

**G-closures of material sets in space-time and perspectives
of dynamic control in the coefficients of linear hyperbolic
equations**

by

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Abstract: The paper is focused on the concept of material mixing in dynamics. This concept introduces the *spatio-temporal composites*, i.e. material microstructures assembled and maintained in space-time.

The knowledge of the relevant sets of effective material parameters (G-closures) is crucial for the purpose of material optimization arising in dynamics of moving media, particularly acoustics and hydroacoustics, as well as in the high-frequency electronic engineering.

The analysis is based on covariant description of the mixing problem in terms of relativistic equations acting in Minkowski space. The specific characterization of a G-closure is given for a special case of a dielectric composite in one-dimensional electromagnetic wave propagation. If such composite is assembled in space-time from two dielectrics with the same ratio E/μ of permittivity E to permeability μ and different otherwise then the ratio E_0/μ_0 of the relevant *effective* parameters will preserve the same value. This is a conservation law for the wave impedance of the medium with respect to one-dimensional wave propagation.

Keywords: mixing in space-time, spatio-temporal composites, hyperbolic G-closure, screening effect

1. Introduction

This paper is concerned with the analysis of dynamic processes governed by linear hyperbolic equations with their senior coefficients variable in *space-time*. The coefficients are acting as controls; since they usually characterize the material properties, we speak about the material control.

The material control is known to be very effective in the elliptic case where the idea of a material mixing on a fine scale with formation of artificially assembled composites has become a key factor. The composites serve to fill the

gap that originally exists in the properties of initially available materials. This gap may be covered once we know the so called *G-closure* of the original set U of materials, i.e. the set GU of *all* composites assembled from the elements of U regardless of the microgeometry of such formations. The *G-closures* have been effectively constructed for several important elliptic operators, Lurie and Cherkaev (1997), but no results are yet known about the *hyperbolic G-closures*. These sets, obviously, preserve their significance in this new environment; as in the elliptic case, they make the relevant material optimization problems well-posed, leave alone a general description of effective properties of spatio-temporal material assemblages. Such assemblages may produce effects otherwise not feasible in the hyperbolic material control, e.g. the phenomenon of complete screening of some extended domain in space-time from the invasion of long wave dynamic disturbances.

The new material features incorporated in spatio-temporal composites are all encoded in the hyperbolic *G-closures*; for this reason, the problem of their explicit description receives a high priority. The efforts towards this goal were hampered until recently by the absence of a general scheme of constructing the hyperbolic *G-closure* as a set of invariant material characteristics of mixtures. Such a scheme has been proposed in Lurie (1998A) for a standard hyperbolic operator of the 2nd order arising in electrodynamics of moving dielectrics. It requires the introduction of a suitable Minkowski space and the parallel analysis of a Maxwell's system associated with the original hyperbolic equation. As a result, it became possible to characterize the *G-closure* in a non-trivial special case of one-dimensional wave propagation.

A solution of the general *G-closure* problem for an arbitrary set of isotropic dielectrics in Minkowski's space is yet to come. Another goal will be to develop the G_m -*closure* for a set of two isotropic dielectrics, i.e. the *G-closure* with the volume fraction of original materials specified. The third goal will be to describe *G-closures* for some selected sets of isotropic materials with respect to hyperbolic differential operators arising in acoustics and elastodynamics. This specification will require analysis of the relevant conceptual schemes not identical with the Maxwell's scheme suitable for electrodynamics.

2 Possible applications

Material mixing in space-time arises in many practical situations.

In electrical engineering, the slow wave circuits and the traveling wave tubes are used to maintain an effective coupling and the transport of energy between the electronic beams and transmission lines. To this end, the specific material characteristics of the line, i.e. its linear inductance and capacitance, are subjected to external *activation* in the form of a "pump wave" traveling along the line of Louisell (1960). This activation supplies the energy needed for an effective coupling. The "wave of linear capacitance" is generated through the use of $p - n$ junction diodes distributed along the line and appropriately activated in

time, Louisell (1960); a similar "wave of inductance" is created through the use of a linear arrangement of non-linear inductances. Both types of activation are maintained electronically at a desired speed. A spatio-temporal material pattern may also be generated in the dielectric continuum by a purely mechanical means, i.e. by bringing the neighboring portions of such continuum into a *relative mechanical motion*. This will occur if we apply a high frequency background mechanical vibration in the form of standing waves. The electromagnetic disturbances with wavelengths much larger than that of a background wave will perceive the medium as a spatio-temporal composite produced as a result of a high-frequency vibration.

