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Hermite-Hadamard type inequalities for h-preinvex mappings via fractional integrals*

by

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Abstract: In this paper, we first establish the Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals for the h-preinvex function. Then, some Hermite-Hadamard type integral inequalities for the fractional integrals are obtained.

Keywords: Riemann-Liouville integrals, Hermite-Hadamard type inequalities, h-preinvex function

1. Introduction and preliminaries

The following definition is well known in the literature: a function $\mathbf{f}:\mathbf{I}\to\mathbf{R}$, $\phi\neq\mathbf{I}\subset\mathbf{R}$, is said to be convex on I if the inequality

$$\mathbf{f}\left(\mathbf{tx}+\left(\mathbf{1}-\mathbf{t}\right)\mathbf{y}\right)\leqslant\mathbf{tf}\left(\mathbf{x}\right)+(\mathbf{1}-\mathbf{t})\mathbf{f}(\mathbf{y}) \tag{1}$$

holds for all $\mathbf{x},\ \mathbf{y}\in\mathbf{I}$, and $\mathbf{t}\in[\mathbf{0},\ \mathbf{1}]$.

Many important inequalities have been established for the class of convex functions, but the most famous among them is the Hermite-Hadamard inequality. This double inequality is formulated as follows:

$$\mathbf{f}\left(\frac{\mathbf{a}+\mathbf{b}}{\mathbf{2}}\right)\leqslant\frac{\mathbf{1}}{\mathbf{b}-\mathbf{a}}\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{f}\left(\mathbf{x}\right)\mathbf{dx}\leqslant\frac{\mathbf{f}\left(\mathbf{a}\right)+\mathbf{f}\left(\mathbf{b}\right)}{\mathbf{2}} \tag{2}$$

where $\mathbf{f}:[\mathbf{a},\ \mathbf{b}]\to\mathbf{R}$ is a convex function.

In 1978, Breckner (Breckner, 1978) introduced an $s-$convex function as a generalization of a convex function. Such a function is defined in the following way: a function $\mathbf{f}:[\mathbf{0},\infty)\to\mathbf{R}$ is said to be $s-$convex in the second sense if

$$\mathbf{f}\left(\mathbf{tx}+\left(\mathbf{1}-\mathbf{t}\right)\mathbf{y}\right)\leqslant\mathbf{t}^{\mathbf{s}}\mathbf{f}\left(\mathbf{x}\right)+(\mathbf{1}-\mathbf{t})^{\mathbf{s}}\mathbf{f}(\mathbf{y}) \tag{3}$$

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holds for all $\mathbf{x}$, $\mathbf{y} \in [\mathbf{0}, \infty)$, $\mathbf{t} \in [\mathbf{0}, \mathbf{1}]$, and for fixed $\mathbf{s} \in [\mathbf{0}, \mathbf{1}]$.

Dragomir and Fitzpatrick (1999) proved the following variant of the Hermite-Hadamard inequality for $s-$convex functions:

$$\mathbf{2^{s-1} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{s+1}}. \tag{4}$$

In 2007, Varošanec (Varošanec, 2007) introduced a large class of non-negative functions, the so-called $h-$convex functions. This class contains several well-known classes of functions, such as non-negative convex functions. The class is defined in the following way: a non-negative function $\mathbf{f} : \mathbf{I} \to \mathbf{R}$, $\phi \neq \mathbf{I} \subset \mathbf{R}$ is an interval, is called $h-$convex if

$$\mathbf{f(tx+(1-t)\,y) \leqslant h(t)f(x)+h(1-t)f(y)} \tag{5}$$

holds for all $\mathbf{x}, \mathbf{y} \in \mathbf{I}$, $\mathbf{t} \in (\mathbf{0}, \mathbf{1})$, where $\mathbf{h} : \mathbf{J} \to \mathbf{R}$ is a non-negative function, $\mathbf{h} \not\equiv \mathbf{0}$ and J is an interval, $(\mathbf{0}, \mathbf{1}) \subseteq \mathbf{J}$.

Sarikaya, Saglam and Yildirim (2008) proved that for an $h-$convex function the following variant of the Hermite-Hadamard inequality is fulfilled:

$$\mathbf{\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx}$$
$$\mathbf{\leqslant [f(a)+f(b)] \cdot \int_0^1 h(t)\,dt}. \tag{6}$$

In 1988, Weir and Mond (1988) introduced the concept of preinvex functions. We now recall some notions from the invexity analysis, which will be used throughout the article.

A set $\mathbf{S} \subseteq \mathbf{R^n}$ is said to be invex with respect to the map $\eta : \mathbf{S} \times \mathbf{S} \to \mathbf{R^n}$, if for every $\mathbf{x}$, $\mathbf{y} \in \mathbf{S}$ and $\mathbf{t} \in [\mathbf{0}, \mathbf{1}]$,

$$\mathbf{y} + \mathbf{t}\,\eta\,(\mathbf{x}, \mathbf{y}) \in \mathbf{S}.$$

Let $\mathbf{S} \subseteq \mathbf{R^n}$ be an invex set with respect to $\eta :\ \mathbf{S} \times \mathbf{S} \to \mathbf{R^n}$. Then, the function $\mathbf{f} : \mathbf{S} \to \mathbf{R}$ is said to be preinvex with respect to η, if for every $\mathbf{x}$, $\mathbf{y} \in \mathbf{S}$ and $\mathbf{t} \in [\mathbf{0}, \mathbf{1}]$,

$$\mathbf{f}(\mathbf{y} + \mathbf{t}\,\eta\,(\mathbf{x}, \mathbf{y})) \leqslant \mathbf{tf}(\mathbf{x}) + (\mathbf{1} - \mathbf{t})\,\mathbf{f}(\mathbf{y})\,. \tag{7}$$

For the further reasoning, we also need the following assumption, regarding the function η, this assumption being due to Mohan and Neogy (Mohan and Neogy, 1995).

