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Control and optimization of abstract continuous time evolution inclusions*

by

Hocine Mokhtar-Kharroubi

Université Oran I Ahmed Benbella, Département de Mathématiques
BP l524, Elmn'aouer Oran 31000, b Algérie
hmkharroubi@yahoo.fr

Abstract: Abstract controlled evolution inclusions are revisited in the Banach spaces setting. The existence of solution is established for each selected control. Then, the input–output (or, control-states) multimap is examined and the Lipschitz continuous well posedness is derived. The optimal control of such inclusions handled in terms of a Bolza problem is investigated by means of the so-called $P_{\mathcal{F}}$ format of optimization. A strong duality is provided, the existence of an optimal pair is given and the system of optimalty is derived. A Fenchel duality is built and applied to optimal control of convex process of evolution. Finally, it will be shown how the general theory we provided can be applied to a wide class of controled integro-differental inclusions.

Keywords: evolution inclusion; well posedness; optimal control; strong duality; system of optimality; Fenchel duality; convex process of evolution; integro-differential inclusions

1. Introduction

A vast literature has been devoted in these last decades to evolution inclusions with a rich variational analysis in control theory and optimization, along with applications in various areas. The set-valuedness arises naturally in the modeling of systems not entirely identified, subject, for instance, to a shortage of information associated with unknown physical constraints or involving random inputs. To cite but a few exemplary studies, see Bressan and Zhang (2012), Fiacca, Papageorgiou and Papalini (1998), Oppezzi and Rossi (1995), Papageorgiou (1987), Peypouquet and Sorin (2009), Vilches and Nguiven (2020), or

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Zagurovsky, Mel'nik and Kasyanov (2011). The renewal of interest in abstract inclusions has been motivated by numerous concrete problems in quasistatic mechanics for the modeling of contact boundary and/or friction problems; see Denkowski, Migórski and Papageorgiu (2003), Han and Sofonea (2003), Kuttler (2019), Kuttler and Shillor (1999), Kuttler and Li (2015), or Migórski, Ochal and Sofonea (2013), where the studies have focused on abstract elliptic ω-parametric inclusions

$$0 \in A(y(\omega), \omega) + u(\omega), \tag{1}$$

covering variational inequalities, and on evolution inclusions under the mold

$$\begin{cases} \frac{d}{dt}(B_t\,(y(\omega))) \in A(y(\omega), \omega) + u(\omega), \\ \qquad B_0\,(y(0)) := B_0 y_0. \end{cases} \tag{2}$$

Here, $\omega \in \Omega$ and Ω is endowed with a σ-algebra Σ.

In applications, the set-valued operator $A(., \omega)$ supposed to be pseudomonotone (see Kuttler and Shillor, 1999), may arise, for instance, as the subgradient map of nonconvex locally Lipschitz functionals, the parameter u as an input datum and $B(t) := B_t$ as a linear operator that may vanish, so that inclusion (2) covers problems of mixed type.

The continuous time evolution inclusions represent the overwhelming proportion of dynamical systems and abound in various fields of mathematics and in many applications. Indeed, the case $\Omega := [0, T]$ with Σ being the sigma-algebra of Lebesgue subsets of Ω covers a wide range of applications in many areas; see, e.g., Bian and Weeb (1999), Han and Sofonea (2003), Kuttler and Li (2015), Kuttler and Shillor (1999), Motreanu and Radulescu (2003), Zagurovsky, Mel'nik and Kasyanov (2011). But, abstract time-evolution inclusions with a parameter $\omega \in \Omega$ dealt mainly with existence of t-measurable solutions; see, Andrews et al. (2019), Kuttler and Shillor(2000, 2019), Kutller, Li and Shillor (2016), Kuttler and Shillor (1999), and, more recently, in Andrews, Kuttler and Li (2020), where Ω is a sample space equipped with a σ-algebra and the studies quoted focused on product measurability or (t, ω)-measurable solution.

For our part, we deal here with control and optimization of abstract evolution inclusions without random character and framed as follows.

$$(x_t, u_t, \frac{d}{dt}x_t) \overset{a.e}{\in} E(t)\ ;\ x_\tau \in \Omega. \tag{3}$$

Throughout, the notations $\overset{a.e}{\in}, \overset{a.e}{=}, \overset{a.e}{\geq}, \overset{a.e}{\to}$ mean that the indicated relations hold $a.e.$ $t \in J := [t_0, T] \subset [0, +\infty[$ for the usual Lebesgue measure dt on J.

For X and U being reflexive Banach spaces, the multimap E from J to $X \times U \times X$ is measurable with closed values and, to keep formulas compact and readable, x_t and u_t stand for the state $x(t) \in X$ and the control $u(t) \in U$. Finally, (τ, x_τ) is fixed in $J \times \Omega$ and the subset $\Omega \subset X$ is convex and closed.