In acoustics, a similar phenomenon may be implemented in the propagation of sound through ducts with variable cross sections. If a duct is filled by a mixture of water and air (or vapor) bubbles, then, under a suitable bubbles' concentration, the effective velocity of sound in the mixture may be made fairly low, of the order of meters per/sec, and the cross section of the duct may be subjected to a mechanical change in the form of a periodic wave traveling along the duct at a comparable speed. The duct will then be acting like a transmission line with variable parameters Morse, and Uno Ingard (1968).

The concept of smart materials Russell (1994) capable of changing their properties both in space and time appears to perfectly fit into this scheme as well.

In all cases, we assume that the senior coefficients of the relevant hyperbolic equations reveal a *chattering* behavior, i.e. they appear to be the fast periodic functions in space-time.

3. Second order hyperbolic equation with chattering coefficient and a screening phenomenon

To give an example illustrating the effect of chattering, consider a model hyperbolic equation

$$(pvt)t - (kvz)z = 0, \quad -\infty < z < \infty, \quad t \geq 0. \quad (1)$$

The coefficients p, k will be assumed fast periodic, i.e., periodic in the argument $(z - Vt)/\delta$ where δ is a small parameter and V is the velocity of the "property wave"

$$p = p\left(\frac{z - Vt}{\delta}\right), \quad k = k\left(\frac{z - Vt}{\delta}\right). \quad (2)$$

At $t = 0$, this wave becomes a periodic pattern

$$p = p(z/\delta), \quad k = k(z/\delta),$$

and (2) may be perceived as this pattern moving with a uniform velocity V along the z -axis.

To make this description more specific, we will assume that:

- (a) at each point (z, t) the controls p and k can take either the values (p_1, k_1) or (p_2, k_2) (we will refer to "material 1" and "material 2", respectively);
- (b) these materials are placed within alternating layers having the slope $dz/dt = V$ in the (z, t) - plane;
- (c) the period of the pattern is combined of two successive layers filled by materials 1 and 2, the volume fractions of these layers being m_1 and m_2 , respectively, ($m_1, m_2 \geq 0, m_1 + m_2 = 1$).

The slope V will be so chosen as to ensure regular transition of continuous disturbance $v(z, t)$ through the interface from one layer to another. In other words, we shall consider smooth solutions with both kinematic and dynamic compatibility conditions observed across the interface.

This condition will be satisfied if we postulate the following relationship between the characteristic slopes (phase velocities) $a_i = \sqrt{k_i/p_i}, i = 1, 2$, and V :

$$\frac{v^2 - a_i^2}{v^2 - V^2} \geq 0. \tag{3}$$

This inequality is obviously satisfied if $V = 0$ or $V = \infty$. Applying the standard procedure of homogenization, Bakhvalov and Panasenko (1997), we arrive at the following differential equation for the function (v) - the value of v averaged over the period of the array, Lurie (1997A,B):

$$-\overline{a^2} \nabla_{\{a\}}^2 [v^2 - k(i)](v) + 2v [p(\{i\}) - (i_2 \dots i_1)](v) + \dots - eG) [v' - \dots] (v),, o. \tag{4}$$

Here, $(v) = m_1 v_1 + m_2 v_2, k = m_1 k_2 + m_2 k_1$, etc.; the subscripts 1 and 2 are related to materials 1 and 2, respectively. Eq. (4) is hyperbolic once ineq. (3) is satisfied. It has no dispersion, i.e. it possesses wave solutions $(v)(z - \lambda_1 t), (v)(z - \lambda_2 t)$. The characteristic velocities λ_1 and λ_2 have the product

$$A_1 A_2 = - \overline{a^2} \overline{p} \left(\frac{1}{k} \frac{v^2 - k_0}{v^2 - k(i)} \right) \tag{5}$$

Assuming that $a_2 = \sqrt{k_2/p_2} \geq a_1 = \sqrt{k_1/p_1}$, we shall consider two cases:

- (a) $k_2 \geq k_1, p_2 \leq p_1$ (regular mode),
- (b) $k_2 \geq k_1, p_2 \geq p_1$ (irregular mode).

