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Dual sufficient optimality conditions for optimal control

by

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Abstract: In this paper we develop the first and second order dual sufficient optimality conditions for a nonlinear optimal control problem. Our conditions are derived from the dual Hamilton-Jacobi approach applied to the generalized problem of Bolza. We do not require neither any convexity on the data, nor that the control set U be polyhedral, nor that the control function be in the interior of U. Instead, we assume the existence of a function which satisfies a certain inequality.

Keywords: optimal control problem, dual Hamilton-Jacobi inequality, dual sufficient optimality conditions, relative minimum.

1. Introduction

Let an interval $[a, b]$, a point r in R^n, a closed subset $U \subset R^m$ and functions $f : [a, b] \times R^n \times R^m \to R^n$, $g : [a, b] \times R^n \times R^m \to R$ and $l^0 : R^n \to R$ be given. We consider in this paper an optimal control problem (C):

$$\text{minimize } J(x, u) := \int_a^b g(t, x(t), u(t))\, dt + l^0(x(b))$$

$$\text{subject to } \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e.} \tag{1}$$

$$x(a) = r \tag{2}$$

$$u(t) \in U \text{ a.e.} \tag{3}$$

where $x : [a, b] \to R^n$ is an absolutely continuous function and $u : [a, b] \to R^m$ is a Lebesgue measurable function. We call the function $H : [a, b] \times R^n \times R^n \to R$ defined by

$$H(t, x, p) := \sup\{\langle p, f(t, x, u)\rangle - g(t, x, u) : u \in U\} \tag{4}$$

the *Hamiltonian* of the problem (C). Any absolutely continuous function $y : [a, b] \to R^n$ is called an *arc*.

It is known from Rockafellar (1971) that, under some nonrestrictive assumptions, the problem (C) can be written as a generalized problem of Bolza (see the proof of Theorem 2.1 in Section 2). Thus, any sufficient conditions obtained for the generalized problem of Bolza can be transferred to the problem (C).

Various sufficient criteria for a general optimal control problem (C) are well known. The earliest involve the Hamilton-Jacobi inequality (HJ inequality). Some other require the Hamiltonian H, defined by (4), to be concave in x and convex in p. In the others it is sometimes assumed that U is a compact polyhedron. We can also find sufficient conditions for simplified problems. For instance, when the functions f and g are convex or when the control function $\overline{u}$ always lies in the interior of U.

In this paper we develop the dual sufficient optimality conditions for (C) by replacing the optimal control problem (C) by the generalized problem of Bolza (P_C) and then by applying to the latter the dual sufficient optimality criterion obtained in Młynarska (2000). The way of deriving results in the paper is the same as in Zeidan (1984b) and the conditions obtained are analogous to the relevant conditions presented there in the primal version. However, unlike in Zeidan (1984b) now they are implied by the dual HJ inequality instead of the HJ inequality and they are formulated in the terminology of the dual theory. Applying dual sufficient optimality criterion of Młynarska (2000) we derive not only the results which are similar to the theorems of Zeidan (1984B), but we also obtain in such a way the new sufficient optimality conditions for (C). We begin with presenting a very important dual sufficient optimality criterion for (C) based on the dual HJ inequality. Using this criterion, we obtain in subsequent sections the sufficient optimality conditions of the first and hence second orders for (C), which are different from those known so far.

2. Dual sufficient optimality criterion

In this section we are interested in finding a sufficient criterion for the existence of a strong relative minimum in the opimal control problem (C). Throughout the paper, we use the following definitions and notations. We say that an arc y *lies* in a set $Y \subset R^{n+1}$ if its graph is contained in this set; we denote by T_x the projection of a given set $T \subset R^{n+1}$ of variables (t, x) onto the space of the variable x. Moreover, for a given arc $\overline{y} : [a, b] \to R^n$ and a positive number ε, we adopt the following notations:

$$N(\overline{y}; \varepsilon) := \{(t, y) \in R^{n+1} : t \in [a, b],\ |y - \overline{y}(t)| < \varepsilon\} \tag{5}$$

$$N_\varepsilon(\overline{y}) := \{y \in R^n : |y - \overline{y}(t)| < \varepsilon \text{ for some } t \in [a, b]\}. \tag{6}$$

DEFINITION 2.1 *Let x be an arc and let $u : [a, b] \to R^m$ be measurable. The pair $(x, u$ is called admissible for* (C) *if it satisfies conditions* (1)–(3).

