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Sufficient approximate optimality condition for the inverse one-phase Stefan problem*

by

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Abstract: We investigate the identification problem for the one-phase Stefan problem. As the inverse Stefan problem is not well posed, an optimal control problem is considered instead. In the paper we develop a dual dynamic programming approach to derive sufficient approximate optimality conditions for that optimal control problem. As a next step we formulate and prove a verification theorem for approximate solution. The verification Theorem 4.1 is the basis for the development of a numerical algorithm. Having the verification theorem we do not need the convergence of our algorithm.

Keywords: sufficient approximate optimality condition, inverse one-phase Stefan problem, computational algorithm, verification theorem

1. Introduction

In the series of papers, Abdulla (2013, 2016), Abdulla, Cosgrove and Goldfarb (2017), Budak and Vasileva (1972, 1973, 1974), Goldman (1997), the inverse problems to the one-phase Stefan problem:

$$(au_x)_x + bu_x + cu - u_t = f, \text{ in } \Omega, \tag{1.1}$$

$$u\,(x, 0) = \phi(x), \; 0 \le x \le s(0) = s_0, \tag{1.2}$$

$$a(0, t)u_x(0, t) = g(t), \; 0 \le t \le T, \tag{1.3}$$

$$a(s(t), t)u_x(s(t), t) + \gamma(s(t), t)s'(t) = \chi(s(t), t),$$

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$$0 \le t \le T, \tag{1.4}$$

$$u(s(t),t) = \mu(t), \; 0 \le t \le T, \tag{1.5}$$

where

$$\Omega = \{(x,t) : 0 < x < s(t), \; 0 < t \le T\}, \tag{1.6}$$

$a, b, c, f, \phi, g, \gamma, \chi, \mu$ are given functions, were investigated. The inverse Stefan problem **(ISP)** is to find a tuple

$$\{u, a, b, c, f, g, s\} \tag{1.7}$$

that satisfy conditions (1.1)-(1.6) and

$$u(x,T) = w(x), \; 0 \le x \le s(T) = s_*,$$

where $w(x)$, a final moment temperature measurement, and s_*, a final moment ablation depth, are given. We also find in those papers the motivations to study **(ISP).** The unknown parameters of the model, such as a, b, c, f, g are very difficult to measure by experiments. Lab experiments pursued on the laser ablation of biological tissues allow to measure the final temperature distribution and final ablation depth. The aim of **(ISP)** is to achieve the identification of some or all of the unknown parameters a, b, c, f, g.

It is worth to note that **(ISP)** is not well posed in the sense of Hadamard (see, e.g., Goldman, 1997), i.e. in general, the solution may not exist; if it exists it may not be unique and we cannot demonstrate continuous dependence on the data. Just those issues caused that earlier variational methods, applied to **(ISP)**, failed and a new method was developed in Abdulla (2013, 2016). That new approach treats the tuple (1.7) as controls and for the quadratic cost functional the optimal control problem is formulated in suitable spaces of functions. In Abdulla (2013, 2016) existence of the optimal control and convergence of the sequence of discrete optimal control problems to the continuous optimal control problem was proven. Next, in Abdulla and Goldfarb (2024) the Fréchet differentiability of the functional and the existence of optimal control were proven in Besov spaces, and also the necessary optimality conditions were presented. We need to stress that just the existence of solutions to (1.1)-(1.6) requires that the coefficients $(a, b, c, f, \phi, g, \gamma, \chi, \mu)$ belong to suitable Besov spaces (see Goldman, 1997).

The aim of this paper is to derive for the control problem from Abdulla, Cosgrove and Goldfarb (2017) sufficient optimality conditions for approximate minimum in the form of a verification theorem and then a construction of a numerical algorithm, allowing to calculate with given $\varepsilon > 0$ an approximate minimum. It is a continuation and extension of the method presented in Lipnicka and Nowakowski (2018b). The precise formulation will be given in the next section, after the formulation of the optimal control problem. Concerning notations of spaces of functions as well as the control problem we follow the paper by Abdulla, Cosgrove and Goldfarb (2017). However, we want to stress that because we are not interested in the existence of the optimal control problem solution, as well as Fréchet differentiability, we weakened some restrictive assumptions from Abdulla, Cosgrove and Goldfarb (2017), concerning the set of controls.

2. The preliminaries and the control problem

2.1. The preliminaries

Denote by U a domain in $\mathbb{R}$ and by $Q_T = (0,1) \times (0,T]$. For $l \in \mathbb{Z}_+$, by $W_p^l(U)$ we mean a Banach space of measurable functions with finite norm

$$\|u\|_{W_p^l(U)} = \left(\int_U \sum_{k=0}^{l} \left| \frac{d^k u}{dx^k} \right|^p dx \right)^{1/p},$$

for $l \notin \mathbb{Z}_+$, and by $B_p^l(U)$ we mean a Banach space of measurable functions with finite norm

$$\|u\|_{B_p^l(U)} = \|u\|_{W_p^l(U)} + [u]_{B_p^l(U)}$$

where

$$[u]_{B_p^l(U)} = \int_U \int_U \frac{\left| \frac{\partial^{[l]} u(x)}{\partial x^{[l]}} - \frac{\partial^{[l]} u(y)}{\partial x^{[l]}} \right|^p}{|x-y|^{1+p(l-[l])}} dxdy.$$