Inequality $a_2 \geq a_1$ that apparently holds for case (a), will be assumed to hold for case (b) as well.

Eqs.(4), (5) show that, Lurie (1997),

$$\begin{aligned} \lambda_1 = -\lambda_2 &= J(\overline{k - 1} - 1(p) - 1) \text{ for } V = 0 \text{ (static spatial pattern),} \\ \lambda_1 = -\lambda_2 &= J(\overline{k})(p - 1) \text{ for } V = \infty \text{ (dynamic temporal pattern).} \end{aligned} \quad (6)$$

It can be seen from these formulas that for the regular mode the long wave disturbances may propagate at the speed that always belongs to the interval $[a_1, a_2]$. For the irregular mode, however, this speed *may lie beyond* this interval. Particularly, if $V = 0$, then always $v = a_1 \leq a_2$; however, either of the inequalities $v = a_1 \geq a_2$ at or $v = a_1 \leq a_2$ may occur. Also, if $V = \infty$, then always $v = a_2$ at, but there may either be $v = a_2 \leq a_1$ or $v = a_2 \geq a_1$. Long wave disturbances are slowed down by a static pattern and may be speeded up by a dynamic pattern.

In the first case, high frequency harmonics are suppressed because of redistribution of energy through the multiple reflections and refractions on the interfaces that do not move. In the second case ($V = \infty$), the situation is different. Energy is then *pumped* into the system each time the value of p decreases, and it is *withdrawn* from it as p increases. The phasing of these events, by a suitable choice of m_1 , affects the phase velocity of the long wave disturbances.

Another phenomenon associated with the irregular mode occurs when $V \neq 0(\infty)$. Eq. (5) shows that $v_1 v_2$ may become positive. This happens when V^2 belongs to either of the following two ranges:

$$V^2 < a_1^2 < a_2^2 \text{ (slow range),}$$

$$a_1 < a_2 < V^2 < k \text{ (!) (fast range).}$$

Both ranges are possible for the irregular mode and do not apply for the regular mode.

With this choice of V^2 , one obtains spatio-temporal arrays (laminates) for which both effective phase velocities v_1, v_2 become of the same sign. This sign may be switched to opposite as we go from V to $-V$.

The relevant waves will become *coordinated*: they will propagate both in the same direction relative to a laboratory frame. We will then refer to the right (left) laminate if both waves travel to the right (left).

The use of such laminates may implement the effect of screening mentioned in section 1. To this end we may place the right laminate in the 1st quadrant $z > 0, t \geq 0$ of the (z, t) -plane, and the left laminate in the 2nd quadrant. Within a sector bounded by the characteristic rays starting at the origin and having slopes $-v_2, v_2$ ($v_1 \geq v_2$), there appears to be a shadow zone protected from

the invasion of long wave disturbances initiated at $t = 0$. Such disturbances will propagate into the rest of the half plane $z > 0$. A similar effect was observed in Lurie (1971) for the first order equation

$$V_t + UV_z = 0, \quad (7)$$

with $u(z, t)$ treated as a control. It was shown that a "control wave" $u(z - Vt)$ may create a "shadow zone" on the (z, t) -plane totally screened from the invasion of *all* dynamic disturbances. This phenomenon is typical in the traffic control (see Whitham, 1974).

Contrary to Lurie (1971), the screening now affects the long-wave disturbances alone; it does not apply in narrow regions near the boundary of the domain in the (z, t) -plane. If the system occupies a segment $[a, b]$ of the z -axis, then the boundary layers will arise close to its endpoints.

Coordinated wave motion will not apply there; waves will be reflected from the endpoints and travel back to the interior of $[a, b]$ but these waves will carry only higher harmonics.