The implicit mold (3) is more adequate for the study of evolution inclusion since, for many situations, the set-valuedness can be due also to the ignorance of the laws relating the state and the eventual parameters and/or controls. In addition, the mold (3) covers (1) and (2) by setting

$$E(t) := gh(A(.,t)),$$

where gh denotes the graph (see definition in (9)). Indeed, (1) and (2) amount to

$$u_t \in E(t)$$

and

$$(\frac{d}{dt}z_t - u_t, y_t) \in E(t) \ ; \ z := By$$

reducing thus the second inclusion to the control of the observation z.

The mold (3) is said to be *convex* (respectively*convex process*; *linear*) if the values of the multimap E are *convex* (respectively *convex cones; linear subspaces*). It will be called *convex-like* if the multimap $F : J \times X \times U \rightrightarrows X$:

$$F(t, a, b) \overset{a.e}{:=} \{c \in X : \ (a, b, c) \in E(t)\} \tag{4}$$

is convex valued and onto in the sense that

$$rg(F(t, ., .)) \overset{a.e}{=} X, \tag{5}$$

where rg denotes the range (see the definition later on in (9)).

However, having that inclusion (3) is equivalent to

$$\frac{d}{dt}x_t \overset{a.e}{\in} F(t, x_t, u_t) \ ; \ x_\tau \in \Omega,$$

it is well known that condition (5) holds if $F(t, ., .)$ is monotone maximal and coercitive (see, e.g., Kuttler, 2019). In this way, the convex-like mold covers almost all of the parabolic inclusions.

This case will be considered in another work, because here, this would increase too much the length of the paper. Nevertheless, let us point out the unified approach, achieved in Mokhtar-Kharroubi (2017) for the control of *discrete time systems* framed by convex-like inclusions and without assuming any monotonicity condition.

Let us now provide a brief summary of basic facts on Bochner integral of functionals for later use. Denote $\int_\tau^t \omega_s ds$ by $\int_\tau^t \omega_s$ and $\int_J \omega_s ds$ by $\int \omega_s$. The

Lebesgue–Bochner space L_X^p of functionals x from J to a reflexive Banach space $(X, \|.\|)$ is endowed with the norm

$$\begin{aligned} \|x\|_{L^p} &= \left(\int \|x_t\|^p \right)^{\frac{1}{p}} \text{ if } p \in [1, +\infty[\,, \\ \|x\|_{L^\infty} &= ess.sup_{t\in J} \|x_t\|\,. \end{aligned}$$

Then, the dual $(L_X^p)^*$ for $p \in [1, +\infty[$ is identified with $L_{X^*}^{p^*} \ : \frac{1}{p} + \frac{1}{p^*} = 1$.

By $\omega \in L_+^p$ we mean that $\omega_t \overset{a.e}{\geq} 0$ and $\int \omega_t^p < +\infty$.

The linear space $M_X^p := X \oplus L_X^p$, equipped with the norm

$$\begin{aligned} &\|(c, v)\|_p := (\|c\|^p + \|v\|_{L^p}^p)^{\frac{1}{p}} \text{ if } p \in [1, +\infty[\,, \\ &\|(c, v)\|_\infty := \max\{\|c\|, \|v\|_{L^\infty}\}\,, \end{aligned}$$

is a Banach space, whose dual for $p \in [1, +\infty[$ is $M_{X^*}^{p^*}$ under the pairing

$$\langle (c, v), (d, w) \rangle := \langle c, d \rangle + \int \langle v_t, w_t \rangle.$$

Let A_X^p be the space of absolutely continuous functions $x : J \to X$, whose derivative $\frac{d}{dt}x_t$ (in the sense of distributions) lies in L_X^p (we denote $\frac{dx}{dt} \in L_X^p$). The norm in A_X^p is

$$\begin{aligned} \|x\|_{A^p} \ &: = \left(\|x_\tau\|^p + \left\| \frac{dx}{dt} \right\|_{L^p}^p \right)^{\frac{1}{p}} \text{ if } p \in [1, +\infty[\,, \\ \|x\|_{A^\infty} \ &: = \max\left\{ \|x_\tau\|, \left\| \frac{dx}{dt} \right\|_{L^\infty} \right\}. \end{aligned}$$

Because $x \to (x_\tau, \frac{dx}{dt})$ is a linear isometry of A_X^p onto M_X^p, then A_X^p is a Banach space, whose dual for $p \in [1, +\infty[$ is identified with $M_{X^*}^{p^*}$, with the pairing given by

$$\langle x, (d, w) \rangle := \langle x_\tau, d \rangle + \int \langle \frac{d}{dt} x_t, w_t \rangle.$$

Let $C_X(J)$ be the Banach space of continuous functions $x : J \to X$ equipped with the maximum norm $\|x\|_C$. Then, the continuous embedding

$$A_X^p \hookrightarrow C_X(J)$$

holds. See, e.g, Barbu and Precupanu (1978) for more materials on absolutely continuous vector-valued functions.

