) (Ekeland, Temam, 1976).

We use the Uzawa method to compute the saddle point of $\mathcal{L}^k$ (Ekeland, Temam, 1976):

- let us give p^0 in H^+;
- with $p^s \in H^+$ being given, find u^s solution of
$$\inf_{u^s \in V} \mathcal{L}^k(u^s, p^s); \quad (11)$$
- set p^{s+1} as:
$$p^{s+1} = (p^s + \rho^s(|\nabla u^s|^2 - \gamma^2))^+, \quad \rho^s > 0. \quad (12)$$

There exists such interval $0 < \alpha < \rho^s < \beta$ that the Uzawa algorithm converges to the saddle point of $\mathcal{L}^k$ when $s \to \infty$:

$$u^s \to u_k \quad \text{strongly in } L^2(\Omega),$$

5. Herskovits' interior point algorithm

By performing the space discretization of problem (10) we arrive at the following mathematical programming problem with non-linear inequality constraints:

$$\begin{cases} \text{minimize} & J(u_h) \\ \text{subject to} & g(u_h) \leq 0. \end{cases} \quad (13)$$

Here the vector $u_h \in R^n$ is the discretization of $u(x)$, $J \in R$ is the discretization of the functional J^k, and function $g_h \in R^m$ describes all inequality constraints.

To simplify the notations we omit in what follows the index k and h. The Karush-Kuhn-Tucker first order optimality conditions of problem (13) can be expressed as follows:

$$\nabla J(u) + \nabla g(u)\lambda = 0, \quad (14)$$

G(v,)>. = 0, (15)

g(v,) :S 0,

$\lambda \geq 0$,

where G(U,) denotes a diagonal matrix such that Gii(U,) = 9i(u).

The Lagrangian of the problem is ,C(v„ A) = J(u) + A^{t}g(u), and its second derivative becomes H(V„ A) = V^{2}J (v,) + $\sum_{i=1}^{m}$ VV^{2}gi (v,).

A Newton iteration for the solution of (14), (15) is defined by the following system:

$$\begin{bmatrix} B & Vg(v,) \\ \Lambda\nabla g^t(u) & G(v,) \end{bmatrix} \begin{bmatrix} u_o - u \\ A_o - A \end{bmatrix} = - \begin{bmatrix} VJ(v,) + Vg(v,)A \\ G(u)A \end{bmatrix}$$

with the matrix B taken equal to a Newton estimate of $H(u,$ A). Here, A is a diagonal matrix with Aii = Ai,

The Herskovits' interior point algorithm solves iteratively the Karush-Kuhn-Tucker system of the minimization problem with inequality constraints, not necessarily convex (Herskovits, 1986, 1993). Given an initial point at the interior 6. of the inequality constraints, a sequence of interior points is obtained. At each iteration, a feasible descent direction is defined and a line search in this direction is performed in order to obtain a new interior point with a reduction of the objective.

The algorithm is defined as follows:

Parameters. a E (0, 1), 77E (0, 1), P > 0 and v E (0, 1).

Data. v, E 6., A > 0, B E R$^{r \times r}$ symmetric and positive definite and positive w E R^m.

Step 1. Computation of a search direction.

(i) Compute d_o and A_o by solving the linear system

$$Bd_o + Vg(v,)A_o = -Vf(u),$$

$$AVl(v,)d_o + G(v,)A_o = 0.$$

If d_o = 0, stop.

(ii) Compute d_l and A_l by solving the linear system

$$Bd_l + Vg(v,)A1 = 0,$$

$$AVl(u)d1 + G(v,)A1 = -Aw.$$

(iii) If df'vJ(u) > 0, set

$$P= \min[ip \| db \|^2; (a - 1)d 'vJ(u)/di'vJ(v,)].$$

Otherwise, set

$$P= IP\| db \|^2 .$$

(iv) Compute the search direction

$$d=do+pd1,$$

and also

$$\bar{\lambda} = \lambda_o + \rho\lambda_1 .$$

Step 2. Line search.

Compute t the first number of the sequence $\{1, v, v2, v^3, ...\}$ satisfying

$J(v.+td)::;$ J(u)+tryd^t'vJ (u)

and

gi(u + td) < 0 if >i 0,

or

gi(u + td)::; gi(v.)

otherwise.

Step 3. Updates.

(i) Set

$$v := u+td$$

and define new values for

$$w>0,$$

$$\lambda > 0$$

and

$$B$$

symmetric and positive definite.