Condition C Let $\mathbf{S} \subseteq \mathbf{R}^{\mathbf{n}}$ be an open invex subset with respect to η. For any $\mathbf{x}, \mathbf{y} \in \mathbf{S}$ and $\mathbf{t} \in [\mathbf{0}, \mathbf{1}]$,

$$\eta(\mathbf{y}, \mathbf{y} + \mathbf{t}\,\eta(\mathbf{x}, \mathbf{y})) = -\mathbf{t}\,\eta(\mathbf{x}, \mathbf{y})$$

$$\eta(\mathbf{x}, \mathbf{y} + \mathbf{t}\,\eta(\mathbf{x}, \mathbf{y})) = (\mathbf{1} - \mathbf{t})\,\eta(\mathbf{x}, \mathbf{y}).$$

In 2009, Noor (2009) proved the Hermite-Hadamard inequality for preinvex functions under the assumption that the Condition C is fulfilled:

$$\mathbf{f}\left(\mathbf{a}+\frac{\mathbf{1}}{\mathbf{2}}\eta(\mathbf{b}, \mathbf{a})\right) \leqslant \frac{\mathbf{1}}{\eta(\mathbf{b}, \mathbf{a})} \int\limits_{\mathbf{a}}^{\mathbf{a}+\eta(\mathbf{b}, \mathbf{a})} \mathbf{f}(\mathbf{x})\,\mathbf{dx} \leqslant \frac{\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{b})}{\mathbf{2}}. \tag{8}$$

The present author introduced in Matłoka (2013) the concept of $h-$preinvex function. Such a function is defined in the following way: a non-negative function $\mathbf{f}$ on the invex set $\boldsymbol{S}$ is said to be $h-$preinvex with respect to η if

$$\mathbf{f}(\mathbf{y} + \mathbf{t}\,\eta(\mathbf{x}, \mathbf{y})) \leqslant \mathbf{h}(\mathbf{1} - \mathbf{t})\,\mathbf{f}(\mathbf{y}) + \mathbf{h}(\mathbf{t})\,\mathbf{f}(\mathbf{x})$$

holds for all $\mathbf{x}, \mathbf{y} \in \mathbf{S}$ and $\mathbf{t} \in [\mathbf{0}, \mathbf{1}]$.

If $\mathbf{h}(\mathbf{t}) = \mathbf{t}^{\mathbf{s}}$, then the function is called s-preinvex.

In the same paper, the present author proved the Hermite-Hadamard inequality for h-preinvex functions:

$$\frac{\mathbf{1}}{\mathbf{2h}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)}\mathbf{f}\left(\mathbf{a}+\frac{\mathbf{1}}{\mathbf{2}}\eta(\mathbf{b}, \mathbf{a})\right) \leqslant \frac{\mathbf{1}}{\eta(\mathbf{b}, \mathbf{a})} \int\limits_{\mathbf{a}}^{\mathbf{a}+\eta(\mathbf{b}, \mathbf{a})} \mathbf{f}(\mathbf{x})\,\mathbf{dx} \leqslant [\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{b})] \cdot \int\limits_{\mathbf{0}}^{\mathbf{1}} \mathbf{h}(\mathbf{t})\,\mathbf{dt}. \tag{9}$$

Sarikaya, Set, Yaldiz and Basak (2013) established the following Hermite-Hadamard's inequalities for the Riemann-Liouville fractional integral:

$$\mathbf{f}\left(\frac{\mathbf{a}+\mathbf{b}}{\mathbf{2}}\right) \leqslant \frac{\boldsymbol{\Gamma}(\alpha+\mathbf{1})}{\mathbf{2}(\mathbf{b}-\mathbf{a})^{\alpha}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^+}\mathbf{f}(\mathbf{b}) + \mathbf{J}^{\alpha}_{\mathbf{b}^-}\mathbf{f}(\mathbf{a})\right] \leqslant \frac{\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{b})}{\mathbf{2}} \tag{10}$$

where $\mathbf{f}$ is a convex function and the symbols $\mathbf{J}^{\alpha}_{\mathbf{a}^+}\mathbf{f}$ and $\mathbf{J}^{\alpha}_{\mathbf{b}^-}\mathbf{f}$ denote, respectively, the left- and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in \mathbf{R}^+$ that are defined by

$$\mathbf{J}^{\alpha}_{\mathbf{a}^+}\mathbf{f}(\mathbf{x}) = \frac{\mathbf{1}}{\boldsymbol{\Gamma}(\alpha)} \int\limits_{\mathbf{a}}^{\mathbf{x}} (\mathbf{x}-\mathbf{t})^{\alpha-\mathbf{1}}\,\mathbf{f}(\mathbf{t})\mathbf{dt}\ (\mathbf{0} \leqslant \mathbf{a} < \mathbf{x} \leqslant \mathbf{b})$$

and

$$\mathbf{J}^{\alpha}_{\mathbf{b}^-}\mathbf{f}(\mathbf{x}) = \frac{\mathbf{1}}{\boldsymbol{\Gamma}(\alpha)} \int\limits_{\mathbf{x}}^{\mathbf{b}} (\mathbf{t}-\mathbf{x})^{\alpha-\mathbf{1}}\,\mathbf{f}(\mathbf{t})\mathbf{dt}\ (\mathbf{0} \leqslant \mathbf{a} \leqslant \mathbf{x} < \mathbf{b}),$$

respectively. Here, $\mathbf{\Gamma}(\bullet)$ is the gamma function.