DEFINITION 2.2 *Let a subset $T \subset R^{n+1}$ and an admissible pair $(\overline{x}, \overline{u})$ such that the arc $\overline{x}$ lies in T be given. We say that $(\overline{x}, \overline{u})$ is a strong minimum for* (C) *relative to all admissible pairs (x, u) such that x lies in T, if, for all admissible pairs (x, u) such that x lies in T, the inequality $J(x, u) \geq J(\overline{x}, \overline{u})$ is satisfied.*

Given a subset $T \subset R^{n+1}$ of variables (t, x), we shall make the following hypothesis:

(H1) functions f and g are measurable on $T \times U$; for each $t \in [a, b]$, a function $(x, u) \to f(t, x, u)$ is continuous and a function $(x, u) \to g(t, x, u)$ is lower semicontinuous on $T_x \times U$.

We shall now formulate and prove the dual sufficient optimality criterion for (C). We require in this criterion the existence of a function $Q(t)$ which, together with Hamiltonian, satisfies a certain inequality. Moreover, in the theorem below and in subsequent theorems the following assumption is made:

(A) function $Q(t)$ is defined on $[a, b]$ and has a derivative almost everywhere in $[a, b]$; for almost all $t \in [a, b]$, $Q(t)$ is an $n \times n$ symmetric matrix; $Q(b)$ is nonsingular.

The criterion formulated below and the sufficient optimality conditions yield the optimality of an admissible pair $(\overline{x}, \overline{u})$ relative to all pairs in the class R where for given $\overline{p}$ and $Q(t)$

R consists of admissible pairs (x, u) for which there is an arc p lying in $N(\overline{p}, \varepsilon)$ and satisfying $x(t) = \overline{x}(t) - Q(t)(p(t) - \overline{p}(t))$ for $t \in [a, b]$.

The set T is defined in Theorem 2.1 in such a way that for each admissible pair (x, u) belonging to R the arc x lies in T.

THEOREM 2.1 *Let $(\overline{x}, \overline{u})$ be a given admissible pair for* (C) *such that $J(\overline{x}, \overline{u})$ is finite. Assume that there exist a positive number ε, an arc $\overline{p}$ and a function $Q(t)$ satisfying* (A). *Next, define the set T by*

$$T := \{(t, x) \in [a, b] \times R^n : x = \overline{x}(t) - Q(t)(p - \overline{p}(t)) \text{ for } p \in N_\varepsilon(\overline{p})\}.$$

Suppose further that (H1) *holds and:*

(i) *for almost all $t \in [a, b]$, for all $p \in N_\varepsilon(\overline{p})$ and for all $u \in U$,*

$$\begin{aligned} &g(t, x(t) - Q(t)(p - \overline{p}(t)), u) - g(t, \overline{x}(t), \overline{u}(t)) \\ &- \langle p, f(t, \overline{x}(t) - Q(t)(p - \overline{p}(t)), u)\rangle \\ &+ \langle \overline{p}(t), f(t, \overline{x}(t), \overline{u}(t))\rangle + \langle \dot{\overline{p}}(t), Q(t)(p - \overline{p}(t))\rangle \\ &\geq -\langle \dot{\overline{x}}(t), p - \overline{p}(t)\rangle + \tfrac{1}{2}\langle p - \overline{p}(t), \dot{Q}(t)(p - \overline{p}(t))\rangle; \end{aligned}$$

(ii) *for all $d \in R^n$ such that $(b, \overline{x}(b) + d) \in T$,*

$$l^0(\overline{x}(b) + d) - l^0(\overline{x}(b)) \geq -\langle \overline{p}(b), d\rangle + \tfrac{1}{2}\langle d, Q^{-1}(b) d\rangle.$$

Then $(\overline{x}, \overline{u})$ is a strong minimum for (C) *relative to all (x, u) in* R.

Proof. The proof bases on the replacement of the problem (C) by the generalized problem of Bolza (P_C) and the application of Theorem 2.1 of Młynarska (2000) to it.