The analysis of Eq. (1) with coefficients periodic in the argument $z - Vt$ has also been carried out with the means of Floquet theory, Lurie (1998B). Eq. (4) then becomes a long wave Floquet approximation related to a low frequency passing band. A shorter wave approximations may also be obtained by this theory.

4. The analytic scheme of constructing the hyperbolic G -closures for dielectric materials. The use of Maxwell's system

A correct setting of the hyperbolic G -closure problem requires the use of Minkowski space. We will illustrate the relevant procedure by an example related to electrodynamics of inverse dielectric medium. The basic differential equation (1) is then perceived as a consequence of the Maxwell's system describing the behavior of a dielectric medium. The variable coefficients p, k then stand, respectively, for parameters μ and $1/\epsilon$ interpreted as the magnetic permeability and inverse dielectric permittivity of the medium. They are incorporated into a single material tensor of the 4th rank entering the material equation which becomes a part of the ultimate Maxwell's system represented in covariant form in Minkowski space.

The material tensor becomes the main object of the future work on G -closures; the elements of these sets are defined as invariants of a G -limiting material tensor.

In the absence of currents and charges, the Maxwell's system combines the equations

$$\text{curl} E = -B t,$$

$$\begin{aligned} \operatorname{div} B &= 0, \\ \operatorname{curl} H &= D_t, \\ \operatorname{div} D &= 0, \end{aligned} \tag{8}$$

with the material relations that take the form

$$D = c E, \quad B = \mu H \tag{9}$$

in a laboratory frame, i.e. the frame immovable with respect to a material medium.

Considering a one-dimensional plane wave

$$E = E(z, t)j, \quad B = B(z, t)i, \quad H = H(z, t)i, \quad D = D(z, t)j, \tag{10}$$

we obtain a system

$$E_z = B_t, \quad H_z = D_t. \tag{11}$$

By introducing the vector and scalar potentials A, A^*, ϕ , and ϕ^* through the formulas

$$\begin{aligned} E &= -\operatorname{grad} \phi - A_t, \quad B = \operatorname{curl} A, \\ H &= -\operatorname{grad} \phi^* + A; \quad D = \operatorname{curl} A^*, \end{aligned} \tag{12}$$

and by taking

$$A = -uj, \quad A^* = vi, \quad \phi = \phi^* = 0, \tag{13}$$

we satisfy Eqs. (11) because

$$E = U_t, \quad B = U_z, \quad H = V_t, \quad D = V_z. \tag{14}$$

The use of (9) now yields

$$V_z = EU_t, \quad U_z = \mu V_t, \tag{15}$$

which is equivalent to (1) with the correspondence $\mu \leftrightarrow 1/c \leftrightarrow k$. The pair $(c(z, t), \mu(z, t))$ is assumed to take at each point (z, t) of space-time only one of two admissible values (cf. condition (a) in section 3)

$$(c(z, t), \mu(z, t)) = \begin{cases} (\mu_1) & \text{"material 1"}, \\ (\mu_2) & \text{"material 2"}. \end{cases} \tag{16}$$

The law of dependence of (c, μ) on z, t is arbitrary, with a sole observation of Ineq. (3).

Starting from Eqs.(15), we may recover the relevant Maxwell's system (8)-(9) applying the above argument in the inverse direction. This step is important because it creates the conceptual basis for a covariant formulation of the G -closure problem. In the next section, we develop the relevant scheme for a general case of a Maxwell's system.

5. The Maxwell's system for a moving dielectric medium. A material conservation law in one-dimensional wave propagation

The pairs of vectors (B, E) and (H, D) are known to generate two skew-symmetric tensors in Minkowski 4-space ($x_1 = x, x_2 = y, x_3 = z, x_4 = ict$)

$$F = (cB, -iE) = \begin{pmatrix} 0 & cB_3 & -cB_2 & -iE_1 \\ -cB_3 & 0 & cB_1 & -iE_2 \\ cB_2 & -cB_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}, \tag{17}$$

$$f = (H, -icD) = \begin{pmatrix} 0 & H_3 & -H_2 & -icD_1 \\ -H_3 & 0 & H_1 & -icD_2 \\ H_2 & -H_1 & 0 & -icD_3 \\ icD_1 & icD_2 & icD_3 & 0 \end{pmatrix}. \tag{18}$$

Here, c denotes the velocity of light *in vacuum*, and B_1, \dots, D_3 denote the relevant vector components.