The aim of this paper is to establish Hermite-Hadamard inequalities for Riemann-Liouville fractional integral, and some other integral inequalities, using identity which is obtained for fractional integrals.

2. Hermite-Hadamard's inequalities via fractional integrals

Hermite-Hadamard's inequalities can be represented in fractional integral forms as follows.

Theorem 1 *Suppose that* $\mathbf{f}:[\mathbf{a},\,\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})]\to\mathbf{R}$ *is an h-preinvex function, Condition C for* η *holds and* $\mathbf{a}<\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})$, $\mathbf{h}\left(\frac{1}{2}\right)>\mathbf{0}$. *Then, the following inequalities hold:*

$$\begin{aligned}&\frac{\mathbf{1}}{\alpha\cdot\mathbf{h}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)}\mathbf{f}\left(\mathbf{a}+\frac{\mathbf{1}}{\mathbf{2}}\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)\\ &\leqslant\frac{\mathbf{\Gamma}(\alpha)}{\eta(\mathbf{b},\,\mathbf{a})^{\alpha}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\left(\mathbf{a}\right)+\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\left(\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})\right)\right]\\ &\leqslant\left[\mathbf{f}\left(\mathbf{a}\right)+\mathbf{f}\left(\mathbf{b}\right)\right]\cdot\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha-\mathbf{1}}\left[\mathbf{h}(\mathbf{1}-\mathbf{t})+\mathbf{h}(\mathbf{t})\right]\mathbf{dt}.\end{aligned}\tag{11}$$

with $\alpha>\mathbf{0}$.

Proof. From the definition of an h-preinvex function and from Condition C for η it follows that:

$$\mathbf{f}\left(\mathbf{a}+\frac{\mathbf{1}}{\mathbf{2}}\,\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\right)$$

$$=\mathbf{f}\left(\mathbf{a}+\mathbf{t}\,\eta\,\left(\mathbf{b},\,\mathbf{a}\right)+\frac{\mathbf{1}}{\mathbf{2}}\eta\left(\mathbf{a}+\left(\mathbf{1}-\mathbf{t}\right)\eta\left(\mathbf{b},\,\mathbf{a}\right),\,\mathbf{a}+\mathbf{t}\,\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\,\right)\right)$$

$$\leqslant\mathbf{h}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)\left[\mathbf{f}\left(\mathbf{a}+\mathbf{t}\,\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{f}\left(\mathbf{a}+\left(\mathbf{1}-\mathbf{t}\right)\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)\right].$$

By multiplying both sides by $\mathbf{t}^{\alpha-\mathbf{1}}$, and then integrating the resulting inequality with respect to t over $[\mathbf{0},\,\mathbf{1}]$, we obtain

$$\begin{aligned}
&\frac{\mathbf{1}}{\alpha\cdot\mathbf{h}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)}\mathbf{f}\left(\mathbf{a}+\frac{\mathbf{1}}{\mathbf{2}}\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)\\
&\leqslant\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha-\mathbf{1}}\mathbf{f}\left(\mathbf{a}+\mathbf{t}\,\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)\mathbf{dt}+\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha-\mathbf{1}}\mathbf{f}\left(\mathbf{a}+\left(\mathbf{1}-\mathbf{t}\right)\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)\mathbf{dt}\\
&=\int_{\mathbf{a}}^{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})}\left(\frac{\mathbf{u}-\mathbf{a}}{\eta\left(\mathbf{b},\,\mathbf{a}\right)}\right)^{\alpha-\mathbf{1}}\mathbf{f}\left(\mathbf{u}\right)\frac{\mathbf{du}}{\eta\left(\mathbf{b},\,\mathbf{a}\right)}+\\
&\int_{\mathbf{a}}^{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})}\left(\frac{\mathbf{a}+\eta\left(\mathbf{b},\,\mathbf{a}\right)-\mathbf{u}}{\eta\left(\mathbf{b},\,\mathbf{a}\right)}\right)^{\alpha-\mathbf{1}}\mathbf{f}\left(\mathbf{u}\right)\frac{\mathbf{du}}{\eta\left(\mathbf{b},\,\mathbf{a}\right)}\\
&=\frac{\mathbf{\Gamma}(\alpha)}{\eta(\mathbf{b},\,\mathbf{a})^{\alpha}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\left(\mathbf{a}\right)+\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\left(\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})\right)\right]
\end{aligned}$$

i.e.

$$\begin{aligned}
&\frac{\mathbf{1}}{\alpha\cdot\mathbf{h}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)}\mathbf{f}\left(\mathbf{a}+\frac{\mathbf{1}}{\mathbf{2}}\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)\\
&\leqslant\frac{\mathbf{\Gamma}\left(\alpha\right)}{\eta\left(\mathbf{b},\,\mathbf{a}\right)^{\alpha}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\left(\mathbf{a}\right)+\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\left(\mathbf{a}+\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)\right].
\end{aligned}$$

For the proof of the second inequality we first note that if $\mathbf{f}$ is an h-preinvex function, then for $\mathbf{t}\in[\mathbf{0},\,\mathbf{1}]$ it yields

$$\mathbf{f}\left(\mathbf{a}+\mathbf{t}\,\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\right)\leqslant\mathbf{h}\left(\mathbf{1}-\mathbf{t}\right)\mathbf{f}\left(\mathbf{a}\right)+\mathbf{h}\left(\mathbf{t}\right)\mathbf{f}\left(\mathbf{b}\right)$$

and

$$\mathbf{f}\left(\mathbf{a}+\left(\mathbf{1}-\mathbf{t}\right)\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\right)\leqslant\mathbf{h}\left(\mathbf{t}\right)\mathbf{f}\left(\mathbf{a}\right)+\mathbf{h}\left(\mathbf{1}-\mathbf{t}\right)\mathbf{f}\left(\mathbf{b}\right).$$