Define the following functions:

$$L(t, x, v) := \inf\{K(t, x, v, u) : u \in R^m\} \tag{7}$$

$$\text{where } K(t, x, v, u) = \begin{cases} g(t, x, u) & \text{when } u \in U \text{ and } v = f(t, x, u) \\ +\infty & \text{otherwise} \end{cases} \tag{8}$$

$$l(x_1,x_2) := \psi_{\{r\}}(x_1) + l^0(x_2)$$
$$\text{where } \psi_{\{r\}}(x) = \begin{cases} 0 & \text{when } x = r \\ +\infty & \text{when } x \neq r. \end{cases} \tag{9}$$

Consider the following generalized problem of Bolza associated with the optimal control problem (C):

($\mathrm{P_C}$) minimize $J_C(x) := \int_a^b L(t, x(t), \dot{x}(t))\,dt + l(x(a), x(b))$,
where L and l are given by formulae (7) and (9), respectively.

Then, the problems ($\mathrm{P_C}$) and (C) have the same Hamiltonians given by formula (4). Condition (i) of Theorem 2.1 implies the inequality

$$\begin{aligned} &\langle \overline{p}(t), f(t, \overline{x}(t), \overline{u}(t))\rangle - g(t, \overline{x}(t), \overline{u}(t)) \\ &\geq \langle \overline{p}(t), f(t, \overline{x}(t), u)\rangle - g(t, \overline{x}(t), u) \end{aligned} \tag{10}$$

for almost all $t \in [a,b]$ and for all $u \in U$. Hence, applying (7) and (8), we get

$$L(t, \overline{x}(t), \dot{\overline{x}}(t)) = g(t, \overline{x}(t), \overline{u}(t)) \text{ a.e.} \tag{11}$$

On the other hand, by definition (7) of the function L, we have that, for any admissible pair (x,u), the following inequality is satisfied:

$$J_C(x) \leq J(x,u).$$

In view of the above, in order to show that $(\overline{x}, \overline{u})$ is a strong relative minimum for (C), it is enough to prove that $\overline{x}$ is a strong relative minimum for ($\mathrm{P_C}$). To this end, we shall make use of Theorem 2.1 of Młynarska (2000).

From Lemma 6 of Rockafellar (1973) we have that the function $(t,x,v,u) \to K(t,x,v,u)$ defined by (8) is measurable and lower semicontinuous. Hence, and from Theorem 1 of Rockafellar (1971) it follows that the function L defined by (7) is measurable. Inequality (10) and property (11) imply condition (i) of Theorem 2.1 of Młynarska (2000) and the equality

$$H(t, \overline{x}(t), \overline{p}(t)) = \langle \overline{p}(t), f(t, \overline{x}(t), \overline{u}(t))\rangle - g(t, \overline{x}(t), \overline{u}(t)) \text{ a.e.}$$

Thus, from (4), the above equality and condition (i) of Theorem 2.1 we obtain that the inequality

$$\begin{aligned} &H(t, \overline{x}(t) - Q(t)(p - \overline{p}(t)), p) - H(t, \overline{x}(t), \overline{p}(t)) \\ &\leq \langle \dot{\overline{x}}(t), p - \overline{p}(t)\rangle + \langle \dot{\overline{p}}(t), Q(t)(p - \overline{p}(t))\rangle - \tfrac{1}{2}\langle p - \overline{p}(t), \dot{Q}(t)(p - \overline{p}(t))\rangle \end{aligned}$$

is satisfied for almost all $t \in [a,b]$ and for all $p \in N_\varepsilon(\overline{p})$. This means that condition (ii) of Theorem 2.1 of Młynarska (2000) holds. Finally, using formula (9) and condition (ii) of Theorem 2.1, we obtain that condition (iii) of the same theorem is satisfied. Consequently, $\overline{x}$ is a strong relative minimum for ($\mathrm{P_C}$). So, on account of the observations made before, the proof is completed. ■

3. First order dual sufficient optimality conditions

In this section we present two types of the first order dual sufficient conditions for optimality in (C). In the first, we require the given functions f and g to be Lipschitz in x while in the second, we assume that f and g are C^1 in (x, u). In both cases, we derive the sufficient conditions directly from Theorem 2.1 obtained in the previous section.