2. Main assumptions and results

The study is conducted *without random character* under the following conditions.

The multimap E has convex values satisfying the *openness property:*

$$int\,(E(t)) \neq \varnothing \text{ for all } t \in J \tag{6}$$

and the *growth estimation :*

$$\exists \omega \in L^{\infty}_{+}, \exists \delta > 0, \quad \exists r \geq 1$$

$s.t$

$$\|c\|_X \overset{a.e}{\leq} \omega_t(1 + \|a\|_X) + \delta\,\|b\|_U^r \quad \text{for all } (a, b, c) \in E(t). \tag{7}$$

The paper is organized in three parts with the following progress.

(I). In the first part some multimap-results with frequent uses are outlined.

(II). The second part focuses on the inclusion. No monotonicity condition is placed on the multimap. For each control $u \in L^r_{\mathbb{R}^m}(J)$ $(r > 1)$ there exists a state x in A^1_X s.t (x, u) satisfies the inclusion (Theorem 2). The solution-map (*control-states*) is examined and the Lipschitz continuous well posedness is derived (Theorem 3 and Proposition 2).

(III). The third part is devoted to a Bolza problem of control over the inclusion. A strong duality and an optimal pair are provided (Theorem 4 and Theorem 5). The system of optimalty is de:rived (Theorem 5). A Fenchel duality is built and applied to optimal control of convex-process of evolution (Proposition 5 and Proposition 6).

Finally, it will be shown briefly how the general theory we provided can handle a wide class of controlled integro-differental inclusions.

3. The preliminaries

Let us fix the notations and basic facts on functions and multimaps. Throughout, ϑ_a is a generic neighborhood (of a) and the abreviations "s.t" and "iff" stand for "such that" and "if and only if".

All the neutral elements are denoted by 0, while $\|.\|$ and $\langle . \rangle$ stand for the norm and the dual pairing. Let Z be a normed vector space (*n.v.s* in short), whose topological dual is Z^*, then Z_w means that Z is endowed with the weak topology. As usual, B_Z and S_Z are the unit ball and the unit sphere of Z and for a subset $C \subset Z$ the closure and the convex hull of C are denoted by $cl(C)$ and $conv(C)$. The dual cone C^+ and the polar cone C^- are given by

$$C^+ := \{c^* \in Z^* : \langle c^*, c \rangle \geq 0 \ \ \forall c \in C\} := -C^-.$$

The upper support function of C is $\sigma_C : Z^* \to \mathbb{R} \cup \{+\infty\}$:

$$\sigma_C(p) := \sup \{\langle p, v\rangle : v \in C\}$$

and the barrier cone of C is the domain of σ_C, or,

$$b(C) := \{p \in Z^* : \sigma_C(p) < +\infty\}. \tag{8}$$

The convex normal cone at $c \in C$ is given by

$$N_C(c) := \{p \in Z^* : \langle p, c\rangle = \sigma_C(p)\}.$$

Let Y be a $n.v.s$ and $\mathcal{F}$ (denote $\mathcal{F} : Z \rightrightarrows Y$) whose domain, range and graph are respectively:

$$\begin{cases} dom(\mathcal{F}) := \{z \in Z : \mathcal{F}(z) \neq \varnothing\}, \\ rg\,(\mathcal{F}) := \cup \{\mathcal{F}(z) : z \in Z\}, \\ gh\,(\mathcal{F}) := \{(z, y) : y \in \mathcal{F}(z)\}. \end{cases} \tag{9}$$

Then, $\mathcal{F}$ is said to be :

· Proper if $dom(\mathcal{F}) \neq \varnothing$ and $gh(\mathcal{F}) \neq Z \times Y$.

· Strict if $dom(\mathcal{F}) = Z$

· Closed (respectively convex) on a closed (respectively convex) subset $D \subset dom(\mathcal{F})$, if

$gh(\mathcal{F}) \bigcap (D \times Y)$ is a closed (respectively convex) subset of $Z \times Y$.

· Upper semi-continuous ($u.s.c$ in short) at $a \in int(dom(\mathcal{F}))$, if for every open subset $\theta \supset \mathcal{F}(a)$, there exists ϑ_a s.t $\mathcal{F}(\vartheta_a) := \cup \{\mathcal{F}(z) : z \in \vartheta_a\} \subset \theta$.

· Lower-semi-continuous ($l.s.c$) at $a \in int(dom(\mathcal{F}))$, if for every open subset $\theta \subset Z$, $\theta \cap \mathcal{F}(a) \neq \varnothing$, there exists ϑ_a s.t $\mathcal{F}(v) \cap \theta \neq \varnothing$ for each $v \in \vartheta_a$.