By adding these inequalities we obtain

$$\mathbf{f}\left(\mathbf{a}+\mathbf{t}\,\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{f}\left(\mathbf{a}+\left(\mathbf{1}-\mathbf{t}\right)\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\right)\leqslant\left[\mathbf{h}\left(\mathbf{1}-\mathbf{t}\right)+\mathbf{h}\left(\mathbf{t}\right)\right]\left[\mathbf{f}\left(\mathbf{a}\right)+\mathbf{f}\left(\mathbf{b}\right)\right].$$

Then, by multiplying both sides by $\mathbf{t}^{\alpha-\mathbf{1}}$ and integrating the resulting inequality with respect to t over $[\mathbf{0},\mathbf{1}]$, we obtain

$$\begin{aligned}
&\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha-\mathbf{1}}\mathbf{f}\left(\mathbf{a}+\mathbf{t}\,\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)\mathbf{dt}+\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha-\mathbf{1}}\mathbf{f}\left(\mathbf{a}+\left(\mathbf{1}-\mathbf{t}\right)\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)\mathbf{dt}\\
&\leqslant\left[\mathbf{f}\left(\mathbf{a}\right)+\mathbf{f}\left(\mathbf{b}\right)\right]\cdot\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha-\mathbf{1}}\left[\mathbf{h}(\mathbf{1}-\mathbf{t})+\mathbf{h}(\mathbf{t})\right]\,\mathbf{dt}
\end{aligned}$$

i.e.

$$\frac{\mathbf{\Gamma}(\alpha)}{\eta(\mathbf{b},\,\mathbf{a})^{\alpha}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\,(\mathbf{a})+\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\,(\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a}))\right]$$

$$\leqslant[\mathbf{f}\,(\mathbf{a})+\mathbf{f}\,(\mathbf{b})]\cdot\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha-\mathbf{1}}\,[\mathbf{h}(\mathbf{1}-\mathbf{t})+\mathbf{h}(\mathbf{t})]\,\mathbf{dt}.$$

The proof is completed. □

Remark 1 If in Theorem 1, we let $\alpha=\mathbf{1}$, then inequality (11) becomes inequality (9), and, moreover:

- inequality (8) if $\mathbf{h}\,(\mathbf{t})=\mathbf{t}$,
- inequality (2) if $\eta\;(\mathbf{b},\,\mathbf{a})=\mathbf{b}-\mathbf{a}$ and $\mathbf{h}\,(\mathbf{t})=\mathbf{t}$,
- inequality (4) if $\eta\;(\mathbf{b},\,\mathbf{a})=\mathbf{b}-\mathbf{a}$ and $\mathbf{h}\,(\mathbf{t})=\mathbf{t}^{\mathbf{s}}$,
- inequality (6) if $\eta\;(\mathbf{b},\,\mathbf{a})=\mathbf{b}-\mathbf{a}$.

Remark 2 If in Theorem 1, we let $\eta\,(\mathbf{b},\,\mathbf{a})=\mathbf{b}-\mathbf{a}$ and $\mathbf{h}\,(\mathbf{t})=\mathbf{t}$, then we get the inequality (10).

Remark 3 If in Theorem 1 $\mathbf{h}\,(\mathbf{t})=\mathbf{t}^{\mathbf{s}}$, then inequality (11) becomes the following inequality for $s-$preinvex function

$$\frac{\mathbf{1}}{\alpha\cdot\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{s}}}\mathbf{f}\left(\mathbf{a}+\frac{\mathbf{1}}{\mathbf{2}}\eta\,(\mathbf{b},\,\mathbf{a})\right)$$

$$\leqslant\frac{\mathbf{\Gamma}(\alpha)}{\eta(\mathbf{b},\,\mathbf{a})^{\alpha}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\,(\mathbf{a})+\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\,(\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a}))\right]$$

$$\leqslant[\mathbf{f}\,(\mathbf{a})+\mathbf{f}\,(\mathbf{b})]\cdot\left[\frac{\mathbf{\Gamma}(\alpha)\mathbf{\Gamma}(\mathbf{s}+\mathbf{1})}{\mathbf{\Gamma}(\alpha+\mathbf{s}+\mathbf{1})}+\frac{\mathbf{1}}{\mathbf{s}+\alpha}\right]$$

where we have used the fact that

$$\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha-\mathbf{1}}\,(\mathbf{1}-\mathbf{t})^{\mathbf{s}}\mathbf{dt}=\frac{\mathbf{\Gamma}\,(\alpha)\,\mathbf{\Gamma}\,(\mathbf{s}+\mathbf{1})}{\mathbf{\Gamma}\,(\alpha+\mathbf{s}+\mathbf{1})}$$

and

$$\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha+\mathbf{s}-\mathbf{1}}\,\mathbf{dt}=\frac{\mathbf{1}}{\alpha+\mathbf{s}}.$$

Remark 4 If in Theorem 1, we let $\eta\,(\mathbf{b},\,\mathbf{a})=\mathbf{b}-\mathbf{a}$ and $\mathbf{h}\,(\mathbf{t})=\mathbf{t}^{\mathbf{s}}$, the following inequality for $s-$convex function is obtained

$$\frac{\mathbf{1}}{\alpha\cdot\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{s}}}\,\mathbf{f}\left(\frac{\mathbf{a}+\mathbf{b}}{\mathbf{2}}\right)\leqslant\frac{\mathbf{\Gamma}(\alpha)}{(\mathbf{b}-\mathbf{a})^{\alpha}}\,[\mathbf{J}^{\alpha}_{\mathbf{b}^{-}}\,\mathbf{f}\,(\mathbf{a})+\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\,(\mathbf{b})]$$

$$\leqslant[\mathbf{f}\,(\mathbf{a})+\mathbf{f}\,(\mathbf{b})]\cdot\left[\frac{\mathbf{\Gamma}\,(\alpha)\,\mathbf{\Gamma}\,(\mathbf{s}+\mathbf{1})}{\mathbf{\Gamma}\,(\alpha+\mathbf{s}+\mathbf{1})}+\frac{\mathbf{1}}{\mathbf{s}+\alpha}\right].$$

3. Hermite-Hadamard type inequalities via fractional integrals

In order to prove our main results we need the following identity.

