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On existence of energy minimizing configurations for mixtures of two imperfectly bonded conductors

by

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Abstract: We consider a domain filled with a suspension of heat conducting spheres of conductivity σ_p embedded in a matrix of lesser conductivity σ_m. It is assumed that there exists a thermal contact resistance at the sphere - matrix interface. The contact resistance is characterized by a scalar β, which has dimensions of conductivity per unit length. A current flux is prescribed on the domain boundary and we seek the energy minimizing configuration among all suspensions satisfying a resource constraint on the total volume of spheres. We establish the existence of an energy minimizing configuration within the class of polydisperse suspensions of spheres. The optimal suspension depends upon the size of the domain and consists of spheres of radii greater than or equal to $R_{cr} = \beta^{-1}(\sigma_m^{-1} - \sigma_p^{-1})^{-1}$ or no spheres at all. Here R_{cr} is the ratio between the interfacial resistance and the mismatch between the resistivity of each phase.

Keywords: Stekloff eigenvalue, optimal design, contact resistance

1. Introduction

We consider suspensions of thermally conducting spheres embedded in a matrix of lesser thermal conductivity. We allow the suspensions to contain spheres of different radii. This class of suspensions is referred to as the class of *polydisperse* suspensions of spheres. The suspension is contained inside a convex domain $D \subset R^3$ with Lipschitz continuous boundary ∂D. The conductivities of the spheres and matrix are assumed isotropic and are specified by σ_p and σ_m respectively, with $\sigma_p > \sigma_m$. We treat the technologically important case when there is an interfacial contact resistance between the two phases. The contact resistance is characterized by a scalar β with dimensions of conductivity per unit length. Experiments show that for small particles, the presence of an interfacial barrier can diminish or even negate the effect of a highly conducting reinforcement, see, Garret and Rosenberg (1974), Every, Tzou, Hasselman and

Raj (1992), and Hashin (1962). This phenomena is in striking contrast to what occurs for perfectly bonded composites where there is no interfacial thermal barrier. Indeed, for perfectly bonded composites it is known, that the addition of highly conducting particles will always increase the effective conductivity independently of particle size. Recent studies focusing on special micromechanical models and dilute monodisperse suspensions of spheres strongly suggest that the effective conductivity decreases with decreasing particle size, see: Chiew and Glandt (1987), Every, Tzou, Hasselman and Raj (1992), and Hasselman and Donaldson (1992). The low volume fraction results of Chiew and Glandt (1987) and the micromechanical models of Every, Tzou, Hasselman and Raj (1992) show that the effective conductivity tends to that of a porous matrix in the limit of infinitesimally small particle size. More generally for periodic suspensions it is shown in Lipton (1997) that the effective property tends to that of a porous matrix in this limit. This behavior is seen in the experimental results of Hasselman and Donaldson (1992). From the perspective of engineering applications, it is of importance to know how to design suspensions with energy dissipation properties at least as good as that of the matrix. Recently it has been shown by Lipton (1996, 1998) that if a reinforcement particle's second Stekloff eigenvalue, p2, is greater than R_{cr} = (3-^{1}/(u;^{1} - u;;;,^{1})-^{1}, then the energy dissipation of the suspension will not decrease when the particle is added to the suspension. For a spherical particle of conductivity *up* this means that the particle will not lower the energy dissipation of the composite when the particle's radius is less than R_en see Theorem 1.1. In light of this, it is evident that minimizing sequences of designs will not consist of arbitrarily fine suspensions. In fact we show that the existence theory for the optimal design becomes a problem of shape optimization. The author recognizes that polydisperse suspensions of spheres do not represent the most general physical or mathematical case, however it is a first step towards a general theory of existence for these problems.

We present the mathematical formulation of the problem. The region occupied by the i^{th} sphere in the suspension is denoted by Bi, and the configuration of spheres given by their union UBi is denoted by A. The two phase interface is denoted by r = U8Bi- We assume that the spheres are strictly contained inside 0, ie., A c ῼ and r n 8 0 = 0. The local resistivity tensor inside the composite is described by u-^{1}(X_A) = u;^{1}x$_A$i$_A$ + u;;;,^{1}(1 - X_A), where X_A equals one in A and zero otherwise. For a prescribed distribution of current g E H-$^{1/2}$(8!1), such that $\int_{80}$ *gds* = 0, the energy dissipated inside the composite is given by E(A, g), where

$$E(A, g) = \min\{C(A,j) : j \ E \ L^2(0)^3, \operatorname{div} j = 0, \ j \cdot n = g \text{ on } 80\} \quad (1)$$

and

$$C(A,j) = \int_\Omega \sigma^{-1}(\chi_A)\, j \cdot j dx + \beta^{-1} \int_\Gamma (j \cdot n)^2 ds. \quad (2)$$

Here div j = 0 holds in the sense of distributions, ds is the element of surface area, and the vector n is the unit normal pointing into the matrix phase. The first term of the functional C(A,j) is associated with bulk energy dissipation, while the second term gives the energy dissipation at the two-phase interface. The minimzer $]_A$ is precisely the heat flux inside the composite. The associated temperature u_A is related to the heat flux through the constitutive law $]_A = cr(x_A)V u_A$ in each phase. The equilibrium equations for the temperature are given in Section 3. Existence of solution for the equilibrium equations follows from the Lax-Milgram lemma: this is easily established along the lines given in Lene and Leguillon (1982).

We consider the problem of minimizing the energy dissipation among polydisperse suspensions, subject to a resource constraint on the total volume occupied by the spheres. We introduce the class C_{0P} of all polydisperse suspensions containing a finite number of spheres, satisfying the resource constraint meas(A) :S 0_pmeas(D). Here 0_P is an upper bound on the volume fraction occupied by the suspension. Note that there are no constraints on the size or number of spheres for configurations in C_{0P}. We suppose that the spheres do not touch each other. To make this requirement precise we consider a suspension in C_{0P} consisting of N spheres and denote the center and radius of the i^{th} sphere by Xi and Ti respectively. We surround the i^{th} sphere by an open ball Si with center Xi and slightly larger radius (1 + .>-)ri, where A is a fixed positive constant. We require that the open balls do not overlap, ie.,

$$Si \cap S_j = 0 \quad i \neq j, \tag{3}$$

and

$$Si \cap 8D = 0 \quad i = 1, 2, \dots N. \tag{4}$$

The class of suspensions in C_{0P} satisfying (3) and (4) is denoted by $C_{0P,>}$.