· Lipschitz on $D \subset dom(\mathcal{F})$, if for some $\rho > 0$ there holds

$$\mathcal{F}(v_1) \subset cl\left[\mathcal{F}(v_2) + \rho \|v_1 - v_2\| B_Y\right] \text{ for all } v_1, v_2 \in D. \tag{10}$$

A selector of $\mathcal{F}$ on $D \subset dom(\mathcal{F})$ is a function

$$f : D \to Y : f(z) \in \mathcal{F}(z) \quad \forall z \in D.$$

The conjugate $\mathcal{F}^* : Y^* \rightrightarrows Z^*$ is defined by the barrier cone as

$$z^* \in \mathcal{F}^*(p) \text{ iff } (z^*, -p) \in b(gh\,(\mathcal{F})). \tag{11}$$

Define the modulus

$$\|\mathcal{F}(z)\| := \sup \{\|y\| : y \in \mathcal{F}(z)\}. \tag{12}$$

Finally, recall that $\varphi : Z \to \mathbb{R} \cup \{+\infty\}$, whose domain and epigraph are

$$\begin{aligned} dom(\varphi) &:= \{z \in Z : |\varphi(z)| < +\infty\}, \\ epi(\varphi) &:= \{(z, \lambda) \in Z \times \mathbb{R} : \varphi(z) \leq \lambda\}, \end{aligned}$$

is proper if $dom(\varphi) \neq \varnothing$ and closed (respectively convex) if $epi(\varphi)$ is closed (respectively convex).

The *Fenchel conjugate* of φ is the function

$$\varphi^* : Z^* \to \mathbb{R} \cup \{+\infty\} : \varphi^*(z^*) = \sup\{\langle z^*, z\rangle - \varphi(z) \ : z \in Z\}.$$

The *convex subdifferential* of φ is the set-valued map $\partial\varphi : Z \rightrightarrows Z^*$:

$$\partial\varphi(z) = \{z^* \in Z^* : \varphi(z) + \varphi^*(z^*) \leq \langle z^*, z\rangle\}.$$

4. The support function of a multimap

Let $\mathcal{F} : Z \rightrightarrows Y$ be proper with closed and convex values. Under the convention that $M + \varnothing = \varnothing$ for all $M \subset Y$, the support function of $\mathcal{F}$ is the function

$$s_{\mathcal{F}} : Z \times Y^* \to \mathbb{R} \cup \{+\infty\}$$

given by

$$s_{\mathcal{F}}(z, p) := \inf\{\langle p, y\rangle : y \in \mathcal{F}(z)\} \text{ if } z \in dom(\mathcal{F}) \text{ and } +\infty \text{ otherwise.} \quad (13)$$

Denote by $s_{\mathcal{F}}(., p)$ the function $z \to s_{\mathcal{F}}(z, p)$. Then,

$$dom(s_{\mathcal{F}}(., p)) = dom(\mathcal{F}) \quad \forall p \in Y^*.$$

But the domain of $s_{\mathcal{F}}$ is considered in the sense of saddle function; or,

$$dom(s_{\mathcal{F}}) := dom_z(s_{\mathcal{F}}) \times dom_p(s_{\mathcal{F}})$$

where

$$\begin{aligned} dom_z(s_{\mathcal{F}}) &:= \{z \in Z : s_{\mathcal{F}}(z, p) > -\infty, \ \forall p \in Y^*\}, \\ dom_p(s_{\mathcal{F}}) &:= \{p \in Y^* : s_{\mathcal{F}}(z, p) < +\infty, \ \forall z \in Z\} \end{aligned}$$

and $s_{\mathcal{F}}$ is said to be proper if $dom(s_{\mathcal{F}}) \neq \varnothing$. For a complete study of this tool with some applications, see Mokhtar-Kharroubi (1987, 2017). But some facts therefrom have later uses and so we bring them in here.

Proposition 1 *Denote by $\partial_z s_{\mathcal{F}}(.,p)$ the convex subdifferential of $s_{\mathcal{F}}(.,p)$. Then,*

(i). Without assuming that $gh(\mathcal{F})$ is convex there hold

$$(z^*, -p\ ) \in N_{gh(\mathcal{F})}(\widehat{z}, \widehat{y}) \text{ iff } -p \in N_{\mathcal{F}(\widehat{z})}(\widehat{y}) \text{ and } z^* \in \partial_z s_{\mathcal{F}}(\widehat{z}, p) \tag{14}$$

(ii). $\mathcal{F}$ is convex iff $s_{\mathcal{F}}(.,p)$ is convex for all $p \in Y^*$

(iii). $\mathcal{F}$ is Lipschitz on $D \subset dom(\mathcal{F})$ iff

$$\{s_{\mathcal{F}}(.,p) : p \in S_{Y^*}\} \text{ is equi-Lipschitz.}$$

(or of the same rank) on D

(iv). Let Y be complete. Then, $s_{\mathcal{F}}$ is proper iff $\mathcal{F}$ is proper and bounded valued.