LEMMA 1 *Let* $\mathbf{f} : [\mathbf{a},\, \mathbf{a}+\eta(\mathbf{b},\, \mathbf{a})] \to \mathbf{R}$ *be a differentiable mapping on* $(\mathbf{a},\, \mathbf{a}+\eta(\mathbf{b},\, \mathbf{a}))$ *with* $\mathbf{a} < \mathbf{a}+\eta\,(\mathbf{b},\, \mathbf{a})$ *If* $\mathbf{f}' \in \mathbf{L}[\mathbf{a},\, \mathbf{a}+\eta(\mathbf{b},\, \mathbf{a})]$*, then*

$$-\frac{\mathbf{1}}{\eta\,(\mathbf{b},\,\mathbf{a})}\left[\mathbf{f}\,(\mathbf{a}+\eta\;(\mathbf{b},\,\mathbf{a}))+\mathbf{f}\,(\mathbf{a})\right]$$

$$+\frac{\mathbf{\Gamma}(\alpha+\mathbf{1})}{(\eta(\mathbf{b},\,\mathbf{a}))^{\alpha+\mathbf{1}}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\;(\mathbf{a}+\eta\;(\mathbf{b},\,\mathbf{a}))+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\,(\mathbf{a})\right]$$

$$=\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\mathbf{t})^{\alpha}-\mathbf{t}^{\alpha}\right]\,\mathbf{f}'(\mathbf{a}+\mathbf{t}\,\eta\,(\mathbf{b},\,\mathbf{a})\,)\mathbf{dt}.$$

PROOF. Integrating by parts

$$\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\mathbf{t})^{\alpha}-\mathbf{t}^{\alpha}\right]\,\mathbf{f}'(\mathbf{a}+\mathbf{t}\,\eta\,(\mathbf{b},\,\mathbf{a})\,)\mathbf{dt}$$

$$=\frac{\mathbf{1}}{\eta\,(\mathbf{b},\,\mathbf{a})}\left[(\mathbf{1}-\mathbf{t})^{\alpha}-\mathbf{t}^{\alpha}\right]\,\mathbf{f}\,(\mathbf{a}+\mathbf{t}\,\eta\,(\mathbf{b},\,\mathbf{a}))\,|_{0}^{1}$$

$$-\frac{\alpha}{\eta\,(\mathbf{b},\,\mathbf{a})}\int_{\mathbf{0}}^{\mathbf{1}}\left[-(\mathbf{1}-\mathbf{t})^{\alpha-\mathbf{1}}-\mathbf{t}^{\alpha-\mathbf{1}}\right]\,\mathbf{f}(\mathbf{a}+\mathbf{t}\,\eta\,(\mathbf{b},\,\mathbf{a})\,)\mathbf{dt}$$

$$=\frac{\mathbf{1}}{\eta\,(\mathbf{b},\,\mathbf{a})}\left[-\mathbf{f}\,(\mathbf{a}+\eta\,(\mathbf{b},\,\mathbf{a}))-\mathbf{f}(\mathbf{a})\right]$$

$$-\frac{\alpha}{(\eta\,(\mathbf{b},\,\mathbf{a}))^{\alpha+\mathbf{1}}}\int_{\mathbf{a}}^{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})}\left[-(\mathbf{a}+\eta\,(\mathbf{b},\,\mathbf{a})-\mathbf{u})^{\alpha-\mathbf{1}}-(\mathbf{u}-\mathbf{a}^{\alpha-\mathbf{1}})\right]\,\mathbf{f}(\mathbf{u})\mathbf{du}$$

$$=\frac{-\mathbf{1}}{\eta\,(\mathbf{b},\,\mathbf{a})}\left[\mathbf{f}\,(\mathbf{a}+\eta\,(\mathbf{b},\,\mathbf{a}))+\mathbf{f}\,(\mathbf{a})\right]$$

$$+\frac{\mathbf{\Gamma}(\alpha+\mathbf{1})}{(\eta(\mathbf{b},\,\mathbf{a}))^{\alpha+\mathbf{1}}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\;(\mathbf{a}+\eta\;(\mathbf{b},\,\mathbf{a}))+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\,(\mathbf{a})\right],$$

which completes the proof. □

Using this lemma, we can obtain the following fractional integral inequalities:

Theorem 2 *Let* $\mathbf{f} : [\mathbf{a}, \mathbf{a} + \eta(\mathbf{b}, \mathbf{a})] \to \mathbf{R}$ *be a differentiable mapping on* $(\mathbf{a}, \mathbf{a} + \eta(\mathbf{b}, \mathbf{a}))$ *with* $\mathbf{a} < \mathbf{a} + \eta(\mathbf{b}, \mathbf{a})$. *If* $|\mathbf{f}'|$ *is h-preinvex on* $[\mathbf{a}, \mathbf{a} + \eta(\mathbf{b}, \mathbf{a})]$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\Big|\frac{1}{\eta(\mathbf{b},\mathbf{a})}[\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{f}(\mathbf{a})] \\ &-\frac{\Gamma(\alpha+1)}{(\eta(\mathbf{b},\mathbf{a}))^{\alpha+1}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^+}\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\mathbf{a})^-}\mathbf{f}(\mathbf{a})\right]\Big| \\ &\leqslant \left[\left|\mathbf{f}'(\mathbf{a})\right|+\left|\mathbf{f}'(\mathbf{b})\right|\right]\left[\int_0^1 \mathbf{t}^{\alpha}\mathbf{h}(\mathbf{t})\,\mathbf{dt}+\int_0^1(\mathbf{1}-\mathbf{t})^{\alpha}\mathbf{h}(\mathbf{t})\,\mathbf{dt}\right]. \end{aligned} \tag{12}$$

Proof. Using Lemma 1 and the h-preinvexity of $|\mathbf{f}'|$, we have

$$\begin{aligned} &\Big|\frac{1}{\eta(\mathbf{b},\mathbf{a})}[\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{f}(\mathbf{a})] \\ &-\frac{\Gamma(\alpha+1)}{(\eta(\mathbf{b},\mathbf{a}))^{\alpha+1}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^+}\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\mathbf{a})^-}\mathbf{f}(\mathbf{a})\right]\Big| \\ &\leqslant \int_0^1 |(\mathbf{1}-\mathbf{t})^{\alpha}-\mathbf{t}^{\alpha}|\ |\mathbf{f}'(\mathbf{a}+\mathbf{t}\,\eta(\mathbf{b},\mathbf{a}))|\mathbf{dt} \\ &\leqslant \int_0^1 [(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}]\ \left[\mathbf{h}(\mathbf{1}-\mathbf{t})\left|\mathbf{f}'(\mathbf{a})\right|+\mathbf{h}(\mathbf{t})\left|\mathbf{f}'(\mathbf{b})\right|\right]\mathbf{dt} \\ &=\left|\mathbf{f}'(\mathbf{a})\right|\int_0^1[(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}]\,\mathbf{h}(\mathbf{1}-\mathbf{t})\,\mathbf{dt}+\left|\mathbf{f}'(\mathbf{b})\right|\int_0^1[(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}]\,\mathbf{h}(\mathbf{t})\mathbf{dt} \\ &=\left[\left|\mathbf{f}'(\mathbf{a})\right|+\left|\mathbf{f}'(\mathbf{b})\right|\right]\left[\int_0^1 \mathbf{t}^{\alpha}\mathbf{h}(\mathbf{t})\,\mathbf{dt}+\int_0^1(\mathbf{1}-\mathbf{t})^{\alpha}\mathbf{h}(\mathbf{t})\,\mathbf{dt}\right], \end{aligned}$$

which completes the proof. □

Remark 5 If we take $\mathbf{h}(\mathbf{t})=\mathbf{t}^{\mathbf{s}}$, which means that the function is $s-$preinvex, then inequality (12) becomes the following inequality:

$$\begin{aligned} &\Big|\frac{1}{\eta(\mathbf{b},\mathbf{a})}[\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{f}(\mathbf{a})] \\ &-\frac{\Gamma(\alpha+1)}{(\eta(\mathbf{b},\mathbf{a}))^{\alpha+1}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^+}\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\mathbf{a})^-}\mathbf{f}(\mathbf{a})\right]\Big| \\ &\leqslant \left[\left|\mathbf{f}'(\mathbf{a})\right|+\left|\mathbf{f}'(\mathbf{b})\right|\right]\cdot\left[\frac{1}{\mathbf{s}+\alpha+\mathbf{1}}+\frac{\Gamma(\alpha+\mathbf{1})\,\Gamma(\mathbf{s}+\mathbf{1})}{\Gamma(\alpha+\mathbf{s}+\mathbf{2})}\right]. \end{aligned}$$

THEOREM 3 *Let* $\mathbf{f} : [\mathbf{a}, \mathbf{a} + \eta(\mathbf{b}, \mathbf{a})] \to \mathbf{R}$ *be a differentiable mapping on* $(\mathbf{a}, \mathbf{a} + \eta(\mathbf{b}, \mathbf{a}))$ *with* $\mathbf{a} < \mathbf{a} + \eta(\mathbf{b}, \mathbf{a})$*. If* $|\mathbf{f}'|^{\mathbf{q}}$*,* $\mathbf{q} \geqslant \mathbf{1}$*, is h-preinvex on* $[\mathbf{a}, \mathbf{a} + \eta(\mathbf{b}, \mathbf{a})]$*, then one has:*

$$\begin{aligned} &|\frac{\mathbf{1}}{\eta(\mathbf{b},\mathbf{a})}\left[\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{f}(\mathbf{a})\right] \\ &-\frac{\mathbf{\Gamma}(\alpha+\mathbf{1})}{(\eta(\mathbf{b},\mathbf{a}))^{\alpha+\mathbf{1}}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^+}\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\mathbf{a})^-}\mathbf{f}(\mathbf{a})\right]| \\ &\leqslant\left(\frac{\mathbf{2}}{\alpha+\mathbf{1}}\right)^{\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}}}\left(\left[\left|\mathbf{f}'(\mathbf{a})\right|^{\mathbf{q}}+\left|\mathbf{f}'(\mathbf{b})\right|^{\mathbf{q}}\right]\cdot\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}\right]\mathbf{h}(\mathbf{t})\,\mathbf{dt}\right)^{\frac{\mathbf{1}}{\mathbf{q}}}. \end{aligned} \tag{13}$$