For a prescribed current flux g E H-112, such that $\int_n$ gds = 0 we consider the problem,

$$\min\{E(A, g) : A \in C_{0_p,>}\} \tag{5}$$

In this paper it is shown that an energy minimizing configuration exists in the class $C_{0P,>}$. Moreover the optimal suspension depends upon the size of the domain D, and consists of spheres with radii greater than or equal to R_{cr}, or no spheres at all: see Theorem 1.4. Here R_{cr} has the dimensions of length and is the ratio between the interfacial thermal resistance and the mismatch between the thermal resistivity of each phase. We remark that the class $C_{0P,>}$ is sufficiently large to allow for the potential appearance of fine structure in minimizing sequences of configurations. In fact the composite sphere assemblages of Hashin (1962) can be approached by sequences of configurations in $C_{0P,>}$. We note that the restriction to suspensions of spheres that do not touch is a technical one and is only used to apply the methods of optimal shape design as presented in Pironneau (1984).

The functional without the interfacial energy term in (2) has been widely studied. Indeed, in the absence of surface energy, it is well known from the fundamental work of Lurie and Cherkaev (1986) and Murat and Tartar (1985) that problems of the type (5) are most often illposed and exhibit minimizing sequences composed of arbitrarily fine mixtures of the two conductors.

Recently Ambrosio and Buttazzo (1993) have considered functionals with bulk energies similar to the first term in (2) augmented by a penalization proportional to the perimeter of the two phase interface. They allow for arbitrary configurations of the two phases, placing a resource constraint on the better conductor. The perimeter penalization used in Ambrosio and Buttazzo's work rules out the appearance of arbitrarily fine mixtures in minimizing sequences by assigning an infinite value to them.

Their penalization gives the necessary compactness and forces the optimal configuration to lie within the class of sets of finite perimeter that are (up to subsets of measure zero) open.

The approach taken here does not use an explicit perimeter penalization, but instead the penalization opposing the formation of fine scale mixtures follows from the thermal contact resistance at the two phase interface. The explicit mechanism by which fine scale minimizing sequences are eliminated is seen in the following inequality established in Lipton (1996, 1998).

THEOREM 1.1 Energy dissipation inequality.
Let B *denote a sphere of radius* a *such that* $A \cup B$ *is a suspension in* C_{0_p} ,>, *then,*

$$E(A \cup B, g) \geq E(A, g), \tag{6}$$

if

$$a \leq R_{cr} \tag{7}$$

for all $g \in H^{-1/2}(\partial\Omega)$ *such that* $\int_{\partial\Omega} g\,ds = 0$.

We remark that this is a special case of a more general inequality that holds for suspensions of particles with Lipschitz boundaries obtained in Lipton (1996, 1998). For completeness we provide a proof of Theorem 1.1 in Section 5.

It is evident from Theorem 1.1 that R_{cr} gives the critical sphere radius for which the benefits of a highly conducting inclusion are spoiled by the contact resistance at the particle surface. It follows that if a suspension contains spheres of radii less than R_{cr} then there is no advantage to keeping them in the suspension.

At this point it is it is necessary to check that there exist suspensions in C_{0_p} ,> for which the corresponding energy dissipation $E(A, g)$ is less than the energy dissipation $E(0, g)$ associated with pure matrix material. To answer this question positively we introduce the following geometric quantity associated with the configuration A and boundary data g given by:

$$T(A, g) = R_{cr} - \int_{A} l]l^2\,dx \Big/ \Big(\int_{\partial A} (] \cdot n)^2\,ds\Big). \tag{8}$$

Here] is the current field generated inside the domain when filled with pure matrix material and subjected to the prescribed boundary data g We now state the following theorem that is proved in Section 6:

THEOREM 1.2 *IfT(A,g)* S O *then* E(A,g) S E(0,g).

For any given configuration A it is evident from (8) that the particle conductivity and interfacial resistance may be chosen to make R_{cr} sufficiently small so that the quantity T(A, g) is negative. Thus for any configuration in C_{0_v},>there is a choice of particle conductivity and interfacial resistance for which the configuration has lower energy dissipation than pure matrix material. Conversely for fixed material properties and boundary data, the condition T(A, g) S Ois seen to be sufficient for a configuration to reduce the energy dissipation below the unreinforced value. As an example we consider boundary data of the form $g=$:J •*n*, where n is the outward directed unit normal vector on the boundary of !1 and J is a constant vector in R^3. For this case the requirement T(A, g) s 0 is equivalent to

$$(a^{-1})^{-1} \geq R_{er}. \tag{9}$$

Here (a-1) is the volume average of the reciprocal radii for a suspension of N spheres with radii $a_1, a_2, \ldots, a_N$ given by:

$$(_a{}^{-1}) = \text{"}\sum_{i=1}^{N} {}_a{}^{:-1} \frac{|Y_i|}{|A|'} \tag{10}$$

where, $|Y_i|$ is the volume of the i^{th} sphere and |A| is the volume of the suspension. Thus it is evident from Theorem 1.2 that if the "harmonic mean" of the sphere radii is greater than R_{cr} then the configuration reduces the energy dissipation below the unreinforced value. The inequality (9) is established Section 6.

We next introduce the subclass SC_{0_v},>, of suspensions in C_{0_v},> consisting only of spheres with radii greater than of equal to R_{cr} · In view of Theorem 1.1, we see that the size of the domain !1 effects the optimal configuration. Indeed, the domain may have physical dimensions for which the class of configurations SC_{0_p},> is empty. Thus, the class C_{0_p},> consists of spheres of radius less than R_{cr}. For this case, it is evident that the optimal configuration is made from pure matrix material with no particles at all. We summarize these observations in the following,

THEOREM 1.3 Necessary conditions of optimality.
If the configuration A *is a minimizer for problem* (5), *then* A E SC_{e_p},>, *or* $A=0$.