Proof The result *(i)* can be checked in a straightforward manner, while results *(ii)* and *(iii)* hold by usual Hahn-Banach separation arguments. Let us prove *(iv)*.

When $\mathcal{F}$ is proper and bounded valued, then, for every $v \in dom(\mathcal{F})$, there exists $\lambda_v > 0$ s.t $\mathcal{F}(v) \subset \lambda_v B_Y$. Hence,

$$s_{\mathcal{F}}(v,p) \geq -\lambda_v \|p\| > -\infty.$$

Conversely, if for some $a \in Z$ there exists an unbounded sequence $(w_l)_{l\in\mathbb{N}} \subset \mathcal{F}(a)$, then, by the uniform boundedness principle (Y is complete), $\lim_{l\to\infty}\langle p, w_l\rangle = -\infty$ for some $p \in Y^*$, in contradiction with the fact that

$$-\infty < s_{\mathcal{F}}(a,p) \leq \langle p, w_l\rangle \text{ for all } l \in \mathbb{N}.$$

■

A result established in Mokhtar-Kharroubi (1987) (in terms of $s_{\mathcal{F}}$) with a frequent use in the present paper is:

Theorem 1 *Mokhtar-Kharroubi (1987) assume that Z and Y are complete, $\mathcal{F} : Z \rightrightarrows Y$ is closed and convex on an open, convex subset $D \subset int(dom(\mathcal{F}))$. Then, $\mathcal{F}$ is locally Lipschitz on D whenever it is bounded valued (on D).*

See also Mokhtar-Kharroubi (2022) for Lipschitz property of multimap under weakened conditions.

5. The control of continuous-time evolution inclusions

5.1. Introduction

We start with a brief summary of basic facts on Lebesgue-Bochner-Aumann integral of multimaps.

Let Φ be a multimap from a complete, σ-finite measure space (Ω, M_Ω, μ) to a separable Banach space W. Then, Φ is said to be measurable if the following real valued function is measurable.

$$\Omega \times W \ni (s, w) \to d(w, \Phi(s)) := \inf\{\|w - v\| : v \in \Phi(s)\}.$$

This is equivalent to graph measurability; or,

$$gh(\Phi) \in M_\Omega \otimes \Sigma(W),$$

where $\Sigma(W)$ is the Borel sigma-algebra of W and $M_\Omega \otimes \Sigma(W)$ is the smallest σ-algebra which contains the product $M_\Omega \times \Sigma(W)$. But, this amounts to the existence of measurable selectors φ_n $(n \in \mathbb{N})$ $s.t$ the Castaing representation (Castaing and Valadier, 1977) holds; or,

$$\forall n \in \mathbb{N} \ \ \varphi_n(s) \overset{a.e}{\in} \Phi(s) \text{ and } \ \Phi(s) \overset{a.e}{=} cl\{\varphi_n(s) : n \in \mathbb{N}\}. \tag{15}$$

Let S^1_Φ be the set of all Bochner integrable selectors of Φ, i.e.,

$$S^1_\Phi := \left\{\varphi \in L^1_X : \varphi(s) \overset{a.e}{\in} \Phi(s)\right\}.$$

The set S^1_Φ may be empty, but it will be nonempty if the function

$$s \to \inf\{\|v\| : v \in \Phi(s)\}$$

lies in

$$L^1_+(\Omega).$$

In particular, S^1_Φ is nonempty and L^1-bounded if the function

$$s \to \sup\{\|v\| : v \in \Phi(s)\}$$

lies in

$$L^1_+(\Omega).$$

When $S^1_\Phi \neq \varnothing$, the Lebesgue-Bochner-Aumann integral of Φ is taken to be

$$\int \Phi(s) d\mu := \left\{\int \varphi(s) d\mu : \varphi \in S^1_\Phi\right\}.$$

5.2. Existence of controlled solutions

The investigation of the inclusion (3) is conducted through a reduction. Define

$$\mathbf{U} := L^r_U \ , \mathbf{X} := A^1_X \ , \mathbf{Y} := L^1_X \times X \text{ and } \mathcal{F} : \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y}$$

s.t

$$\mathcal{F}(x,u) := \left\{ \Gamma(x,u) - \frac{dx}{dt} \right\} \times \{\Omega - x_\tau\}, \tag{16}$$

where $\Gamma : L^1_X \times L^r_U \rightrightarrows L^1_X$ is given by

$$\Gamma(x,u) := \left\{ v : (x_t, u_t, v_t) \overset{a.e}{\in} E(t) \right\}, \tag{17}$$

so that the inclusion (3) is reduced to the so-called $\mathcal{F}$ -inclusion

$$0 \in \mathcal{F}(x,u).$$

Next is the main existence result.