PROOF. By using Lemma 1, h-preinvexity of $|\mathbf{f}'|^{\mathbf{q}}$, and the well known power mean inequality, we have

$$\begin{aligned} &|\frac{\mathbf{1}}{\eta(\mathbf{b},\mathbf{a})}\left[\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{f}(\mathbf{a})\right] \\ &-\frac{\mathbf{\Gamma}(\alpha+\mathbf{1})}{(\eta(\mathbf{b},\mathbf{a}))^{\alpha+\mathbf{1}}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^+}\mathbf{f}(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\mathbf{a})^-}\mathbf{f}(\mathbf{a})\right]| \\ &\leqslant\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}\right]\left|\mathbf{f}'(\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))\right|\mathbf{dt} \\ &\leqslant\left(\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}\right]\mathbf{dt}\right)^{\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}}} \\ &\cdot\left(\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}\right]\left|\mathbf{f}'(\mathbf{a}+\mathbf{t}\,\eta(\mathbf{b},\mathbf{a}))\right|^{\mathbf{q}}\mathbf{dt}\right)^{\frac{\mathbf{1}}{\mathbf{q}}} \\ &=\left(\frac{\mathbf{2}}{\alpha+\mathbf{1}}\right)^{\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}}}\left(\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}\right]\left|\mathbf{f}'(\mathbf{a}+\mathbf{t}\,\eta(\mathbf{b},\mathbf{a}))\right|^{\mathbf{q}}\mathbf{dt}\right)^{\frac{\mathbf{1}}{\mathbf{q}}} \\ &\leqslant\left(\frac{\mathbf{2}}{\alpha+\mathbf{1}}\right)^{\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}}}\left(\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}\right]\left[\mathbf{h}(\mathbf{1}-\mathbf{t})\left|\mathbf{f}'(\mathbf{a})\right|^{\mathbf{q}}+\mathbf{h}(\mathbf{t})\left|\mathbf{f}'(\mathbf{b})\right|^{\mathbf{q}}\right]\mathbf{dt}\right)^{\frac{\mathbf{1}}{\mathbf{q}}} \\ &=\left(\frac{\mathbf{2}}{\alpha+\mathbf{1}}\right)^{\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}}}\left(\left[\left|\mathbf{f}'(\mathbf{a})\right|^{\mathbf{q}}+\left|\mathbf{f}'(\mathbf{b})\right|^{\mathbf{q}}\right]\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\mathbf{t})^{\alpha}+\mathbf{t}^{\alpha}\right]\mathbf{h}(\mathbf{t})\,\mathbf{dt}\right)^{\frac{\mathbf{1}}{\mathbf{q}}}, \end{aligned}$$

which completes the proof. □

Remark 6 If the function $|\mathbf{f}'|^{\mathbf{q}}$ is $s-$preinvex, i.e. $\mathbf{h}\left(\mathbf{t}\right)=\mathbf{t}^{\mathbf{s}}$, then inequality (13) reduces to the following inequality:

$$\left|\frac{\mathbf{1}}{\eta\left(\mathbf{b},\,\mathbf{a}\right)}\left[\mathbf{f}\left(\mathbf{a}+\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{f}\left(\mathbf{a}\right)\right]\right.$$

$$\left.-\frac{\mathbf{\Gamma}\left(\alpha+\mathbf{1}\right)}{\left(\eta(\mathbf{b},\,\mathbf{a})\right)^{\alpha+\mathbf{1}}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\,\left(\mathbf{a}+\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\left(\mathbf{a}\right)\right]\right|$$

$$\leqslant\left(\frac{\mathbf{2}}{\alpha+\mathbf{1}}\right)^{\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}}}\left(\left[\left|\mathbf{f}^{'}\left(\mathbf{a}\right)\right|^{\mathbf{q}}+\left|\mathbf{f}^{'}\left(\mathbf{b}\right)\right|^{\mathbf{q}}\right]\cdot\left[\frac{\mathbf{\Gamma}\left(\alpha+\mathbf{1}\right)\mathbf{\Gamma}\left(\mathbf{s}+\mathbf{1}\right)}{\mathbf{\Gamma}\left(\alpha+\mathbf{s}+\mathbf{2}\right)}+\frac{\mathbf{1}}{\mathbf{s}+\alpha+\mathbf{1}}\right]\right)^{\frac{\mathbf{1}}{\mathbf{q}}}.$$

Theorem 4 *Let* $\mathbf{f}:[\mathbf{a},\mathbf{a}+\eta(\mathbf{b},\mathbf{a})]\to\mathbf{R}$ *be a differentiable mapping on* $(\mathbf{a},\mathbf{a}+\eta(\mathbf{b},\mathbf{a}))$ *with* $\mathbf{a}<\mathbf{a}+\eta\left(\mathbf{b},\mathbf{a}\right)$. *If* $|\mathbf{f}'|^{\mathbf{q}}$, $\mathbf{q}>\mathbf{1}$, *is h-preinvex on* $[\mathbf{a},\mathbf{a}+\eta(\mathbf{b},\mathbf{a})]$, *then the following inequality for fractional integrals holds:*

$$\left|\frac{\mathbf{1}}{\eta\left(\mathbf{b},\,\mathbf{a}\right)}\left[\mathbf{f}\left(\mathbf{a}+\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{f}\left(\mathbf{a}\right)\right]\right.$$
$$\left.-\frac{\mathbf{\Gamma}\left(\alpha+\mathbf{1}\right)}{\left(\eta(\mathbf{b},\,\mathbf{a})\right)^{\alpha+\mathbf{1}}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\,\left(\mathbf{a}+\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\left(\mathbf{a}\right)\right]\right|$$
$$\leqslant\frac{\mathbf{2}}{\left(\alpha\mathbf{p}+\mathbf{1}\right)^{\frac{\mathbf{1}}{\mathbf{p}}}}\left(\left[\left|\mathbf{f}^{'}\left(\mathbf{a}\right)\right|^{\mathbf{q}}+\left|\mathbf{f}^{'}\left(\mathbf{b}\right)\right|^{\mathbf{q}}\right]\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{h}\left(\mathbf{t}\right)\mathbf{dt}\right)^{\frac{\mathbf{1}}{\mathbf{q}}},\tag{14}$$

where $\frac{\mathbf{1}}{\mathbf{p}}+\frac{\mathbf{1}}{\mathbf{q}}=\mathbf{1}$.