We now assert the existence of an energy rn.inimizing configuration in the class C_{e_p},>

THEOREM 1.4 Exsistence of an energy minimizing suspension.
There exists a minimizing configuration A for problem (5) and $A \in SC_{0_p,>}$ *or* $A=0$.

Clearly, Theorem 1.3 implies that all energy minimizing configurations must lie in the class $SC_{0_p,>}$ or consist of no spheres at all. Thus for domains for which the class $SC_{0_p,>}$ is not empty, our analysis focuses on proving the existence of minimizers within the class $SC_{0_p,>}$. Existence is proved through the direct method of the calculus of variations and we may use the methods discussed in Pironneau (1984). It is shown in Section 2 by means of elementary arguments that the set of characteristic functions associated with all configurations in $SC_{0_p,>}$ is closed and compact with respect to strong L^1 convergence. In Section 3, the functionals $E(A,g)$ are shown to be continuous with respect to L^1 convergence and existence follows. Section 4 provides a derivation of the energy estimates used in section 3. In this Section, a higher regularity result for the trace of the temperature on either side of the two phase boundary is obtained. This is used to establish a Poincare like inequality from which uniform bounds on the energy dissipation follow. We establish Theorem 1.1 in Section 5 and conclude the paper by proving Theorem 1.2 and inequality 9 in Section 6.

2. Compactness of the design space

We consider a sequence $\{A^\nu\}_{\nu=1}^{\infty}$ of configurations in $SC_{0_p,>}$ and state the following:

THEOREM 2.1 *Given* $\{A^\nu\}_{\nu=1}^{\infty}$ *such that* $A^\nu \in SC_{0_p,>}$, $\nu = 1, 2, \ldots$, *there exsists a subsequence also denoted by* $\{A^\nu\}_{\nu=1}^{\infty}$ *and a configuration* $A \in SC_{0_p,>}$ *for which* $meas(A^\nu \Delta A) \to 0$. *Equivalently we have* $\chi_{A^\nu} \to \chi_A$, *strongly in* $L^1(D)$.

Proof:
We first note that any configuration A^ν of "p" particles can be represented by a vector $\underline{y}^\nu$ of length $4p$ consisting of the radius and center of each sphere in the suspension. The maximum length of any such vector is attained by the vector describing the configuration associated with packing D with spheres of radius R_{cr}. The number of spheres in this packing is denoted by ℓ. We see that all configurations in $SC_{0_p,>}$ correspond to a closed bounded set of vectors in R^{4P}. The compactness of the design space in $L^1(D)$ follows immediately.

Remark. Since the minimum sphere size is bounded below by R_{cr} it is clear that any convergent sequence $\{A^\nu\}_{\nu=1}^{\infty}$ contains the same number of spheres for sufficiently large indices ν.

3. Continuity of the energy dissipation functional

In this section we establish the continuity of the functional $E(A,g)$. We state the following theorem:

THEOREM 3.1 Continuity of the energy dissipation.
Given a sequence $\{A^\nu\}_{1}$ *in* SC_{0_p} *and a set* A *in* SC_{0_p} *such that* $meas(A^\nu \Delta A) \to 0$, *then*

$$\lim_{\nu \to \infty} E(A^\nu, g) = E(A, g) \tag{11}$$

Proof:
With out loss of generality, may suppose that the limit configuration A consists of "p" spheres. From the remark following Theorem 2.1 we assume that we are far enough out in the sequence, so that each configuration A^ν consists of p spheres.

For the configuration A^ν we introduce the characteristic function χ_i associated with the i^{th} sphere. The characteristic functions of the spheres are related to the characteristic function of the configuration A^ν by $\chi_{A^\nu} = \sum_{i=1}^{p} \chi_i^\nu$. Moreover, the convergence of $\{A^\nu\}_1$ to A imply $\chi_i^\nu \to \chi_i$ strongly in L^1 where χ_i is the characteristic function of the i^{th} sphere in the configuration A. The proof is facilitated by introducing the equations of state solved by the temperature u_{A^ν} for the configuration A^ν. The heat flux is related to the gradient of the temperature by the constitutive law: $j_{A^\nu} = \sigma^\nu(x)\nabla u_{A^\nu}$ and

$$div(\sigma^\nu(x)\nabla u_{A^\nu}) = 0 \text{ in } \Omega \setminus \Gamma^\nu. \tag{12}$$

Across the interface one has

$$[j_{A^\nu} \cdot n] = 0 \text{ on } \Gamma^\nu, \tag{13}$$

and

$$j_{A^\nu} \cdot n_{|_p} = -\beta[u_{A^\nu}] \text{ on } \Gamma^\nu, \; \sigma_m \nabla u_{A^\nu} \cdot n = g \text{ on } \partial\Omega. \tag{14}$$

Here Γ^ν is the two phase interface $\sigma^\nu(x) = \sigma_p \chi_{A^\nu} + \sigma_m(1 - \chi_{A^\nu})$, and $[u_{A^\nu}] = u_{A^\nu}|_p - u_{A^\nu}|_m$, where the subscripts indicate the side of the interface where the trace is taken. The requirement, $\int_{\partial\Omega} g ds = 0$ is the solvability condition for the equation of state, and the temperature $u_{A^\nu} \in H^1(\Omega \setminus \Gamma^\nu)$ is determined uniquely up to a constant. Exsistence of solution for the boundary value problem (3.2)-(3.4), follows from the Lax-Milgram lemma; this is easily established along the lines given in Lene and Leguillon (1982). We provide a usefull weak formulation of the boundary value problem (12) - (14). Introducing the space of vector fields $\underline{w} = (w_0, w_1, \ldots, w_p)$ belonging to $H^1(\Omega)^{p+1}$, the weak formulation is given by:

$$\sum_{i=1}^{p} \int_\Omega (\chi_i^\nu \sigma_p \nabla u_{A^\nu} \cdot \nabla w_i) dx + \int_\Omega (1 - \chi_{A^\nu}) \sigma_m \nabla u_{A^\nu} \cdot \nabla w_0 dx +$$