Given a subset $T \subset R^{n+1}$ of variables (t, x), the following assumption will be made:

(H2) functions f and g are measurable on $T \times U$; for each $t \in [a, b]$ and x lying in T, a function $u \to f(t, x, u)$ is continuous and a function $u \to g(t, x, u)$ is lower semicontinuous on U; functions $x \to f(t, x, u)$ and $x \to g(t, x, u)$ are Lipschitz on T_x uniformly for $t \in [a, b]$ and $u \in U$.

THEOREM 3.1 *Let* (x, u) *be a given admissible pair for* (C) *such that* $J(\overline{x}, \overline{u})$ *is finite. Assume that there exist a positive number* ε*, an arc* $\overline{p}$ *and a function* $Q(t)$ *satisfying* (A). *Next, define the set* T *by*

$$T := \{(t, x) \in [a, b] \times R^n : x = \overline{x}(t) - Q(t)(p - \overline{p}(t)) \ for\ p \in N_\varepsilon(\overline{p})\}.$$

Besides, suppose that (H2) *holds and:*

(a) *for almost all* $t \in [a, b]$ *and for all* $u \in U$,
$$\langle \overline{p}(t), f(t, \overline{x}(t), \overline{u}(t))\rangle - g(t, \overline{x}(t), \overline{u}(t)) \geq \langle \overline{p}(t), f(t, \overline{x}(t), u)\rangle - g(t, \overline{x}(t)u);$$

(b) *for almost all* $t \in [a, b]$, *for all* $p \in N_\varepsilon(\overline{p})$, *for all* $u \in U$, *for all* $\alpha \in \partial_x^T f(t, \overline{x}(t) - Q(t)(p - \overline{p}(t)), u)$ *and for all* $\beta \in \partial_x g(t, \overline{x}(t) - Q(t)(p - \overline{p}(t)), u)$,
$$\langle -Q(t)\alpha p + Q(t)\beta + f(t, \overline{x}(t) - Q(t)(p - \overline{p}(t)), u) + \dot{Q}(t)(p - \overline{p}(t)) - \dot{\overline{x}}(t) - Q(t)\dot{\overline{p}}(t), p - \overline{p}(t)\rangle \leq 0;$$

(c) *for all* $d \in R^n$ *such that* $(b, \overline{x}(b) + d) \in T$,
$$l^0(\overline{x}(b) + d) - l^0(\overline{x}(b)) \geq -\langle \overline{p}(b), d\rangle + \tfrac{1}{2}\langle d, Q^{-1}(b)d\rangle.$$

Then $(\overline{x}, \overline{u})$ *is a strong minimum for* (C) *relative to all* (x, u) *in* R.

Proof. Using Theorem 2.1, it suffices to show that condition (i) of that theorem is satisfied. To do so, we shall make use of Lemma 4.1 of Zeidan (1984a).

For the functions $\overline{x}$, $\overline{p}$ and Q given in Theorem 3.1, we define

$$\begin{aligned} F(t, u, p) := {} & \langle pf(t, \overline{x}(t) - Q(t)(p - \overline{p}(t)), u)\rangle \\ & - g(t, \overline{x}(t) - Q(t)(p - \overline{p}(t)), u) + \tfrac{1}{2}\langle p - \overline{p}(t), \dot{Q}(t)(p - \overline{p}(t))\rangle \end{aligned} \tag{12}$$

where $t \in [a, b]$ a.e., $p \in N_\varepsilon(\overline{p})$ and $u \in U$. Then, the function $p \to F(t, u, p)$ is Lipschitz on $N_\varepsilon(\overline{p})$. Calculating the generalized gradient $\partial_p F(t, u, p)$ from

Sections 1.11 and 1.14 of Clarke (1981) we have

$$\begin{aligned}&\partial_p F(t,u,p) \subset f(t,\overline{x}(t) - Q(t)(p-\overline{p}(t)),u)\\&- Q(t)\partial_x^T f(t,\overline{x}(t) - Q(t)(p-\overline{p}(t)),u)p\\&+ Q(t)\partial_x g(t,\overline{x}(t) - Q(t)(p-\overline{p}(t)),u) + \dot{Q}(t)(p-\overline{p}(t)).\end{aligned}$$