Theorem 2 *Let* $F : J \times X \times U \rightrightarrows X$ *be given by*

$$F(t,a,b) \overset{a.e}{:=} \{c \in X : \ (a,b,c) \in E(t)\} \tag{18}$$

and assume that E *satisfies the conditions (6)-(7). Then:*

(i). For every $(x,u) \in L^1_X \times L^r_U$ *the set* $S^1(x,u)$ *of integrable selectors of the superposition*

$$J \ni t \rightrightarrows F(t, x_t, u_t) \subset X \tag{19}$$

is nonempty, convex and weakly compact in L^1_X.

(ii). For every $u \in L^r_U$ *there exists* $x \in A^1_X$ *such that*

$$0 \in \mathcal{F}(x,u). \tag{20}$$

Proof The proof is given by means of integrable selectors (Papageorgiou, 1987, Theorem 2.3, p. 308). Clearly, $gh(F(t,.)) = E(t)$ is convex and closed and by the estimation (7) $F(t,.)$ is bounded valued. We claim that

$$int(dom(F(t,.)) \neq \varnothing \text{ for all } t \in J.$$

Indeed, let Q be the projector

$$Q : X \times U \times X \to X \times U : Q(a,b,c) := (a,b).$$

Then, $dom(F(t,.)) = Q(E(t))$ and by the openness condition (6) the open mapping Theorem proves the claim. Thus (by Theorem 1), $F(t,.)$ is locally Lipschitz; hence, continuous (i.e., $u.s.c$ and $l.s.c$) on $intdom(F(t,.))$. In this way, F is Caratheodory, or F is t-measurable and $F(t,.)$ is convex continuous. Then, the superposition (19) has a measurable graph. So, by Aumann's Theorem

(Wagner, 1977, Theorem 5.101) there exists a measurable selector. On the other hand, since Ω is bounded, then, by condition (7), for some $\rho > 0$ there holds

$$\|x_t\| \overset{a.e}{\leq} \mu_t + \int_\tau^t \omega_s \|x_s\| \text{ with, } \mu_t :\overset{a.e}{=} \rho + \delta \|u\|_{L^r}^r + \int_\tau^t \omega_s$$

and by Gronwall's lemma we get

$$\|x_t\| \leq \mu_t + \int_\tau^t \mu_s e^{(t-s)} \text{ for all } t \in J. \tag{21}$$

Again, condition (7) leads to

$$\|F(t, x_t, u_t)\| \overset{a.e}{\leq} \omega_t(1 + \|x_t\|) + \delta \|u_t\|,$$

or,

$$\|F(t, x_t, u_t)\| \overset{a.e}{\leq} \omega_t \left[1 + \mu_t + \int_\tau^t \mu_s e^{(t-s)}\right] + \delta \|u_t\|^r. \tag{22}$$

Then, the superposition $J \ni t \rightrightarrows F(t, x_t, u_t)$ is integrably bounded by

$$\psi \in L_+^1 \;:\; \psi_t :\overset{a.e}{=} \omega_t \left[1 + \mu_t + \int_\tau^t \mu_s e^{(t-s)}\right] + \delta \|u_t\|^r \tag{23}$$

and by Papageorgiou's Theorem (Papageorgiou, 1987, Theorem 2.1, p. 307) the subset $S^1(x, u)$ of the integrable selectors is nonempty, convex and weakly compact in L_X^1.

(ii). For each fixed $u \in L_U^r$, define

$$F^u : J \times X \rightrightarrows X : F^u(t, a) :\overset{a.e}{=} F(t, a, u_t).$$

Then, the same arguments invoked for $F(t, .)$ work for $F^u(t, .)$. Indeed, $F^u(t, .)$ is convex, closed and bounded valued. Thus, $F^u(t, .)$ is locally Lipschitz on $int(dom(F^u(t, .)))$. Hence, F^u is Caratheodory or F^u is t-measurable and $F^u(t, .)$ is convex continuous. Then, for every $x \in L_X^1$, the superposition

$$J \ni t \rightrightarrows F^u(t, x_t) \text{ is measurable.}$$

In addition, F^u is $u.s.c$ from X_w to X_w and integrably bounded (by (23)). Again (by Papageorgiou, 1987, Theorem 2.1, p. 307) there exists $x \in A_X^1$ s.t $x_\tau \in \Omega$ and

$$\frac{d}{dt} x_t \overset{a.e}{\in} F^u(t, x_t) \;;$$

i.e.

$$(x_t, u_t, \frac{d}{dt} x_t) \overset{a.e}{\in} E(t).$$

■

5.3. Stability and well-posedness of the inclusion

Theorem 3 *Under the conditions (6)-(7) for the multimap E, the following holds:*

(i). The multimaps $\mathcal{F}$ and Γ, given by (16) *and* (17)*, are strict, closed, convex, bounded valued and locally Lipschitz.*

(ii). For each $d \in \Omega$, $u \in L^r_U$ and $v \in L^1_X$ there exists $x \in A^1_X$ such that

$$(x_t,\ u_t,\ v_t + \frac{d}{dt}x_t) \overset{a.e}{\in} E(t)\ ;\quad x_\tau = d.$$

In this way, the controllability condition

$$0 \in int(rg(\mathcal{F}))$$

holds true.