(ii) Go to Step 1.

The global convergence of the algorithm is proved in Herskovits (1986, 1993).

The present method is simple to code, strong and efficient. It does not involve penalty functions, active set strategies or quadratic programming subproblems. It merely requires the solution of two linear systems with the same matrix at each iteration, followed by an inexact line search. In practical applications, advantage of the structure of the problem and particularities of the functions in it, can be taken of in order to improve calculus efficiently.

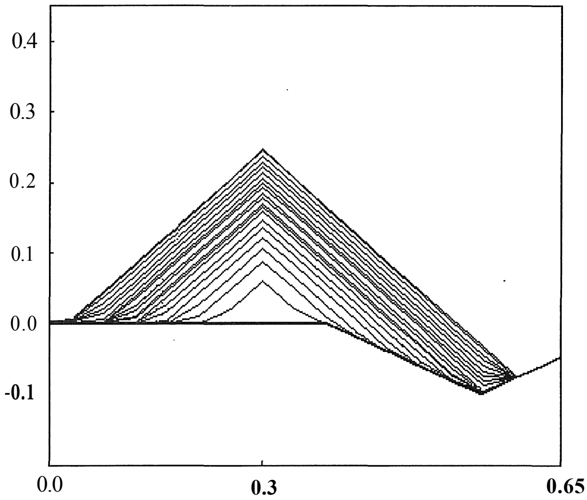

Figure 1.

6. Numerical results

For the test examples below both of the presented algorithms were used. In the case of the interior point algorithm the numerical code developed by Professor J. Herskovits was applied.

The space discretization was performed by the exterior approximation method (Glowinski, Lions, Tremolieres, 1976). Moreover, in the Lagrangian algorithm the point relaxation method to solve problem (11) was used.

As it was expected, the interior point technique proved to be more efficient for this kind of problem. In each time step 5-10 iterations were required to obtain the numerical solution by the interior point algorithm instead of 100-200 iterations when using the Lagrangian technique. We present here 1-D example calculated by the interior point algorithm and 2-D examples obtained by the Lagrangian techniques.

Example 1 *(1-D case) We take* O = [0, 1] *and the initial surface having inclined parts. The sv,bstance angle of repose* a = 45°. *The space discretization was performed with the step* $h = 0.02$. *In the time discretization we have* /::it = 0.01. *The external source of intensity* f = 10 *was taken at the point* $x = 0.3$. *The form of the pile for* 20 (I' = 0.2) *time iterations is presentedin Fig. 1.*

Example 2 *(2-D case) We choose* O = [0, 1] x [0, 1]. *In space discretization we have* $h = 1/60$ *and in time discretization we have* /::it = 0.01. *Two sources of intensity* J i = 2.0 *and* h = 1.0 *are given at the points* (1/2, 1/2) *and* (2/3, 2/3),

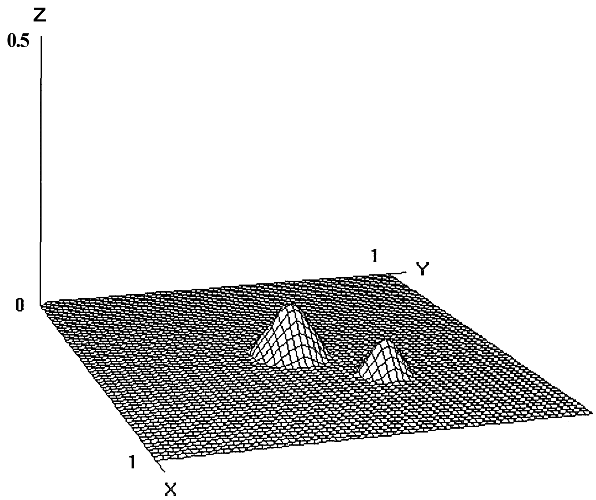

Figure 2.