Hence, condition (b) of Theorem 3.1 implies

$$\langle w - \dot{\overline{x}}(t) - Q(t)\dot{\overline{p}}(t), p - \overline{p}(t)\rangle \le 0$$

for almost all $t \in [a,b]$, for all $p \in N_\varepsilon(\overline{p})$ and for all $w \in \partial_p F(t,u,p)$. The above inequality means that, for a function $h : [a,b] \to R^n$ given by the formula

$$h(t) := \dot{\overline{x}}(t) + Q(t)\dot{\overline{p}}(t),$$

condition (4.2) of Lemma 4.1 of Zeidan (1984a) is satisfied. Thus, applying this Lemma to functions $p \to F(t,u,p)$ and h, we obtain that, for almost all $t \in [a,b]$, for all $p \in N_\varepsilon(\overline{p})$ and for all $u \in U$,

$$F(t,u,p) - F(t,u,\overline{p}(t)) \le \langle \dot{\overline{x}}(t), p - \overline{p}(t)\rangle + \langle Q(t)\dot{\overline{p}}(t), p - \overline{p}(t)\rangle.$$

On the other hand, condition (a) of Theorem 3.1 can be written down, using (12), in the form

$$F(t,u,\overline{p}(t)) \le F(t,\overline{u}(t),\overline{p}(t))$$

for almost all $t \in [a,b]$ and for all $u \in U$. The last two inequalities imply that

$$F(t,u,p) - F(t,\overline{u}(t),\overline{p}(t)) \le \langle \dot{\overline{x}}(t), p - \overline{p}(t)\rangle + \langle \dot{\overline{p}}(t), Q(t)(p - \overline{p}(t))\rangle$$

for almost all $t \in [a,b]$, for all $p \in N_\varepsilon(\overline{p})$ and for all $u \in U$. By (12), this means that condition (i) of Theorem 2.1 is satisfied, which ends the proof. ■

Applying the criterion from the previous section, we have thus obtained the first order sufficient optimality conditions for (C) in case of Lipschitz functions f and g, other than those known so far. We shall now consider the case when functions f and g are C^1 and show that condition (b) of Theorem 3.1 can be replaced by another first order condition.

Given a subset $T \subset R^{n+1}$ of variables (t,x), the following hypothesis will be made:

(H3) functions f and g are measurable on $T \times U$; for each $t \in [a,b]$, functions $(x,u) \to f(t,x,u)$ and $(x,u) \to g(t,x,u)$ are C^1 on $T_x \times U$.

THEOREM 3.2 *Let U be a convex set and let $(\overline{x},\overline{u})$ be a given admissible pair for* (C) *such that $J(\overline{x},\overline{u})$ is finite. Assume that there exist a positive number ε, an arc $\overline{p}$ and a function $Q(t)$ satisfying* (A). *Next, define the set T by*

$$T := \{(t,x) \in [a,b] \times R^n : x = \overline{x}(t) - Q(t)(p - \overline{p}(t)) \ \mathit{for}\ p \in N_\varepsilon(\overline{p})\}.$$

Suppose further that (H3), *conditions* (a), (c) *of Theorem* 3.1 *are satisfied, and that, for almost all* $t \in [a,b]$, *for all* $p \in N_\varepsilon(\overline{p})$ *and for all* $u \in U$, *the inequality*

$$\begin{aligned}
&\langle -Q(t)f_x^T(t,\overline{x}(t)-Q(t)(p-\overline{p}(t)),u)p \\
&+Q(t)g_x(t,\overline{x}(t)-Q(t)(p-\overline{p}(t)),u) \\
&+f(t,\overline{x}(t)-Q(t)(p-\overline{p}(t)),u) \\
&+\dot{Q}(t)(p-\overline{p}(t))-\dot{\overline{x}}(t)-Q(t)\dot{\overline{p}}(t),p-\overline{p}(t)\rangle \\
&+\langle f_u^T(t,\overline{x}(t)-Q(t)(p-\overline{p}(t)),u)p-f_u^T(t,\overline{x}(t),\overline{u}(t))\overline{p}(t) \\
&-g_u(t,\overline{x}(t)-Q(t)(p-\overline{p}(t)),u)+g_u(t,\overline{x}(t),\overline{u}(t)),u-\overline{u}(t)\rangle \le 0
\end{aligned} \tag{13}$$

holds. Then $(\overline{x},\overline{u})$ *is a strong minimum for* (C) *relative to all* (x,u) *in* R.