(iii). The input-output, or solution map $\Psi : L^r_U \rightrightarrows A^1_X$, given by

$$x \in \Psi(u)\ \textit{iff}\ \ (x_t, u_t, \frac{d}{dt}x_t) \overset{a.e}{\in} E(t)\ \textit{and}\ \ x_\tau \in \Omega, \tag{24}$$

is strict and locally Lipschitz.

Proof *(i).* Since E is convex valued then Γ and $\mathcal{F}$ have convex graph with bounded values in view of estimation (7). We claim that Γ is closed. Indeed, let a sequence $(x^l, u^l, v^l)_{l\in\mathbb{N}} \subset gh(\Gamma)$ be convergent to some (x, u, v). Thus, for some $N \subset \mathbb{N}$ one has $(x^l, u^l, v^l)_{l\in N} \overset{a.e}{\mapsto} (x, u, v)$ and then $(x_t, u_t, v_t) \overset{a.e}{\in} E(t)$, since $E(t)$ is closed.

For the closedness of $\mathcal{F}$ let a sequence $(y^l, x^l, u^l, a^l)_{l\in\mathbb{N}} \subset gh(\mathcal{F})$ be strongly convergent to some (y, x, u, a). Then,

$$(u^l)_{l\in\mathbb{N}} \to u \text{ (strongly) in } L^r_U \text{ and } (\frac{d}{dt}x^l)_{l\in\mathbb{N}} \to \varpi \text{ (strongly) in } L^1_X.$$

Having $(x^l) \to x$ in A^1_X we obtain $\varpi = \frac{dx}{dt}$, and since $A^1_X \hookrightarrow C_X(J)$, we get

$$(x^l)_{l\in\mathbb{N}} \to x \text{ in } C_X(J) \text{ and } (x^l_\tau)_{l\in\mathbb{N}} \longmapsto x_\tau \text{ in } X.$$

On the other hand, for every $l \in \mathbb{N}$ there exists $\eta^l \in \Gamma(x^l, u^l)$ and $a^l \in \Omega\, s.t$

$$y^l = (\eta^l - \frac{dx^l}{dt}, a^l - x^l_\tau)$$

and since $\left(y^l\right)_{l\in\mathbb{N}}$ and $(\frac{dx^l}{dt})_{l\in\mathbb{N}}$ converge strongly, then $\left(\eta^l, a^l\right)_{l\in\mathbb{N}}$ converges as well to some $(\eta, a) \in L^1_X \times \Omega$ and

$$(y^l)_{l\in\mathbb{N}} \to y := (\eta - \frac{dx}{dt}, a - x_\tau) \in \mathcal{F}(x, u).$$

Let F be the multimap given by (18), i.e.

$$F(t,a,b) := \{c \in X : \ (a,b,c) \in E(t)\}.$$

Then (by Theorem 3), for every $(v,u) \in L^1_X \times L^r_U$, the set $S^1(v,u)$ of selectors of

$$J \ni t \rightrightarrows F(t, v_t, u_t) \subset X$$

is nonempty in L^1_X. Hence, $(v,u) \in dom(\Gamma)$ and Γ is strict.

(ii). Let $d \in \Omega,\ u \in L^r_U$ and $v \in L^1_X$. Define

$$F^v : J \times X \rightrightarrows X : F^v(t,a) := F(t,a,u_t) + v_t.$$

Again, F^v is t-measurable, $F^v(t,.)$ is $u.s.c$ from X_w to X_w, and integrably bounded in view of (23). Then (by Papageorgiou, 1987, Theorem 2.3), the inclusion

$$\frac{d}{dt} x_t \overset{a.e}{\in} F^v(t,\ x_t)\ ;\ x_\tau = d, \tag{25}$$

admits a solution. In this way, for every $(v,u,d) \in \mathbf{Y} := L^1_X \times L^r_U \times X$ there exist $(x,u) \in \mathbf{X} \times \mathbf{U}$ such that $(v,u,d) \in \mathcal{F}(x,u)$. Hence,

$$0 \in int(rg(\mathcal{F}))$$

holds true.

In addition, for $v = 0$ we get $\mathcal{F}(x,u) \neq \varnothing$ and then, $\mathcal{F}$ is strict.