Proof. From Lemma 1 and using the well known Hölder inequality, we have

$$\left|\frac{\mathbf{1}}{\eta\left(\mathbf{b},\,\mathbf{a}\right)}\left[\mathbf{f}\left(\mathbf{a}+\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{f}\left(\mathbf{a}\right)\right]\right.$$

$$\left.-\frac{\mathbf{\Gamma}\left(\alpha+\mathbf{1}\right)}{\left(\eta(\mathbf{b},\,\mathbf{a})\right)^{\alpha+\mathbf{1}}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^{+}}\,\mathbf{f}\,\left(\mathbf{a}+\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^{-}}\,\mathbf{f}\left(\mathbf{a}\right)\right]\right|$$

$$\leqslant\int_{\mathbf{0}}^{\mathbf{1}}\left[\left(\mathbf{1}-\mathbf{t}\right)^{\alpha}+\mathbf{t}^{\alpha}\right]\left|\mathbf{f}^{'}\left(\mathbf{a}+\mathbf{t}\,\eta(\mathbf{b},\,\mathbf{a})\right)\right|\mathbf{dt}$$

$$=\int_{\mathbf{0}}^{\mathbf{1}}\left(\mathbf{1}-\mathbf{t}\right)^{\alpha}\left|\mathbf{f}^{'}\left(\mathbf{a}+\mathbf{t}\,\eta(\mathbf{b},\,\mathbf{a})\right)\right|\,\mathbf{dt}+\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{t}^{\alpha}\left|\mathbf{f}^{'}\left(\mathbf{a}+\mathbf{t}\,\eta(\mathbf{b},\,\mathbf{a})\right)\right|\,\mathbf{dt}$$

$$\leqslant\left(\int_{\mathbf{0}}^{\mathbf{1}}\left(\mathbf{1}-\mathbf{t}\right)^{\alpha\,\mathbf{p}}\mathbf{dt}\right)^{\frac{\mathbf{1}}{\mathbf{p}}}\left(\int_{\mathbf{0}}^{\mathbf{1}}\left|\mathbf{f}^{'}\left(\mathbf{a}+\mathbf{t}\,\eta(\mathbf{b},\,\mathbf{a})\right)\right|^{\mathbf{q}}\mathbf{dt}\right)^{\frac{\mathbf{1}}{\mathbf{q}}}$$

$$+\left(\int_0^1 \mathbf{t}^{\alpha\,\mathbf{p}}\mathbf{dt}\right)^{\frac{1}{\mathbf{p}}}\left(\int_0^1 \left|\mathbf{f}'\left(\mathbf{a}+\mathbf{t}\,\eta(\mathbf{b},\,\mathbf{a})\right)\right|^{\mathbf{q}}\mathbf{dt}\right)^{\frac{1}{\mathbf{q}}}$$

$$\leqslant\frac{\mathbf{2}}{(\alpha\mathbf{p}+\mathbf{1})^{\frac{1}{\mathbf{p}}}}\left(\left[\left|\mathbf{f}'\left(\mathbf{a}\right)\right|^{\mathbf{q}}+\left|\mathbf{f}'\left(\mathbf{b}\right)\right|^{\mathbf{q}}\right]\int_0^1 \mathbf{h}\left(\mathbf{t}\right)\mathbf{dt}\right)^{\frac{1}{\mathbf{q}}},$$

which completes the proof. □

Remark 7 If $\mathbf{h}(\mathbf{t})=\mathbf{t}^{\mathbf{s}}$, i.e. $|\mathbf{f}'|^{\mathbf{q}}$ is an $s-$preinvex function, then inequality (14) becomes the following inequality

$$|\frac{\mathbf{1}}{\eta\left(\mathbf{b},\,\mathbf{a}\right)}\left[\mathbf{f}\left(\mathbf{a}+\eta\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{f}\left(\mathbf{a}\right)\right]$$

$$-\frac{\mathbf{\Gamma}\left(\alpha+\mathbf{1}\right)}{(\eta(\mathbf{b},\,\mathbf{a}))^{\alpha+\mathbf{1}}}\left[\mathbf{J}^{\alpha}_{\mathbf{a}^+}\,\mathbf{f}\,\left(\mathbf{a}+\eta\,\left(\mathbf{b},\,\mathbf{a}\right)\right)+\mathbf{J}^{\alpha}_{\mathbf{a}+\eta(\mathbf{b},\,\mathbf{a})^-}\,\mathbf{f}\left(\mathbf{a}\right)\right]|$$

$$\leqslant\frac{\mathbf{2}}{(\alpha\mathbf{p}+\mathbf{1})^{\frac{1}{\mathbf{p}}}\;\;(\mathbf{s}+\mathbf{1})^{\frac{1}{\mathbf{q}}}}\left[\left|\mathbf{f}'\left(\mathbf{a}\right)\right|^{\mathbf{q}}+\left|\mathbf{f}'\left(\mathbf{b}\right)\right|^{\mathbf{q}}\right]^{\frac{1}{\mathbf{q}}}.$$

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