Proof. We shall demonstrate that condition (i) of Theorem 2.1 is satisfied.

Consider the function F defined by (12). Now, the function $(u,p) \to F(t,u,p)$ is C^1 on $U \times N_\varepsilon(\overline{p})$ with the gradient $\nabla_{u,p}F(t,u,p)$. Then, inequality (13) is equivalent to

$$\begin{aligned}
&\langle \nabla_{u,p}F(t,u,p)-(F_u(t,\overline{u}(t),\overline{p}(t)),\dot{\overline{x}}(t)+Q(t)\dot{\overline{p}}(t)), \\
&(u-\overline{u}(t),p-\overline{p}(t))\rangle \le 0
\end{aligned}$$

for almost all $t \in [a,b]$, for all $p \in N_\varepsilon(\overline{p})$ and for all $u \in U$. Reasoning similarly to Lemma 4.1 of Zeidan (1984a), by convexity of the set U we get

$$\begin{aligned}
&F(t,u,p)-F(t,\overline{u}(t),\overline{p}(t)) \\
&\le \langle F_u(t,\overline{u}(t),\overline{p}(t)),u-\overline{u}(t)\rangle+\langle \dot{\overline{x}}(t)+Q(t)\dot{\overline{p}}(t),p-\overline{p}(t)\rangle
\end{aligned}$$

for almost all $t \in [a,b]$, for all $p \in N_\varepsilon(\overline{p})$ and for all $u \in U$. On the other hand, condition (a) of Theorem 3.1 implies that

$$\langle F_u(t,\overline{u}(t),\overline{p}(t),u-\overline{u}(t)\rangle \le 0 \text{ a.e.}$$

From the last two inequalities it follows that condition (i) of Theorem 2.1 is satisfied, which completes the proof. ■

It can be seen from the last two proofs that inequality (13) and condition (a) of Theorem 3.1 imply condition (i) of the dual sufficient optimality criterion for (C) presented in the previous section. We have thus obtained new sufficient optimality conditions of the first order for (C) in case when functions f and g are smooth.

4. Second order dual sufficient optimality conditions

In this section we present the second order dual sufficient optimality conditions for (C), obtained from Theorem 3.2.

Given a subset $T \subset R^{n+1}$ of variables (t, x), we shall make the following assumption:

(H4) functions f and g and their partial derivatives up to the second order with respect to (x, u) exist and are continuous on $T \times U$.

Moreover, we define the following:

$$
\begin{aligned}
W(t,u) &:= g_{xx}(t,\overline{x}(t),u) - f_{xx}^T(t,\overline{x}(t),u)\overline{p}(t)\\
S(t,u) &:= g_{ux}(t,\overline{x}(t),u) - f_{ux}^T(t,\overline{x}(t),u)\overline{p}(t)\\
R(t,u) &:= g_{uu}(t,\overline{x}(t),u) - f_{uu}^T(t,\overline{x}(t),u)\overline{p}(t)\\
A(t,u) &:= f_x(t,\overline{x}(t),u)\\
B(t,u) &:= f_u(t,\overline{x}(t),u).
\end{aligned}
$$

THEOREM 4.1 *Let U be a compact convex set and let $(\overline{x}, \overline{u})$ be a given admissible pair for* (C) *such that $J(\overline{x}, \overline{u})$ is finite. Assume that there exist a positive number ε, an arc $\overline{p}$ and a Lipschitz function $Q(t)$ defined on $[a, b]$, such that, for almost all $t \in [a, b]$, $Q(t)$ is an $n \times n$ symmetric matrix, $Q(b)$ is nonsingular. Next, define the set T by*

$$T := \{(t,x) \in [a,b] \times R^n : x = \overline{x}(t) - Q(t)(p - \overline{p}(t)) \ \mathit{for}\ p \in N_\varepsilon(\overline{p})\}.$$

Besides, suppose that (H4), *conditions* (a), (c) *of Theorem* 3.1 *are satisfied and:*