$$+ \beta \sum_{i=1}^{p} \int_{\partial B_i^\nu} [u_{A^\nu}] (w_i - w_0) ds - \int_{\partial\Omega} w_0 g ds = 0, \tag{15}$$

for all $w_i \in H^1(\Omega)^{p+1}$. Here ∂B_i^ν denotes the boundary of the i^{th} sphere in the configuration A^ν. One can establish the existence of a sequence of constants $\{c^\nu\}_{\nu=1}^{\infty}$ such that the normalized sequence of temperatures $\{u_{A^\nu} - c^\nu\}_{\nu=1}^{\infty}$ is uniformly bounded, ie.,

$$\sup_{\nu} \|u_{A^\nu} - c^\nu\|_{H^1(\Omega\setminus\Gamma^\nu)} < \infty \tag{16}$$

This estimate is derived in Section 4: see Theorem 4.2. The normalized temperature is a solution of the equation of state (12)-(14) and for the remainder of this Section we continue to denote it by u_{A^ν}. Next we observe that there is a uniform bound on the Lipschitz constant associated with the boundary of any sphere B_i^ν, that holds independently of the indices i and ν. Thus we may apply the Theorem of D. Chenais (1975) to assert the exsistence of a positive number K and p + 1 linear and continuous extension operators, $M_0^\nu, M_1^\nu, \ldots, M_p^\nu$ such that for all $A^\nu = \cup_{i=1}^{p} B_i^\nu \in SC_{e_p,A}$

$$M_0^\nu : H^1(D \setminus (A^\nu \cup \Gamma^\nu)) \to H^1(R^3), \tag{17}$$

$$M_i^\nu : H^1(B_i^\nu) \to H^1(R^3), \tag{18}$$

for i = 1, 2, ... ,p, where

$$\|M_0^\nu\| \le K \text{ and } \|M_i^\nu\| \le K \tag{19}$$

for i = 1, 2, ... ,p.

It is evident from (16) and (19) that

$$\sup_{\nu} \|M_0^\nu u_{A^\nu}\|_{H^1(\Omega)} < \infty \tag{20}$$

and

$$\sup_{\nu} \|M_i^\nu u_{A^\nu}\|_{H^1(\Omega)} < \infty, \tag{21}$$

for i = 1, 2, ... ,p.

From (20) and (21) we may pass to a subsequence if necessary to find that there exists functions $u_0^\infty, u_1^\infty, \ldots, u_p^\infty$ all in $H^1(\Omega)$ such that:

$$M_0^\nu u_{A^\nu} \to u_0^\infty, \text{ weakly in } H^1(\Omega) \tag{22}$$

and

$$M_i^\nu u_{A^\nu} \to u_i^\infty, \text{ weakly in } H^1(\Omega), \tag{23}$$

for i = 1, 2, ... ,p.

It follows from the weak formulation (15) that the choise $w_0 = M_0^\nu u_{A^\nu}$, $w_i = M_i^\nu u_{A^\nu}$, i = 1, 2, ... ,p, gives:

$$E(A^\nu, g) = \int_{\partial\Omega} M_0^\nu u_{A^\nu} g ds = \int_{\partial\Omega} u_{A^\nu} g ds. \tag{24}$$

From the weak convergence (22), it follows that:

$$\lim_{\nu \to 00} E(A^{v}, g) = \int_{\partial D} u0gds. \tag{25}$$

Next we let XA denote the characteristic function of A and XA = :Z::t:Xi, where Xi is the characteristic function of the i^{th} sphere in the limit configuration A. It is evident from (25) that the theorem follows once we show that, u^{00} = u0(1 - XA) + I:!:'.f Xiuf° is the solution of:

$$\sum_{i=I} \int_{\Omega} (xia_p 'vu'(° \cdot 'vwi)dx + \int_{\Omega} (1 - n) a_m 'vu0 \cdot 'vwadx +$$

$$+ \beta \sum_{i=I} \int_{\partial B_i} (u'f - u'o)(wi - wa)ds - \int_{\partial \Omega} wagds = 0, \tag{26}$$

for all w in H^1(D)P+l, where 8Bi is the boundary of the i^{th} sphere in the limit configuration. To establish this we will pass to the limit in the weak formulation (15) to show that it agrees with (26).

We observe first that the weak convergence 'vNI[u_{A^v} - >'vuf°, i = 1, 2, ... ,P and 'vMou$_{A^v}$ - 'vu';" together with the strong convergence of X_i^ν to Xi, implies that the first two terms of (15) converge to the first two terms of (26). To expedite the presentation we denote the differences uf° - u0 and Wi - w_0, defined everywhere on D, by [u^{00}]i and δ_i respectively. We consider the difference between the third terms of (15) and (26), given by:

$$/3 \{ t \ laB'\{ [u_{A^v}]15ids - _t \ laB_; [u^{00}]iδids \}$$

$$= /3 \{ t \ \{laB'\{ ([uA v] - [u^{00}]i)δids$$

$$+ (\int_{laBr} [u^{00}]iδids - \int_{laB;} [u^{00}]iδids)\}\}. \tag{27}$$

We show for each term in the sum (27) that the difference

$$\int_{\partial B_i^\nu} ([u_{A^\nu}] - [u^\infty]_i)\delta_i ds \to 0, \text{ for all } \delta_i \in H^1(\Omega). \tag{28}$$