If $l \in \mathbb{Z}_+$, the seminorm $[u]^p_{B_p^l(U)}$ is given by

$$[u]_{B_p^l(U)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\left| \frac{\partial^{l-1} u(x)}{\partial x^{l-1}} - 2\frac{\partial^{l-1} u(\frac{x+y}{2})}{\partial x^{l-1}} + \frac{\partial^{l-1} u(y)}{\partial x^{l-1}} \right|^p}{|x-y|^{1+p}} dydx.$$

For $1 \leq p < \infty$, $0 < l_1, l_2$, the Besov space $B_{p,x,t}^{l_1,l_2}(Q_T)$ is defined as the closure of the set of smooth functions under the norm

$$\|u\|_{B_{p,x,t}^{l_1,l_2}(Q_T)} = \left(\int_0^T \|u(x,t)\|^p_{B_p^{l_1}[0,1]} dt \right)^{1/p} + \left(\int_0^1 \|u(x,t)\|^p_{B_p^{l_2}[0,1]} dx \right)^{1/p}.$$

The Hölder space $C_{x,t}^{\alpha,\alpha/2}(Q_T)$ is the set of continuous functions with $[\alpha]$ x-derivative and $[\alpha/2]$ t-derivatives, and for which the highest order x-derivatives satisfy Hölder conditions of order $\alpha - [\alpha]$ and $\alpha/2 - [\alpha/2]$, respectively. Put $D = \{(x,t) : 0 \le x \le l,\ 0 \le t \le T\}$, where l is always chosen in such a way that $s(t)$, occurring in (1.2), (1.4), satisfies $s(t) \le l$. Define, for fixed $0 < \alpha < \alpha^*$,

$$\begin{aligned} H =& C_{x,t}^{3/2+2\alpha^*,3/4+\alpha^*}(D) \times C_{x,t}^{1/2+2\alpha^*,1/4+\alpha^*}(D) \times \\ & C_{x,t}^{1/2+2\alpha^*,1/4+\alpha^*}(D) \times B_{2,x,t}^{1,1/4+\alpha}(D) \times B_2^{1/2+\alpha}[0,T] \times B_2^2[0,T] \end{aligned} \tag{2.1}$$

with the norm

$$\begin{aligned} \|(a,b,c,f,g,s)\| = \max\Big(& \|a\|_{C_{x,t}^{3/2+2\alpha^*,3/4+\alpha^*}(D)}, \|b\|_{C_{x,t}^{1/2+2\alpha^*,1/4+\alpha^*}(D)}, \\ & \|c\|_{C_{x,t}^{1/2+2\alpha^*,1/4+\alpha^*}(D)}, \|f\|_{B_{2,x,t}^{1,1/4+\alpha}(D)}, \|g\|_{B_2^{1/2+\alpha}[0,T]}, \|s\|_{B_2^2[0,T]} \Big). \end{aligned}$$

2.2. The control problem

The optimal control problem that we investigate is the following one: for given $\beta_0, \beta_1, \beta_2 \ge 0$

minimize the functional

$$\begin{aligned} J(v) =& \beta_0 \int_0^{s(T)} |u(x,T;v) - w(x)|^2 \, dx \\ & + \beta_1 \int_0^T |u(s(t),t;v) - \mu(t)|^2 \, dt + \beta_2 |s(T) - s_*|^2 \end{aligned} \tag{2.2}$$

over the control set $\mathbf{V} = \{v = (a,b,c,f,g,s) \in H\}$ subject to the state constraints

$$(au_x)_x + bu_x + cu - u_t = f, \text{ in } \Omega, \tag{2.3}$$

$$u(x,0) = \phi(x),\ 0 \le x \le s(0) = s_0, \tag{2.4}$$

$$a(0,t)u_x(0,t) = g(t),\ 0 \le t \le T, \tag{2.5}$$

$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t),\ 0 \le t \le T. \tag{2.6}$$

We would like to stress that the above control problem differs from that considered in Abdulla, Cosgrove and Goldfarb (2017) in the definition of the control set $\mathbf{V}$. We do not impose any restrictions on the set $\mathbf{V}$, as this is done in Abdulla, Cosgrove and Goldfarb (2017). In that paper, Fréchet differentiability is studied and, as a corollary, the necessary optimality conditions are derived. Those investigations require well-posedness of the Neumann problem (1.1)-(1.4). We deal with sufficient optimality of first order and well-posedness of (1.1)-(1.4) is not essential. This is also the reason why we have to define a solution to (2.3)-(2.6). For given $v \in \mathbf{V}$ we call a function $u \in W_2^{2,1}(\Omega)$ (Sobolev space of functions) a solution to (2.3)-(2.6) if it satisfies them pointwise and almost everywhere. The set of pairs (v, u), satisfying, (2.3)-(2.6), is called admissible and denoted by Ad. We shall assume that the values of all controls $v \in \mathbf{V}$ are contained in a set $\mathbb{V} = \mathbb{A} \times \mathbb{B} \times \mathbb{C} \times \mathbb{F} \times \mathbb{G} \times \mathbb{S} \subset \mathbb{R}^6$, $\mathbb{S} = [s_0, s^*]$, $s_0 > 0$. We do not assume that $\mathbb{V}$ is closed or that it has nonempty interior. However, boundedness of $\mathbb{V}$ is to some extent a necessary and natural assumption and we introduce it because of the dynamic programming inequality.