(1) $-\dot{\overline{p}}(t) = f_x^T(t,\overline{x}(t),\overline{u}(t))\overline{p}(t) - g_x(t,\overline{x}(t),\overline{u}(t))$ *a.e.;*

(2) $R(t,u) > 0$ *for all $t \in [a,b]$ and for all $u \in U$;*

(3) *for all $t \in [a,b]$, for all $u \in U$ and for all $\eta(t) \in \partial Q(t)$,*

$$
\begin{aligned}
M(t,\eta(t),u) &:= -\eta(t) + A(t,u)Q(t) + Q(t)A^T(t,u)\\
&+ Q(t)W(t,u)Q(t) - K^T(t,u)R(t,u)K(t,u) > 0
\end{aligned}
$$

where $K(t,u) := R^{-1}(t,u)(S(t,u)Q(t) + B^T(t,u))$.

Then, $(\overline{x}, \overline{u})$ is a strong minimum for (C) *relative to all (x, u) in* R.

Proof. By Theorem 3.2, it is sufficient to show that inequality (13) is satisfied. Conditions (2) and (3) are equivalent to the statement that the $(n+m) \times (n+m)$-matrix

$$N(t,\eta(t),u) := \begin{pmatrix} M(t,\eta(t),u) & O_{n\times m} \\ O_{m\times n} & R(t,u) \end{pmatrix}$$

is positive definite for all $t \in [a, b]$, for all $u \in U$ and for all $\eta(t) \in \partial Q(t)$. Define another $(n+m) \times (n+m)$-matrix

$$C(t,u) := \begin{pmatrix} I_{n\times n} & O_{n\times m} \\ -K(t,u) & I_{m\times m} \end{pmatrix}.$$

Since the matrix $C(t, u)$ is nonsingular, conditions (2) and (3) are equivalent to the fact that the $(n+m) \times (n+m)$-matrix

$$D(t,\eta(t),u) := C^T(t,u)N(t,\eta(t),u)C(t,u)$$
$$= \begin{pmatrix} -\eta(t) + A(t,u)Q(t) + Q(t)A^T(t,u) & -K^T(t,u)R(t,u) \\ +Q(t)W(t,u)Q(t) & \\ -R(t,u)K(t,u) & R(t,u) \end{pmatrix} \tag{14}$$

is positive definite for all $t \in [a,b]$, for all $u \in U$ and for all $\eta(t) \in \partial Q(t)$.

Consider the function $F(t,u,p)$ defined by (12). Then, for almost all $t \in [a,b]$ and for all $u \in U$, having computed the Jacobian $\nabla^2_{p,u}F(t,u,\overline{p}(t))$, we have

$$\nabla^2_{p,u}F(t,u,\overline{p}(t)) = -D(t,\dot{Q}(t),u),$$

where the matrix D is given by (14). Since $D(t,\eta(t),u)$ is positive definite and U is compact, we can find a positive number γ such that

$$\nabla^2_{p,u}F(t,u,p) \le 0$$

for almost all $t \in [a,b]$, for all $p \in N_\gamma(\overline{p})$ and for all $u \in U$. Thus, for almost all $t \in [a,b]$, for all $p \in N_\gamma(\overline{p})$ and for all $u \in U$, the inequality

$$\langle \nabla_{u,p}F(t,u,p) - \nabla_{u,p}F(t,\overline{u}(t),\overline{p}(t)), (u-\overline{u}(t), p-\overline{p}(t))\rangle \le 0 \tag{15}$$

holds. Using condition (1) of Theorem 4.1, we obtain that (15) is equivalent to the inequality

$$\langle \nabla_{u,p}F(t,u,p) - (F_u(t,\overline{u}(t),\overline{p}(t)), \dot{\overline{x}}(t) + Q(t)\dot{\overline{p}}(t)),$$
$$(u-\overline{u}(t), p-\overline{p}(t))\rangle \le 0$$

which, in turn, as we know from the proof of Theorem 3.2, is equivalent to (13). This completes the proof. ■

The above proof shows how important is the role played by the dual sufficient optimality criterion for (C) formulated in Section 2. Namely, conditions (1)–(3) of Theorem 4.1 imply inequality (13) which, together with (a) of Theorem 3.1, gives condition (i) of Theorem 2.1. We have thus obtained sufficient optimality conditions of the second order for (C), other than those known so far.

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