To do this we observe,

$$\int_{\partial B_i^\nu} ([u_{A^\nu}] - [u^\infty]_i)\delta_i ds = \int_{\partial B_i^\nu} ([u_{A^\nu}] - [u^\infty]_i)\delta_i n^\nu \cdot n^\nu ds, \tag{29}$$

where n^ν is the unit normal pointing out of ∂B_i^ν. Extending the normal inside B_i^ν we apply the divergence theorem to find:

$$
\begin{aligned}
&\int_{\partial B_i^\nu} ([u_{A^\nu}] - [u^\infty]_i)\delta_i \, ds \\
&= \int_{\partial B_i^\nu} ((M(u_{A^\nu}) - M_0 u_{A^\nu}) - [u^\infty]_i)\delta_i n^\nu \cdot n^\nu \, ds \\
&= \int \chi_i^\nu \{(\nabla M(u_{A^\nu}) - \nabla u^\infty) + (\nabla u_0 - \nabla M_0 u_{A^\nu})\} \cdot n^\nu \delta_i \, dx \qquad (30)\\
&+ \int \chi_i^\nu \{(M(u_{A^\nu}) - u^\infty) + (u_0 - M_0 u_{A^\nu})\}(\operatorname{div} n^\nu \delta_i + n^\nu \cdot \nabla \delta_i)\, dx.
\end{aligned}
$$

For the i^{th} sphere, the extension of the unit normal vector is simply $(x - x_i^\nu)(a_i^\nu)^{-1}$ and $\operatorname{div} n^\nu = 3(a_i^\nu)^{-1}$, where a_i^ν and x_i^ν are the radii and center of the i^{th} sphere in the configuration A^ν. Clearly the products $\chi_i^\nu n^\nu = \chi_i^\nu(x - x_i^\nu)(a_i^\nu)^{-1}$ and $\chi_i^\nu \operatorname{div} n^\nu$ converge strongly in L^2 to $\chi_i n$ and $\chi_i \operatorname{div} n$, where n is the extended normal associated with the limit configuration. Thus from the weak convergence (22) and (23) of the extended fields, it follows from (30) that the difference (28) vanishes in the limit. Last, we observe that for each term in the sum (27) the difference:

$$\int_{\partial B_i^\nu} [u^\infty]_i \delta_i ds - \int_{\partial \breve{B}_i} [u^\infty]_i \delta_i ds \qquad (31)$$

vanishes in the limit; this follows from the continuity of the trace, see Lions and Magenes (1972). Thus passing to the limit in (15) we recover (26) and the Theorem follows.

4. The Poincare inequality and the energy estimate

In this Section we provide the energy estimates for the temperature satisfying the equations of state given by (3.2) – (3.4). In view of the discussion in Sections 2 and 3 we will only consider potentials associated with configurations in the class $SC_{\sigma_p,\kappa}$ consisting of at most p spheres. For these configurations there exsists a boundary layer Δ of thickness $\frac{1}{2}\kappa_{cr}$ in which no sphere of conductivity σ_p is present. We let $\hat\Omega$ denote the subset of Ω obtained by removing the boundary layer Δ from Ω, ie., $\hat\Omega = \Omega \setminus \Delta$.

We start with an elementary observation on the regularity of the temperature on the two phase interface. The trace of the temperature on the two phase interface is denoted by $u_{A^\nu}|_p$ and $u_{A^\nu}|_m$ where the subscripts m and p denote the side of the interface where the trace is taken. We give the following:

LEMMA 4.1 *The traces $u_{A^\nu}|_p$ and $u_{A^\nu}|_m$ lie in $C^0(\Gamma^\nu)$ for all $\nu = 1, 2, \ldots$*

Proof:

We observe from (3.2) - (3.4) that in each phase the solution u_{A^ν} satisfies the following set of Neumann problems:

$$\int_{B_i^\nu} \nabla u_{A^\nu} \cdot \nabla \varphi \, dx = -\beta \, {}_{H^{-1/2}(\partial B_i^\nu)} \langle [u_{A^\nu}], \varphi \rangle_{H^{1/2}(\partial B_i^\nu)}, \qquad (32)$$

for all $\varphi \in H^1(B_i^\nu)$, $i = 1, 2, \ldots, P$, and

$$\int_{\Omega \setminus A^\nu} \nabla u_{A^\nu} \cdot \nabla \varphi \, dx = -\beta \, {}_{H^{-1/2}(\Gamma^\nu)} \langle [u_{A^\nu}], \varphi \rangle_{H^{1/2}(\Gamma^\nu)}$$
$$+ \; {}_{H^{-1/2}(\partial\Omega)} \langle g, \varphi \rangle_{H^{1/2}(\partial\Omega)}, \qquad (33)$$

for all $\varphi \in H^1(\Omega \setminus A^\nu)$.

Observing that $u_{A^\nu} \in H^1(\Omega \setminus \Gamma^\nu)$ we see that the jump $[u_{A^\nu}]$ lies in $H^{1/2}(\Gamma^\nu)$ and we appeal to the regularity theory for the Neumann problem, (see Grisvard, 1980) to conclude that $u_{A^\nu} \in H^2(\Omega \setminus \Gamma^\nu)$. The Lemma now follows from the Sobelev imbedding theorem.

Remark. Since Γ^ν is of class C^∞, we can iterate the procedure used to prove Lemma 4.1 to find that $u_{A^\nu}|_p$ and $u_{A^\nu}|_m$ lie in $C^\infty(\Gamma^\nu)$.

Letting d be the diameter of the domain Ω and setting w_3 equal to the volume of the unit sphere in three dimensions we state the following Poincare like inequality:

THEOREM 4.1 *For any sequence of configurations in the class SC_{θ_p}, consisting of p spheres, there exists a constant $M = 2(\frac{1}{\beta} \ldots)^2 (3w_3)^2 |\Omega|^{2} d^3$, such that for all temperatures u_{A^ν} satisfying the equations of state:*

$$\|u_{A^\nu} - c^\nu\|^2_{L^2(\Omega)} \leq pM\Big(\int_\Omega |\nabla u_{A^\nu}|^2 dx + \int_{\Gamma^\nu} ([u_{A^\nu}])^2 ds\Big), \qquad (34)$$

where $c^\nu = \frac{1}{|L_\lambda|} \int_{L_\lambda} u_{A^\nu} dx$.

The proof of Theorem 4.1 proceeds in two steps, first we introduce the Riesz and double layer potentials given by:

$$V_{1/3}(|\nabla u_{A^\nu}|)(x) = \int_\Omega |y - x|^{-2} |\nabla u_{A^\nu}(y)| dy \qquad (35)$$

and

$$P^\nu(|[u_{A^\nu}]|)(x) = \int_{\Gamma^\nu} |[u_{A^\nu}(y)]| \, |\partial_{n_y} E(x,y)| ds_y, \qquad (36)$$

where n is the outward pointing normal and $E(x, y)$ is the Newtonian potential $|y - x|^{-1}$.