3. ε-optimality

We follow the same dual approach as in Lipnicka and Nowakowski (2018a) for constructing the ε-dual dynamic programming theory for the ε-optimal problem (compare Lipnicka and Nowakowski, 2022a,b, 2023). The main idea of dual dynamic programming is to extend the primal space Ω to a larger space and carry over all calculations, related to dynamic programming, to that new space, including value function. The essential point in the dual dynamic programming approach is that we do not deal directly with a value function, but with some auxiliary function, defined in a dual set, satisfying ε-dual dynamic inequality and then we derive sufficient ε-optimality conditions for a primal ε-value function.

Let $\boldsymbol{P} \subset \mathbb{R}^2$ be an open set of the variables $p = (y^0, y)$, $y^0 < 0$, $y \in \mathbb{R}$. Let $P \subset \mathbb{R}^4$ be an open set of the variables (t, x, p), $(x, t) \in \Omega$, $p \in \boldsymbol{P}$, i.e.

$$P = \{(t, x, p) \in \mathbb{R}^4 : \ (x, t) \in \Omega, \ p \in \boldsymbol{P}\}. \tag{3.1}$$

Denote by $W^{1:2}(P)$ the specific Sobolev space of functions with real values of the variables (t, x, p), having the first order weak or generalized derivative (in the sense of distributions) with respect to t, x and up to the second order weak derivatives with respect to the variable p. The primal and dual variables are independent and the functions in the space $W^{1:2}(P)$ enjoy different properties of continuity and differentiability with respect to t, x and p. Let $V(t, x, p)$ of $W^{1:2}(P)$ be a (auxiliary) real valued function, defined on P, and satisfying the following condition:

$$V(t, x, p) = y^0 V_{y^0}(t, x, p) + y V_y(t, x, p) = p V_p(t, x, p), \tag{3.2}$$

for $(t, x, p) \in P$. Here, V_{y^0}, V_y and V_p denote the partial derivatives with respect to the dual variables y^0, y and $p = (y^0, y)$, respectively. Since we consider Neumann problem in a weak form, we formulate the inequality of ε-dual dynamic programming also in a weak form, i.e. to be satisfied pointwise and almost everywhere. Let us fix $\varepsilon > 0$ and any $y^0_\varepsilon < 0$. We require that the function $V(t, x, p)$ satisfy the second order partial differential inequality of ε-dual dynamic programming in a weak form, for $(x,t) \in \Omega,\ p \in \boldsymbol{P}$:

$$\begin{aligned}\varepsilon y^0_\varepsilon \leq \frac{\partial}{\partial t} V(t,x,p) - \sup_{v\in\mathbb{V}}\{(aV_x(t,x,p))_x + ybV_x(t,x,p) - ycV_y(t,x,p) - yf \\ -y^0(1/T)\left|-V_y(T,x,p) - w(x)\right|^2\} - y^0 \sup_{s\leq l}\{\beta_1(1/l)\left|-V_y(t,s,p) - \mu(t)\right|^2 \\ +(1/l)(1/T)\beta_2\left|s - s_*\right|^2\} \leq 0,\end{aligned} \tag{3.3}$$

$$V_{y^0}(0,x,p) = 0,\ 0 \leq x \leq s_0,\ p \in \boldsymbol{P}, \tag{3.4}$$

$$V_{y^0 x}(t,s,p) = 0,\ 0 \leq t \leq T,\ s \in \mathbb{S}, p \in \boldsymbol{P}. \tag{3.5}$$

Denote by $p(t,x)$, $(x,t) \in \Omega$, the dual trajectory, while $u(t,x)$, $(x,t) \in \Omega$ stands for the primal trajectory. Let us put

$$\boldsymbol{u}(t,x,p) = -V_y(t,x,p) \text{ for } (t,x,p) \in P. \tag{3.6}$$

Using the function $\boldsymbol{u}$ it is possible to come back from the dual trajectories $p(t,x)$, lying in P, to the primal trajectories $u(t,x)$, $(x,t) \in \Omega$. Further, we confine ourselves only to those admissible trajectories $u(\cdot)$, for which there exist functions $p(t,x) = (y^0, y(t,x))$, $(t,x,p(t,x)) \in P$, $y(\cdot) \in H^1(\Omega)$, $y(T,x) = 0$, $(x,T) \in \Omega$, such that $u(t,x) = \boldsymbol{u}(t,x,p(t,x))$ for $(x,t) \in \Omega$. Thus, denote

$$\begin{aligned}Ad_{\boldsymbol{u}} = \{(v,u) \in Ad :\ &\text{there exist } p(t,x) = (y^0, y(t,x)), \quad y(\cdot) \in H^1(\Omega), \\ &y(T,x) = 0,\ (x,T) \in \Omega,\ (T,x,p(T,x)) \in P, \\ &(x,T) \in \Omega,\ \psi : \mathbb{R}^3 \mapsto \mathbb{R}^2,\ y(0,x) = \psi(0,x), \\ &(0,x,y^0,\psi(0,x)) \in P,\ (0,x) \in \Omega,\ \psi(0,\cdot) \in H^1(\Omega) \\ &\text{such that } u(t,x) = \boldsymbol{u}(t,x,p(t,x)),\ (x,t) \in \Omega\}.\end{aligned} \tag{3.7}$$