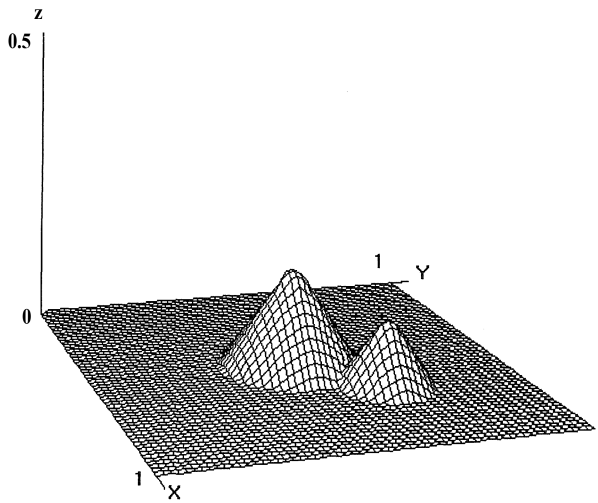

Figure 3.

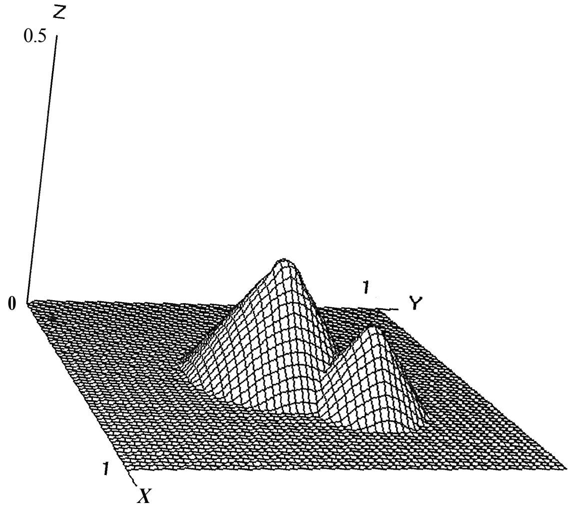

Figure 4.

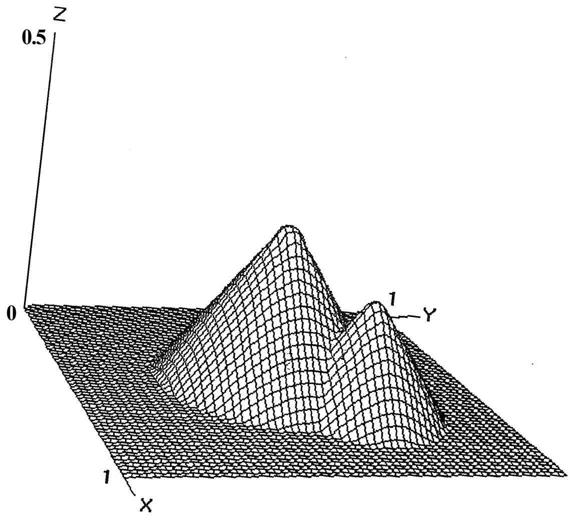

Figure 5.

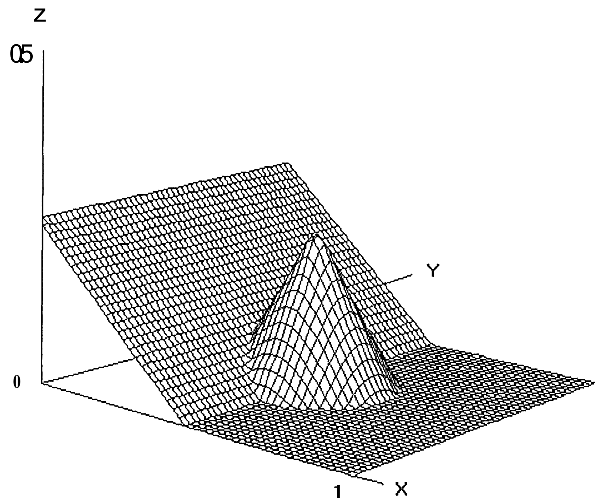

Figure 6.

respectively. The angle of repose of the sv,bstance is $a = 45^\circ$. *Fig. 2 presents the form of the pile at the time* $T = 100 \times t$; *Fig. 3,* $T = 500 \times t$; *Fig. 4,* $T = 1000 \times t$; *Fig. 5,* $T = 2000 \times t$.

Example 3 *(2-D case) We choose* $D = [0,1] \times [0,1]$. *In space discretization we have* $h = 1/50$ *and in time discretization we have* $t = 0.01$. *The source of intensity* $f = 3.0$ *is given at the point* $(1/2, 1/2)$. *The angle of repose is* $a = 45^\circ$. *The angle of gradient of the initial sv,rface is* $a/2$. *The form of the pile at the time* $T = 500 \times t$ *is presented in Fig. 6.*

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