We state the following:

LEMMA 4.2

$$|u_{A^\nu}(x) - c^\nu| \leq \frac{d3}{3IL,\backslash I}\{V1;3(l'vuAvl)(x) + pv(l[uA^v]l)(x)\}. \quad (37)$$

Proof:
For x and y in n we write:

UAv (x) - uAv (y) = - Jh$^{lx_yl}_{0}$ D$_r$ UAv (x + rw)dr - L$^{c}_{j=1}$ [u Av (x + rj w)], (38)

where 0 ::; £ ::; p, w = (y - x)/IY - xl. Here x + TjW lies on the intersection of the / h interface and the line segment connecting the points x and y, and [] indicates the jump of uAv across the surface of a spherical particle. Next we integrate (38) with respect to the y variable over the boundary layer L,\ to obtain:

IL,\ l(uAv (x) - cv) = - JL>- dy Jo^{lx_yl} D$_r$ UAv (x + rw)dr

− ∫$_{L>}$ dy (I:$^{£}_{j=1}$ [uAv (x + TjW)]). (39)

We write

IL,\ ll(uAv (x) - c^V)I = I JL>- dy Jo^{lx_yl} D$_r$ UAv (x + rw)dr l

+ I} dy (t, [uAv (x + rj w)])I. (40)

Proceeding as in Gilbarg and Trudinger (1983), the first term in (40) is estimated above by:

| ∫$_{L>}$ dy ∫$_{O}^{lx\_yl}$ D$_r$ uAv (x + rw)dr l ::; -$\frac{d3}{3}${Vi;3(l'vuAv l)(x). (41)

From the convexity of the domain n we may integrate the second term in (40) using the polar coordinates dy = p^2 dpdw , to obtain

$$|\int_{L_\lambda} dy(\sum_{j=1}^{\ell}[u_{A^\nu}(x+r_j\omega)])| \leq \frac{d^3}{3}\sum_{i=1}^{p}\int_{\Omega_i}|[u_{A^\nu}(x+r_j(\omega)\omega)]|d\omega, \quad (42)$$

where Di is the solid angle subtended by the i^{th} sphere and x + rj(w)w ranges over all points on its surface. We apply a standard change of variables, (cf., Jackson, 1975), to obtain:

tii l[uAv (x + rj (w)w)]ldw = tJ t l[uAv]l lan$_y$E (x,y)ldsy, (43) and the Lemma follows.

Application of Lemma 4.2 and Cauchy's inequality gives:

$$\begin{aligned} \|u_{A^\nu} - c^\nu\|_{L^2(\Omega)} &\le \frac{\sqrt{2}}{3}\frac{d^3}{|L_\lambda|}\Big(\int_\Omega |V_{1/3}(|\nabla u_{A^\nu}|)|^2 dx \\ &+ \int_\Omega |P^\nu(|[u_{A^\nu}]|)|^2 dx\Big)^{1/2}. \end{aligned} \tag{44}$$

From Gilbarg and Trudinger (1983) we have:

$$\int_\Omega |V_{1/3}(|\nabla u_{A^\nu}|)|^2 dx \le (3\omega_3^{2/3})^2|\Omega|^{2/3}\|\nabla u_{A^\nu}\|^2_{L^2(\Omega)}. \tag{45}$$

Next we estimate the surface layer term on the right-hand side of (44). We write,

$$P^\nu(|[u_{A^\nu}]|) = \sum_{i=1}^{p} P_i^\nu(|[u_{A^\nu}]|), \tag{46}$$

where $P_i^\nu(|[u_{A^\nu}]|) = \int_{\partial B_i^\nu} |[u_{A^\nu}(y)]|\,|\partial_{n_y}E(x,y)|\,ds_y$, Application of Cauchy's inequality gives:

$$\int_\Omega |P^\nu(|[u_{A^\nu}]|)|^2 dx \le p\sum_{i=1}^{p}\int_\Omega |P_i^\nu(|[u_{A^\nu}]|)|^2 dx. \tag{47}$$

The Poincare inequality now follows from (44), (45), (47) and the following estimate:

LEMMA 4.3

$$\begin{aligned} &\int_\Omega |P_i^\nu(|[u_{A^\nu}]|)|^2 dx \\ &\le (3\omega_3/3)^2|\Omega|^{2/3} h \int_{\partial B_i^\nu} ([u_{A^\nu}(y)])^2 ds_y, \end{aligned} \tag{48}$$

Proof:
Let $K(x,y) = |\partial_{n_y}E(x,y)|$, write $|[u_{A^\nu}]|K = (|[u_{A^\nu}]|^2K)^{1/2}K^{1/2}$ and apply Cauchy's inequality to obtain:

$$|P_i^\nu(|[u_{A^\nu}]|)|^2 \le \Big(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y\Big)\Big(\int_{\partial B_i^\nu} K\, ds_y\Big). \tag{49}$$

Elementary estimates, (cf., Jackson, 1975), show that:

$$\|\int_{\partial B_i^\nu} K(x,y)\, ds_y\|_{L^\infty(\Omega)} \le 4\pi, \tag{50}$$

thus,

$$\int_{\Omega} |P_i^{\nu}(|[u_{A^\nu}]|)|^2 dx \leq 4\pi \int_{\Omega} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx. \tag{51}$$

Let $W \subset D$, be a thin shell containing the boundary ∂B_i^ν then,

$$\begin{aligned} \int_{\Omega} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx &= \int_{\Omega \setminus W} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx \\ &+ \int_{W} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx. \end{aligned} \tag{52}$$