Actually, this means that we are going to study the problem (2.2) possibly in some smaller set $Ad_{\boldsymbol{u}}$, which is determined by the function (3.6). Therefore, in order to define a dual optimal value $S_D^{\bar{\boldsymbol{u}}}$, we need a function $\bar{\boldsymbol{u}}(t,x,p) = -\bar{V}_y(t,x,p)$, where $\bar{V}$ is a solution to (3.3) with the second inequality replaced by equality. Then, it is necessary to change the definition $Ad_{\boldsymbol{u}}$ to $Ad_{\bar{\boldsymbol{u}}}$ by replacing in it $\boldsymbol{u}(t,x,p(t,x))$ by $\bar{\boldsymbol{u}}(t,x,p(t,x))$. Hence, a dual optimal value $S_D^{\bar{\boldsymbol{u}}}$ for problem (2.2) is defined by the formula

$$\begin{aligned}S_D^{\bar{\boldsymbol{u}}} := \inf_{(v,u)\in Ad_{\bar{\boldsymbol{u}}}}\{ -y^0(\beta_0 \int_0^{s(T)} \left|u(x,T;v) - w(x)\right|^2 dx \\ + \beta_1 \int_0^T \left|u(s(t),t;v) - \mu(t)\right|^2 dt + \beta_2\left|s(T) - s_*\right|^2)\}.\end{aligned} \tag{3.8}$$

We named $S_D^{\bar{\mathbf{u}}}$ dual optimal value in contrast to the optimal value

$$S = \inf_{(v,u)\in Ad} J(v)$$

as $S_D^{\bar{\mathbf{u}}}$ depends strongly upon the dual trajectories $p(t,x)$, which, in fact, determines the set $Ad_{\bar{\mathbf{u}}}$. Moreover, an essential point is that the set $Ad_{\bar{\mathbf{u}}}$ is, in general, smaller than Ad i.e. $Ad_{\bar{\mathbf{u}}} \subset Ad$ and thus the dual optimal value $S_D^{\bar{\mathbf{u}}}$ is greater than the optimal value S, i.e. $S_D^{\bar{\mathbf{u}}} \geq (-\bar{y}^0)S$ ($\bar{y}^0$ corresponds to optimal dual trajectory). In our problem of finding the set $Ad_{\bar{\mathbf{u}}}$, first we must find the function V, i.e., solve equations (3.3) with (3.2), and then define the set of admissible dual trajectories (see below). It is not an easy work, but then we will have a possibility of finding that a suspected trajectory is really optimal with respect to all trajectories lying in $Ad_{\bar{\mathbf{u}}}$. Of course, one can wonder whether we are able to find $\bar{V}$ or is the set $Ad_{\bar{\mathbf{u}}}$ nonempty? The answer is not simple. In some cases we can solve that problem, in many cases we cannot do it, similarly as in the classical calculus of variation with the Weierstrass approach. That is one of the motivations for investigating the approximate minimum. An approximate minimum always exists.

Let us fix $y_\varepsilon^0 < 0$, $\varepsilon > 0$, $\boldsymbol{u}$, and denote the corresponding to them suitable value of (3.8) by $S_D^{\boldsymbol{u},y_\varepsilon^0}$. We shall call each value $S_{\varepsilon D}^{\boldsymbol{u},y_\varepsilon^0}$, satisfying inequality below, for the above fixed $y_\varepsilon^0 < 0$ and $\varepsilon > 0$,

$$S_D^{\boldsymbol{u},y_\varepsilon^0} \leq\ S_{\varepsilon D}^{\boldsymbol{u},y_\varepsilon^0} \leq\ S_D^{\boldsymbol{u},y_\varepsilon^0} - \varepsilon y_\varepsilon^0, \tag{3.9}$$

the dual ε-optimal value for (2.2) This means that we are looking for such a control v_ε, which will lead state u_ε to endowing the cost functional $J(v)$ with such a value that

$$J(v_\varepsilon) \leq J(v) + \varepsilon \tag{3.10}$$

for all $(v,u) \in Ad_{\boldsymbol{u}}$. Then, we call the pair $(v_\varepsilon, u_\varepsilon) \in Ad_{\boldsymbol{u}}$ ε-optimal relative to all $(v,u) \in Ad_{\boldsymbol{u}}$. We would like to stress that when taking into account the definition (3.9) of $S_{\varepsilon D}^{\boldsymbol{u},y_\varepsilon^0}$ we need to assume in the definition of dual optimal value the same function $\boldsymbol{u}$ (which generates $Ad_{\boldsymbol{u}}$) and the same fixed y_ε^0, as in the ε-optimal value. The sets, over which we take infimum, must be the same. Otherwise, we do not get inequality (3.10).