Eqs. (8) now obtain the covariant tensor form

$$\frac{\partial F_{ik}^*}{\partial x_k} = 0, \quad \frac{\partial f_{ik}}{\partial x_k} = 0, \tag{19}$$

where F_{ik} is a tensor dual to F_{ik} , i.e.

$$\bar{F}_{ik}^* = \frac{1}{2\epsilon_{iklm}} \bar{F}_{lm}, \tag{20}$$

with ϵ_{iklm} denoting a completely antisymmetric tensor of the 4th rank.

In the space of skew-symmetric tensors of the 2nd rank in four dimensions we introduce an orthogonal basis defined as a set of skew-symmetric 2nd rank tensors a_{ik} ($i, k = 1, 2, 3, 4$) specified by the formulas

$$\begin{aligned} a_{12} &= 1/\sqrt{2}(i_1 i_2 - i_2 i_1), & a_{13} &= 1/\sqrt{2}(i_1 i_3 - i_3 i_1), & a_{14} &= 1/\sqrt{2}(i_1 i_4 - i_4 i_1), \\ a_{23} &= 1/\sqrt{2}(i_2 i_3 - i_3 i_2), & a_{24} &= 1/\sqrt{2}(i_2 i_4 - i_4 i_2), \\ & & a_{34} &= 1/\sqrt{2}(i_3 i_4 - i_4 i_3), \end{aligned}$$

with $i_a, a = 1, 2, 3, 4$ being the orthonormal basis of unit vectors pointing along the X_a -axes; we see that

$$a_{ik} : a_{lm}^T = \begin{cases} 1, & i = l, k = m, \\ 0 & \text{otherwise.} \end{cases}$$

The material equations (9) are now incorporated in a single tensor relationship

$$f = s : F \tag{21}$$

where the 4th rank material tensor s is given in a laboratory frame by the formula, Lurie (1998A),

$$s = -\frac{1}{\mu c}(a_{12}a_{12} + a_{13}a_{13} + a_{23}a_{23}) - \epsilon c(a_{14}a_{14} + a_{24}a_{24} + a_{34}a_{34}). \quad (22)$$

Eqs. (17)-(21) constitute a standard system required for constructing the G -closure. This system possesses the same structure as in the relevant elliptic case: it includes two original entities - the skew symmetric tensors F and f in Minkowski space; these tensors are linked through a linear material equation (21). When homogenization is applied to this system, it reveals a G -limiting material tensor, and invariants of this tensor become the elements of a G -closure.

Eq. (22) shows that a dielectric medium that is assumed isotropic in space is *anisotropic in space-time* unless $c^2 = 1/E\mu$. This latter possibility holds for a vacuum which is the only medium isotropic in Minkowski space. Disregarding this case, we may single out the following three types of problems of mixing the materials isotropic in a conventional sense.

First, there is a case of materials differing in the eigenvalues $1/\mu c$, \mathbb{E} of their s -tensors, i.e. in the values of their (E, μ) -pairs. These pairs may address different values in the spatio-temporal domains with interfaces consistent with Ineq. (3); the layers of multiple rank matching this condition may serve as examples of such a situation. As to the eigentensors, they will be assumed the same for both materials in this case. This means that the materials are not subjected to a relative motion which is the only factor responsible for the difference in eigentensors. The difference in material properties is implemented in these circumstances through the mechanism of *activation* described in section 2.

The second type of mixing problems arises when the original materials differ in their eigentensors alone. This difference will not exist if both materials are isotropic and identical in a conventional sense, and if *they are not exposed to a relative motion*. This motion alone makes the materials different from one another in this situation because it creates the difference in their eigentensors. When such motion occurs, we arrive at what may be called a *spatio-temporal polycrystal* assembled in space-time from the conventionally isotropic materials.