For $x \in D \setminus W$ we have $K(x,y) \leq |x-y|^{-2}$ and application of Fubini's theorem gives:

$$\begin{aligned} \int_{\Omega \setminus W} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx &\leq \int_{\partial B_i^\nu} \left(\int_{\Omega \setminus W} |x-y|^{-2} dx \right) ([u_{A^\nu}])^2 ds_y \\ &\leq \sup_{y \in \Omega} \{ V_{1/3}(1)(x) \} \int_{\partial B_i^\nu} ([u_{A^\nu}])^2 ds_y \\ &\leq (3\omega_3^{1/3})^2 |\Omega|^{2/3} \int_{\partial B_i^\nu} ([u_{A^\nu}])^2 ds_y. \end{aligned} \tag{53}$$

To estimate the second term on the right-hand side of (52) we recall from Lemma 4.1, that $[u_{A^\nu}]$ lies in $C^0(\Gamma^\nu)$ thus $\|[u_{A^\nu}]\|_{L^\infty(\partial B_i^\nu)} < \infty$ and

$$\int_{W} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx \leq 4\pi |W| \, \|([u_{A^\nu}])^2\|_{L^\infty(\partial B_i^\nu)}. \tag{54}$$

Lemma 4.3 follows upon choosing W such that its volume tends to zero.

We conclude this Section by providing a uniform estimate on the normalized temperature fields given by:

THEOREM 4.2 *Given a sequence of suspensions consisting of p spheres in the class $SC_{0}p$, the associated temperature u_{A^ν} satisfies:*

$$\sup_{\nu} \|u_{A^\nu} - c^\nu\|_{H^1(\Omega \setminus \Gamma^\nu)} < \infty, \tag{55}$$

where

$$c^\nu = |L_\lambda|^{-1} \int_{L_\lambda} u_{A^\nu}(y) dy. \tag{56}$$

Proof:

We observe that $u_{A^\nu} - c^\nu$ is a solution to the equations of state (3.2)-(3.4) and:

$$\begin{aligned} E(A^\nu, g) &= \int_{\partial\Omega} g u_{A^\nu} ds = \int_{\partial\Omega} g(u_{A^\nu} - c^\nu) ds \\ &\leq \|g\|_{H^{-1/2}(\partial\Omega)} \|u_{A^\nu} - c^\nu\|_{H^{1/2}(\partial\Omega)} \\ &\leq \|g\|_{H^{-1/2}(\partial\Omega)} \|u_{A^\nu} - c^\nu\|_{H^1(\Omega \setminus \Gamma^\nu)}. \end{aligned} \tag{57}$$

From the Poincare inequality (34), it follows that

$$\|u_{A^\nu} - c^\nu\|^2_{H^1(\Omega\setminus\Gamma^\nu)} \le (1+pM)\Big(\int_\Omega |\nabla u_{A^\nu}|^2 dx + \int_{\Gamma^\nu} ([u_{A^\nu}])^2 ds\Big). \tag{58}$$

Next, set a = min{a_p, a_m,,6} and make the substitution 1Q= (UAv, UAv, •. . , UAv) in (3.5) to obtain:

$$\begin{aligned} E(A^\nu, g) &= \int_\Omega \sigma^\nu(x)|\nabla u_{A^\nu}|^2 dx + \beta \int_{\Gamma^\nu} ([u_{A^\nu}])^2 ds \\ &\ge \alpha\Big(\int_\Omega |\nabla u_{A^\nu}|^2 dx + \int_{\Gamma^\nu} ([u_{A^\nu}])^2 ds\Big). \end{aligned} \tag{59}$$

It is evident from (57), (58), and (59) that

sup ll ∫ln lv'uA"|²dx ⩽ oo, sup ll ∫lrv ([uA"])²ds ⩽ oo, (60)

and the theorem follows in view of (58) and (60).

5. Proof of the energy dissipation inequality

We start by introducing the second Stekloff eigenvalue p2 of a particle E. When I: has Lipschitz continuous boundary the variational formulation for the second Stekloff eigenvalue is given by:

P2 = rmin div(σpVφ)=D (f aE (apVφ · n)² ds) / (JE σpVφ · Vφdx), (61)

cf. Kuttler and Sigillito (1968). Separation of variables shows that for a sphere of radius a and conductivity a_p that P2 = a_p/a.

We now give the proof. For any g E H-¹¹²(8D) for which $f_a$ngds = 0, we write the difference .6.E = E(A UE, g) - E(A, g) as

·.6.E = C(A,]) - C(A,]) + D(E,]), (62)

where]= argmin{C(A UE,j)}, 3 = argmin{C(A,j)}, and D(E,]) is given by

D(E,3) = ,S-¹{ l{aE (3 · n)²ds - J{E ,B(a;;.¹ - a;;¹)] ·]dx}. (63)

Noting that the feild] is an admissible trial for the variational principle (1.1), we have:

C(A,3) - C(A,]) 0. (64)

Thus

.6.E D(E,J). (65)

Now, the equations of state for the temperature u E H¹ (!1\(r U8:E)) imply that] = a_p Vu in :E, [a(xA)Vu · n] = 0 on 8:E, and J ·nl$_2$ = $\bar{a}_p$ Vu· nl$_2$, on 8:E. Thus from (63) and (65) we obtain:

$$\text{b.E;::,:,B-1}\{ \int_{la} (a_p \text{Vu} \cdot n)^2 ds - \int ,B(\bar{a};,,1-a;^{1})a_p \text{Vu} \cdot a_p \text{Vudx}\}. \quad (66)$$

From (61), it follows that:

$$\int_{\partial\Sigma} (\sigma_p \nabla\varphi \cdot n)^2 ds - \rho_2 \int_{\Sigma} \sigma_p \nabla\varphi \cdot \nabla\varphi dx \geq 0 \quad (67)$$

for all φ E H³¹² (:E) such that div (a_p V_{cp}) = 0, in :E.

Comparing the right-hand side of (66) with (67), we discover that

$$\Delta E \geq 0 \quad (68)$$

for

$$\sigma_p \beta(\sigma_m^{-1} - \sigma_p^{-1})\sigma_p \leq \sigma_p \rho_2. \quad (69)$$

Theorem 1.1 follows noting that p_2 = a_p/a for a sphere of radius a.