For the above fixed $y_\varepsilon^0 < 0$, $\varepsilon > 0$, V and $Ad_{\boldsymbol{u}}$ we define the set of dual trajectories

$$\begin{aligned}\mathcal{P} =&\{p(t,x) = (y_\varepsilon^0, y(t,x)), (t,x) \in \Omega;\\ &(t,x,p(t,x)) \in P,\ y(\cdot) \in H^1(\Omega) \text{ exist }\ (v,u) \in Ad_{\boldsymbol{u}},\\ &u(t,x) = -V_y(t,x,p(t,x)),\ (t,x) \in \Omega,\ y(T,x) = 0,\ (T,x) \in \Omega\}.\end{aligned}$$

Having the above notions and inequality, in the next section we formulate and prove our main theorem, the so called verification theorem, being, in fact, equivalent to sufficient ε-optimality conditions for our problem (2.2).

4. ε-optimality - the verification theorem

Assume that there exists a $W^{1:2}$ solution $V(t,x,p)$ of (3.3) on P, such that (3.2) holds and

$$\boldsymbol{u}(t,x,p) = -V_y(t,x,p),\ (t,x,p) \in P.$$

This means that we have, for fixed $y_\varepsilon^0 < 0$, $\varepsilon > 0$, well defined $Ad_{\boldsymbol{u}}$ and $\mathcal{P}$. We assume that for each $(v,u) \in Ad_{\boldsymbol{u}}$ there exists $p \in \mathcal{P}$. This is not anyhow a restrictive assumption as we can achieve that by taking smaller $Ad_{\boldsymbol{u}}$.

Theorem 4.1 *Let $(v_\varepsilon, u_\varepsilon) \in Ad_{\boldsymbol{u}}$, $p_\varepsilon(t,x) = (y_\varepsilon^0, y_\varepsilon(t,x))$, $y_\varepsilon(\cdot) \in H^1(\Omega)$, $p_\varepsilon \in \mathcal{P}$, $y_\varepsilon(T,x) = 0$, $(T,x) \in \Omega$, be a function such that $u_\varepsilon(t,x) = -V_y(t,x,p_\varepsilon(t,x))$ for $(t,x) \in \Omega$. Suppose that*

$$\begin{aligned}
&\frac{d}{dt}V(t,x,p_\varepsilon(t,x)) - (a_\varepsilon(x,t)V_x(t,x,p_\varepsilon(t,x)))_x \\
&+ y_\varepsilon(t,x)b_\varepsilon(x,t)V_x(t,x,p_\varepsilon(t,x)) - y_\varepsilon(t,x)c_\varepsilon(x,t)V_y(t,x,p_\varepsilon(t,x)) \\
&- y_\varepsilon(t,x)f_\varepsilon(x,t) - y_\varepsilon^0(1/T)\left|-V_y(T,x,p_\varepsilon(T,x)) - w(x)\right|^2 \\
&- y_\varepsilon^0\beta_1(1/l)\left|-V_y(t,s_\varepsilon(t),p_\varepsilon(t,s_\varepsilon(t))) - \mu(t)\right|^2 \\
&- y_\varepsilon^0\beta_2(1/l)(1/T)\left|s_\varepsilon(T) - s_*\right|^2 \le 0.
\end{aligned} \tag{4.1}$$

Then $(v_\varepsilon, u_\varepsilon)$ is an ε-optimal pair relative to all $(v,u) \in Ad_{\boldsymbol{u}}$.

Proof The idea of the proof is similar to that of the verification theorem from Lipnicka and Nowakowski (2018a), although we deal with completely different equations (2.3)-(2.6) and (3.3). This is why we follow here with all details.

Take any $p \in \mathcal{P}$ and a pair $(v,u) \in Ad_{\boldsymbol{u}}$ with $u(t,x) = \boldsymbol{u}(t,x,p(t,x)) = -V_y(t,x,\, p(t,x))$. By applying transversality condition (3.2) we derive that

$$\begin{aligned}
&\frac{d}{dt}V(t,x,p(t,x)) - (a(x,t)(V_x(t,x,p(t,x)))_x = y_\varepsilon^0(\frac{d}{dt}V_{y^0}(t,x,p(t,x)) \\
&- (a(x,t)(V_{y^0x}(t,x,p(t,x)))_x y(t,x)(\frac{d}{dt}V_y(t,x,p(t,x)) \\
&- (a(x,t)(V_y(t,x,p(t,x))))_x
\end{aligned} \tag{4.2}$$

and using the fact that $u(t,x) = -V_y(t,x,p(t,x))$, $(t,x) \in \Omega$, we obtain

$$-\frac{d}{dt}V_y(t,x,p(t,x)) + (a(x,t)(V_y(t,x,p(t,x))))_x \quad (4.3)$$
$$+ b(x,t)(V_y(t,x,p(t,x)))_x + c(x,t)V_y(t,x,p(t,x)) = -f(x,t).$$