The required type of motion may be created as a high frequency background mechanical vibration imposed on a dielectric continuum in the form of standing waves. This mechanism, also mentioned in section 2, makes the long wave electromagnetic disturbances perceive the medium as a spatio-temporal polycrystal of the above type.

The third type of mixing problems arises when both factors - activation and motion - are acting simultaneously, generating the difference in the ultimate material properties in space-time. In this case, both the eigenvalues and eigentensors of the relevant s -tensors become different for the original materials involved.

A similar classification applies when the original materials possess a spatial anisotropy, i.e., they are anisotropic in a conventional sense.

In one spatial dimension, the G-closure of a set of two conventionally isotropic materials allows for an important characterization when the determinant E/μ of the s-tensor possesses the same value for both materials; otherwise, these materials may be arbitrary and particularly exposed to a relative motion. Namely, the determinant E_0/μ_0 of the effective tensor s_0 appears in these circumstances to be equal to $E_1/\mu_1 = c_2/\mu_2$, i.e. the common value of the original determinant, Lurie (1998A,B). This conservation law represents a hyperbolic analog of the statement that holds for a similar elliptic 2nd order situation in a plane (cf. Lurie and Cherkaev, 1997); particularly, a hyperbolic polycrystal preserves, like its planar elliptic analog, the value of the determinant $dets_0$ of its effective material tensor.

6. The problem of material bounds

In the case of a plane elliptic polycrystal, the sole fact that the determinant $dets_0 = \lambda_1, \lambda_2$ of an effective material tensor remains equal to the determinant $dets$ of the original material tensor, does not completely characterize the G-closure, i.e. the set of all possible plane polycrystals. This characterization becomes complete after we add the inequality $I_i(s_0) \leq f_i(s)$ indicating that the mixing operation may only reduce the original material anisotropy. This latter inequality appears to be a consequence of the Reuss-Voigt estimates of the eigenvalues of an effective tensor s_0 , and these estimates come up as a consequence of the minimal variational principle that generates the basic differential equation of elliptic type.

The hyperbolic equation (1) is also generated by a variational principle, but this principle establishes just stationarity, not minimality of the relevant functional of action. For this reason, the point (λ_1, λ_2) in the plane of eigenvalues of the s_0 -tensor for a hyperbolic polycrystal may move along the hyperbola $dets_0 = \lambda_1, \lambda_2 = dets$ either towards or away from the diagonal, thus making the effective tensor s_0 either less or more anisotropic compared to the original tensor s . The reason for this is because the estimates of Reuss-Voigt type are no more in effect since the functional of action itself is not positive (or negative) definite, and, consequently, it is neither lower nor the upper bounded.

We may expect to regain control over the eigenvalues once we deliberately restrict the set of admissible tensors F (or \mathcal{D}) to make the functional of action bounded on this new set. How to specify such restrictions, still remains an open question.

7. Acoustics, elastodynamics

The general scheme of constructing the G-closures developed above for electro-dynamics of moving dielectrics cannot be directly extended to acoustics and

elastodynamics.

In both cases, it will be necessary to single out entities that will enter the scheme as analogs of the tensors F and f appearing in electrodynamics of dielectrics. Material tensors (the analogs of s) that link those entities should also obtain a clear conceptual characterization. These objects may be adequately identified on the basis of the relevant relativistic equations.

8. Conclusion

Over the last two decades, structural design has become *optimal* to a large extent. Optimization is understood in this context primarily as an appropriate layout of constructive materials *in space*, i.e. throughout the body of a structure. This design is necessarily *static*, i.e. it works well in a static environment, and is less appropriate whenever the construction is exposed to dynamic environment. A suitable dynamic response requires that the material medium possesses characteristics variable both *in space and time*, and conventional optimization technique must be adjusted to fit into this new environment. The results may find applications through the design of devices aimed to effectively command the dynamic behavior of distributed systems; particularly, these devices may either suppress undesired vibrations, or, contrary to that, maintain and control vibrations whenever they should come into being.

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