We observe that strict inequality in(68) follows from strict inequality in (69), provided that Vu is not identically equal to zero on :E.

6. A proof of Theorem 1.2

We write the energy dissipation due to a distribution of current on the domain boundary as:

$$E(A, g) = \min\{C(A,j) : j \text{ E } 1^2 (!1)^3, \text{div } j = 0, \ j \cdot n = g \text{ on } 8!1\} \quad (70)$$

where

$$C(A, j) = \int_{\Omega} \{\sigma_m^{-1} - \chi_A(\sigma_m^{-1} - \sigma_p^{-1})\} \, j \cdot j dx + \beta^{-1} \int_{\Gamma} (j \cdot n)^2 ds. \quad (71)$$

Next we write the energy dissipated in the unreinforced domain as:

$$E(0,g) = \min\{C : j \text{ E } L^2 (n)^3, \text{div } j = 0, \ j \cdot n = g \text{ on } 8!1\} \quad (72)$$

where

$$C = h \ a;,,,1 \ j \cdot j dx. \quad (73)$$

We set] = argminE(0, g) and observe that it is an admissible trial field for (70). Substitution *of]* into (70) gives the estimate:

$$E(A, g) \leq E(0, g) + (a;;/ - ai;-^{1})(\int_{JaA} (J. n)^2 ds) T(A, g) \quad (74)$$

where $T(A, g)$ is given by (1.8) and Theorem 1.2 follows.

Lastly, we establish the inequality given by (1.9). Indeed, for the choice $g = \bar{j} \cdot n$ the quantity T(A, g) is given by.

$$T(A,g) = R_{cr} - \int_{A} |\bar{j}|^2 dx / \int_{\partial A} (\bar{j} \cdot n)^2 ds. \tag{75}$$

The inequality follows from $T(A,g) \leq 0$ upon noting that

$$\int_A |\bar{j}|^2 dx = |\bar{j}|^2 \times |A| \tag{76}$$

and

$$\int_{\partial A} (\bar{j} \cdot n)^2 ds = |\bar{j}|^2 \times \sum_{i=1}^{N} a_i^{-1} |Y_i|. \tag{77}$$

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References

AMBROSIO, L. and BUTTAZZO, G. (1993) An optimal design problem with perimiter penalization. *Calc. Var.,* 1, 55-69.

CHENAIS, D. (1975) On the exsistence of a solution in a domain identification problem. *Jour. Math. Anal. and Applications,* 52, 189-219.

CHENAIS, D. (1980) Homeomorphisme entre ouverts lipschitziens. *Annali di Matematica Pura ed Applicata,* 118, 343-398.

CHIEW, Y.C. and GLANDT, E.D. (1987) Effective conductivity of dispersions: the effect of resistance at the particle surfaces. *Chem. Eng. Sci.,* 42, 2677-2685.

EVERY, A.G., Tzou, Y., HASSELMAN, D.P.H. and RAJ, R. (1992) The effect of particle size on the thermal conductivity of ZnS/diamond composites. *Acta Metall. Mater.,* 40, 123-129.

GARRET, K.W. and ROSENBERG, H.M. (1974) The thermal conductivity of epoxy-resin powder composite materials. *J. Phys. D. Appl. Phys.,* 7, 1247-1258.

GILBARG, D. and TRUDINGER, N.S. (1983) *Elliptic Partial Differential Equations of Second Order.* (Grundlehren der math. Wiss., vol 224) Berlin Heidleberg New York: Springer.

GRISVARD, P. (1980) Boundary Value Problems in Non-Smooth Domains. *Univ. Maryland, Dept. of Math. Lecture Notes.* 19.

HASHIN, Z. (1983) The elastic moduli of heterogeneous materials. *J. Appl. Mech.,* 29, 143.

HASSELMAN, D.P.H. and DONALDSON, K.Y. (1992) Effect of reinforcement particle size on the thermal conductivity of a particulate-silicon carbide-reinforced aluminum matrix composite. *J. Am. Ceram. Soc.,* 75, 3137-3139.

HASSELMAN, D.P.H. and JOHNSON, L.F. (1987) Effective thermal conductivity of composites with interfacial thermal barrier resistance. *J. Compos. Mater.,* 21, 508-515.

JACKSON, J.D. (1975) *Classical Electrodynamics.* John Wiley and Sons. New York.

KUTTLER, J.R. and SIGILLITO, V.G. (1983) Inequalities for membrane and Stekloff eigenvalues. *lour. Math. Anal. and Applications,* 23, 148-160.

LENE, F. and LEGUILLON, D. (1982) Homogenized constitutive law for a partially cohesive composite material. *Int. J. Solids Structures,* 18, 433-458.

LIONS, J.L. and MAGENES, E. (1972) *Non-Homogeneous Boundary Value Problems and Applications I.* (Grundlehren der math. Wiss., vol 181) Berlin Heidleberg New York: Springer.

LIPTON, R. (1998) The Second Stekloff Eigenvalue and Energy Dissipation Inequalities for Functionals with Surface Energy. *Institute for Math and Applications Preprint,* 1410, July (1996). *SIAM J. on Analysis,* 29, 637-680.

LIPTON, R. (1996) The Second Stekloff Eigenvalue of an Inclusion and New Size Effects for composites with interface thermal barriers. *J. Appl. Phys.,* 80, 5583-5586.

LIPTON, R. (1998) Heat conduction in fine scale mixtures with interfacial contact resistance. *SIAM J. on Applied Mathematics,* 58, 55-72.

LURIE, K.A. and CHERKAEV, A.V. (1986) Effective characteristics of composite materials and the optimal design of structural elemets. *Uspekhi Mekaniki, Advances in Mechanics,* 9, 2.

MURAT, F. and TARTAR, L. (1985) Calcul des Variations et homogeneisation. In: *Les Methodes de l'Homogeneisation: Theorie et Applicatons en Physique Coll. de la Dir. des etudes et Reserches de Electr. del. France; Eyrolles, Paris,* 319-370.

PIRONNEAU, 0. (1984) *Optimal Shape Design for Elliptic Systems.* Springer-Verlag. Berlin.