Joining (4.2) and (4.3), we come to

$$\frac{\partial}{\partial t}V(t,x,p(t,x)) - (a(x,t)(V_x(t,x,p(t,x)))_x + y(t,x)b(x,t)V_x(t,x,p(t,x)) = \quad (4.4)$$
$$y^0_\varepsilon(\frac{d}{dt}V_{y^0}(t,x,p(t,x)) - (a(x,t)(V_{y^0x}(t,x,p(t,x)))_x$$
$$+ y(t,x)c(x,t)V_y(t,x,p(t,x)) + y(t,x)f(x,t).$$

Next, we apply to (4.4) the inequality (3.3) along $p(t,x)$:

$$y^0_\varepsilon \varepsilon \le y^0_\varepsilon(\frac{d}{dt}V_{y^0}(t,x,p(t,x)) \ - (a(x,t)(V_{y^0x}(t,x,p(t,x)))_x \quad (4.5)$$
$$- y^0_\varepsilon(1/T)\left|-V_y(T,x,p(T,x)) - w(x)\right|^2$$
$$- y^0_\varepsilon\beta_1(1/l)\left|-V_y(t,s(t),p(t,s(t)))\right.$$
$$\left.-\mu(t)\right|^2 - y^0_\varepsilon\beta_2(1/l)(1/T)\left|s(t) - s_*\right|^2.$$

Proceeding as above, but using (4.1), we get the inequality

$$- y^0_\varepsilon(1/T)\left|-V_y(T,x,p_\varepsilon(T,x)) - w(x)\right|^2 \quad (4.6)$$
$$-y^0_\varepsilon\beta_1(1/l)\left|-V_y(t,s_\varepsilon(t),p_\varepsilon(t,s_\varepsilon(t))) - \mu(t)\right|^2$$
$$-y^0_\varepsilon\beta_2(1/l)(1/T)\left|s_\varepsilon(t) - s_*\right|^2 + y^0_\varepsilon(\frac{d}{dt}V_{y^0}(t,x,p_\varepsilon(t,x))$$
$$- (a_\varepsilon(x,t)(V_{y^0x}(t,x,p_\varepsilon(t,x)))_x \le 0.$$

By integrating (4.5) over $[0,T] \times \Omega$ (remember that Ω is defined by $s(t)$ and take into account (3.4), (3.5)) we come to

$$y^0_\varepsilon \varepsilon \le - y^0_\varepsilon \int_0^{s(T)} \left|-V_y(T,x,p(T,x)) - w(x)\right|^2 dx \quad (4.7)$$
$$- y^0_\varepsilon \int_0^T \beta_1(s(T)/l)\left|-V_y(t,s(t),p(t,s(t))) - \mu(t)\right|^2 dt$$
$$- y^0_\varepsilon\beta_2(s(T)/l)\left|s(T) - s_*\right|^2 + y^0_\varepsilon \int_\Omega V_{y^0}(T,x,p(T,x))dx.$$

Similarly, from (4.6) we obtain

$$\begin{aligned}
&-y_\varepsilon^0 \int_0^{s_\varepsilon(T)} \left|-V_y(T,x,p_\varepsilon(T,x)) - w(x)\right|^2 dx \\
&-y_\varepsilon^0 \int_0^T \beta_1(s_\varepsilon(T)/l) \left|-V_y(t,s_\varepsilon(t),p_\varepsilon(t,s_\varepsilon(t))) - \mu(t)\right|^2 dt \\
&-y_\varepsilon^0 \beta_2(s_\varepsilon(T)/l) \left|s_\varepsilon(T) - s_*\right|^2 + y_\varepsilon^0 \int_\Omega V_{y^0}(T,x,p_\varepsilon(T,x))dx \le 0.
\end{aligned} \tag{4.8}$$

Taking into account the definition of $\mathcal{P}$ both (4.7) and (4.8) imply

$$\begin{aligned}
&-\varepsilon y_\varepsilon^0 - y_\varepsilon^0 \int_0^{s(T)} \left|-V_y(T,x,p(T,x)) - w(x)\right|^2 dx \\
&-y_\varepsilon^0 \int_0^T \beta_1(s(T)/l) \left|-V_y(t,s(t),p(t,s(t))) - \mu(t)\right|^2 dt \\
&-y_\varepsilon^0 \beta_2(s(T)/l) \left|s(T) - s_*\right|^2 \ge \\
&\quad -y_\varepsilon^0 \int_0^{s_\varepsilon(T)} \left|-V_y(T,x,p_\varepsilon(T,x)) - w(x)\right|^2 dx \\
&\quad -y_\varepsilon^0 \int_0^T \beta_1(s_\varepsilon(T)/l) \left|-V_y(t,s_\varepsilon(t),p_\varepsilon(t,s_\varepsilon(t))) - \mu(t)\right|^2 dt \\
&\quad -y_\varepsilon^0 \beta_2(s_\varepsilon(T)/l) \left|s_\varepsilon(T) - s_*\right|^2 .
\end{aligned}$$

Thus, $(u_\varepsilon, v_\varepsilon)$ is an ε-optimal pair relative to all $(u,v) \in Ad_{\boldsymbol{u}}$. ■

From (4.8) we infer that knowing the auxiliary function V, satisfying (3.3), we have the explicit formulae for an ε-optimal value of J in terms of V_{y^0} and $p_\varepsilon(0,x) = (y_\varepsilon^0, \psi(0,x))$, i.e. we come to the following

COROLLARY 4.2 *Assume the same as in Theorem 4.1. Then, the ε-optimal value of the functional J may be expressed as*

$$-y_\varepsilon^0 J(v_\varepsilon) = -y_\varepsilon^0 \int_\Omega V_{y^0}(0,x,y_\varepsilon^0,\psi(0,x))dx.$$

5. Numerical algorithm

In order to illustrate the above theory, we use the following numerical algorithm.

1. Define two natural numbers N_d and N_p that determine how many times we will repeat some steps of the calculation procedure, namely
   (a) N_d defines how many different domains we will consider (see step 4.).

   (b) N_p defines how many different parameters we will consider for a given domain (see step 6.).

2. Define an empty set M.
   We start the algorithm with the empty set. We will add elements to it during calculations.
3. Repeat steps (4) - (9) N_d times (for N_d different domains).
4. Define the shape of a domain

$$\Omega = \{(x,t) : 0 < x < s(t),\ 0 < t \leq T\} :$$

   (a) Define the value T. This parameter is used to specify the interval in which the variable t is defined.

   (b) Define the function $s(t)$. This function defines one of the edges of Ω. The shape of $s(t)$ is very important in the implementation of the numerical procedure

5. Repeat steps (6) - (9) N_p times (for N_p different parameters).
6. Let the set of parameters for the problem (2.3) - (2.6): be as follows:

   (a) Take functions a, b, c. These are the parameters of the equation (2.3).

   (b) Take function f. This function defines the right hand side of the equation (2.3).

   (c) Take function ϕ and s. These functions define the boundary condition (2.4).

   (d) Take function g. This function defines the condition (2.5).

   (e) Take functions γ, χ. These functions define the condition (2.6).

7. Define the control $v = (a, b, c, f, g, s) \in H$, where H is defined by (2.1).
8. Solve the problem (2.3) - (2.6) for the above specifications.
9. Save all parameters from the above calculations (steps 6 and 8) in the set M.
10. Define the functional (2.2) by:

   (a) the numbers $\beta_0, \beta_1, \beta_2 \geq 0$,

   (b) the functions $w(x)$, $\mu(t)$ and s_*.

11. Calculate (2.2) for all elements from M. The value of (2.2) informs us about the type of solution of the main problem.
12. Denote the controls of the problem (2.3) - (2.6), for which the value of (2.2) is minimal as $\hat{v}$.
   Start computations in the dual space.
13. Define $\varepsilon > 0$. This is the accuracy of approximation.
14. Define $y^0 < 0$.
15. Define the set $\boldsymbol{P}$, where $\boldsymbol{P} = \{p = (y^0, y), y^0 < 0, y \in \mathbb{R}\}$.

16. Define $P \subset \mathbb{R}^4$; it is an open set of variables (t, x, p), $(x, t) \in \Omega$, $p \in \boldsymbol{P}$, i.e. $P = \{(t, x, p) \in \mathbb{R}^4 : \ (x, t) \in \Omega, \ p \in \boldsymbol{P}\}$.
17. For $\hat{v}$ calculate the solution of (2.3). Suspect that this solution is optimal.
18. Solve equation (3.2) in order to find V.
19. Solve inequality (3.3) in order to define the relationship between the set $\boldsymbol{P}$ and the function V with the given ε.
20. For the set $Ad_{\boldsymbol{u}}$, defined by (3.7), take $u_\varepsilon(t, x) = -V_y(t, x, p_\varepsilon(t, x))$ for $(t, x) \in \Omega$. Check whether $u_\varepsilon(t, x)$ and $v_\varepsilon(t, x)$ satisfy (4.1).
    (a) If yes - $(v_\varepsilon, u_\varepsilon)$ is an ε-optimal pair relative to all $(v, u) \in Ad_{\boldsymbol{u}}$.
    (b) If not - repeat the algorithm once again.

6. Numerical example

In this section one concrete numerical example is given. This example is intended not to show a solution to a real problem, but rather to clarify all the steps, needed to complete the computations in accordance with our algorithm.

6.1. The example

Taking the above algorithm, the calculation procedure, for a concrete case, is as follows.

Ad step 1. Let $N_d = 5$ and $N_p = 100$.

Ad step 2. Let $M = \emptyset$.

Ad step 3. Repeat steps (4) - (9) $N_d = 5$ times.

Ad step 4. Define the domain Ω.

In our case, the domain is defined in the following way.

Take $T = 1$ and the curve $s(t)$. We define $s(t)$ as a polygonal chain, specified by a sequence of points $((0, s_1), (0.25, s_2), (0.5, s_3), (0.75, s_4), (1, 1))$, where $s_1, s_2, s_3, s_4 \in (0, 1)$. For example, let $s(t)$ be the polygonal chain, specified by $((0, 0.4339), (0.25, 0.2787), (0.5, 0.3577), (0.75, 0.5431), (1, 1))$ (see Fig. 1).

Ad step 5. Repeat steps (6) - (9) $N_p = 100$ times.

Ad step 6. Define all parameters as random values and constant or linear functions with random coefficients.

In this example, the parameters are defined as $a = 0.6323$, $b = 0.0975$, $c = 0.2784$ and $f = x + t$.

Ad step 7. Assume the control $v = (a, b, c, f, g, s) \in H$ generated in previous steps.

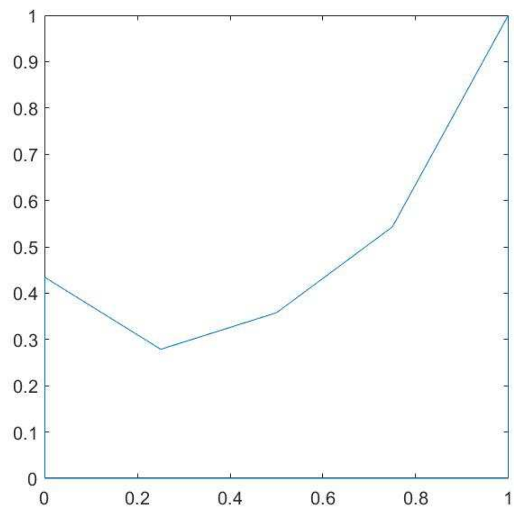

Figure 1. Domain Ω

Ad step 8. Solve the problem (2.3) - (2.6). The solution u is illustrated in Fig. 2.

Ad step 9. Save all parameters from steps 6 and 8 in the set M.

Ad step 10. Let numbers β_0, β_1 and β_2 be random positive numbers. For example, let $\beta_0 = 1$, $\beta_1 = 0.25$, $\beta_2 = 0$. Also, let functions $w(x) = 0$, $\mu(t) = -(t-1)(t+19)$, and $s_* = 0$.

Ad step 11. Calculate the value of the functional for all elements from M.

Ad step 12. In this step, we calculate the minimal value of the functional. We denote by $\hat{v}$ the controls of the problem (2.3) - (2.6), for which the value of (2.2) is minimal. In our example, this is satisfied for $a = 0.6323$, $b = 0.0975$, $c = 0.2784$, $f = x + t$, and s defined in step 4.

Define the dual space. To this end, realize the following steps:

Ad step 13. Define $\varepsilon > 0$: $\varepsilon = 0.01$.

Ad step 14. Define $y^0 < 0$: $y^0 = -0.5$.

Ad step 15. Define the set $\boldsymbol{P} = \{p = (y^0, y), y \in \{-2.5, -0.4, 0.25\}\}$.

Ad step 16. Define $P = \{(t, x, p) \in \mathbb{R}^4 : \ (x, t) \in \Omega, \ p \in \boldsymbol{P}\}$. Points (t, x) we define as a set of nodes on mesh, which was generated for the domain Ω.

Ad step 17. We calculate the solution of (2.3). It is the best solution,

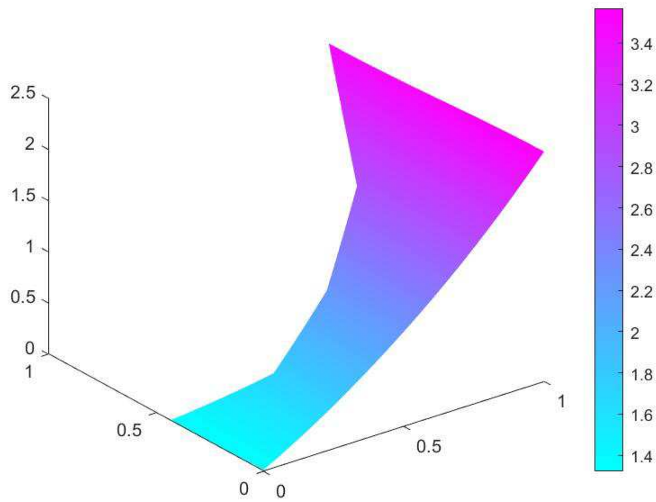

Figure 2. Solution u of the problem (2.3) - (2.6)

taking into account the value of the functional. We want to know, whether this solution satisfies the sufficient conditions from the verification theorem. In other words, is this solution an optimal solution for the main problem?

Ad step 18. Solve equation (3.2). We obtain the solution V.

Ad step 19. Solve inequality (3.3). We obtain the relationship between the set $\boldsymbol{P}$ and the function V with the given ε.

Ad step 20. The last step gives us an answer, as to whether our minimal (considering the value of the functional) solution is optimal. For the set $Ad_{\boldsymbol{u}}$ we obtain that the theory is satisfied and our solution is an ε-optimal solution.

7. Conclusions

We apply the dual dynamic programming to the construction of sufficient approximate optimality conditions for solving the problem of identification of the unknown parameters a, b, c, f, g in the one-phase Stefan problem (2.3) - (2.6). The obtained verification Theorem 4.1 is the basis for developing a numerical algorithm. We calculate numerically the set of parameters (a, b, c, f, g, s). If the set of parameters, together with the auxiliary function V, satisfy the conditions of Theorem 4.1, they represent an approximate solution of the identification problem. Having the verification theorem, we do not need the convergence of our algorithm.

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