(iii). We check easily that Ψ, given by (24), is convex, closed and bounded valued and $dom(\Psi) = L^r_U$ (by the result *(ii)*). Then, by Theorem 1, Ψ is locally Lipschitz. ∎

PROPOSITION 2 *The underlying evolution inclusion is Lipschitz-continuous well-posed. Namely, the solution map Ψ given by (24) is s.t*

$\forall v \in L^r_U,\ \exists l_v > 0,\ \exists \delta_v > 0$

and

$$Haus(\Psi(u_1), \Psi(u_2)) \leq l_v(\|u_1 - u_2\|_{L^r_U})\ \forall u_1, u_2 \in v + \delta_v B_{L^r_U}, \tag{26}$$

where $B_{L^r_U}$ is the unit ball (of L^r_U) and $Haus$ stands for the Hausdorff distance.

Further, Ψ admits a continuous selector, or a continuous functional

$$\psi : L^r_U \to A^1_X \quad s.t \ \ 0 \in \mathcal{F}(\psi(u), u)\ \forall u \in L^r_U.$$

Proof Ψ is locally Lipschitz; then, (26) holds and Lipschitz-continuous well-posedness follows. By Michael's Theorem (Aubin and Cellina, 1984, Theorem 1, p. 82), there exists a continuous selector of Ψ, or a continuous functional $\psi : L^r_U \to A^1_X$ *s.t*

$$0 \in \mathcal{F}(\psi(u), u) \text{ for all } u \in L^r_U.$$

■

6. The optimal control problem

Throughout, $\overline{\lim}$ and $\underline{\lim}$ denote the upper and the lower limit, while $sol(Q)$ and $val(Q)$ stand for the set of optimal solutions and the value of the problem indicated by (Q). We deal here with a Bolza problem of control, written as

$$(P_E) : \begin{cases} inf \ \left[g(x_\tau, x_T) + \int f(t, x_t, u_t)\right] \\ (x, u) \in A^1_X \times L^r_U \ \text{ s.t} \\ (x_t, u_t, \frac{d}{dt}x_t) \overset{a.e}{\in} E(t) \ ; \ x_\tau \in \Omega. \end{cases} \tag{27}$$

We assume that

$$g : \quad X \times X \to \mathbb{R} \cup \{+\infty\} \ \text{ is proper, convex and } l.s.c,$$
$$f : \quad J \times X \times U \to \mathbb{R} \cup \{+\infty\} \ \text{ is convex-Caratheodory}$$

and, additionally, the following conditions hold.

Condition **(g)**. (See V. Barbu, 1994).

$g(a_1, a_2) \geq g_1(a_1) + g_2(a_2)$ for all $(a_1, a_2) \in X \times X$, where

$g_1,\, g_2 : X \to \mathbb{R} \cup \{+\infty\}$ are convex, proper, l.s.c and satisfy

$$\begin{array}{l} \lim_{\|a_1\| \to +\infty} \left(\frac{g_1(a_1)}{\|a_1\|}\right) = +\infty, \\ \liminf_{\|a_2\| \to +\infty} \left(\frac{g_2(a_2)}{\|a_2\|}\right) > -\infty. \end{array} \tag{28}$$

Condition **(f)**. (See Aubin and Clarke, 1979).

$f(.,0) \in L^1_{\mathbb{R}}$ and there exist $\gamma > 0, \rho > 0,\ r > 1$ and $\widehat{\omega} \in L^1_+$ s.t

$$\rho \|b\|^r - \widehat{\omega}_t \overset{a.e}{\leq} |f(t,a,b)| \overset{a.e}{\leq} |f(t,0)| + \gamma(\|a\| + \|b\|^r) \ \ \forall (a,b) \in X \times U. \tag{29}$$

Observe that condition (29) holds for the usual tracking objective, given by

$$\int f(t, x_t, u_t) := \|x - \widetilde{x}\|_{L^1} + \|u - \widetilde{u}\|^r_{L^r_m}.$$

6.1. Existence of an optimal pair

THEOREM 4 *The optimal control problem* (P_E) : *(27) admits a solution pair* $(\widehat{x},\widehat{u}) \in L^1_X \times L^r_U$ *whenever conditions (6)-(7) hold for the multimap* E *and the objective satisfies the conditions (28)-(29).*

PROOF The condition (29), combined with the fact that $t \to f(t,x_t,u_t)$ is measurable, ensure that the functional

$$\psi : L^1_X \times L^r_U \to \overline{\mathbb{R}} \ : \psi(x,u) := \int f(t,x_t,u_t)$$

is everywhere finite, or

$$-\infty < \psi(x,u) < +\infty \text{ for all } (x,u) \in L^1_X \times L^r_U$$

and clearly, by condition (29), ψ is u-coercitive on L^r_U in the sense that

$$\text{for every fixed } x \in L^1_X \ , \lim_{\|u\|_{L^r} \to +\infty} \psi(x,u) = +\infty.$$

But, arguing by contradiction, we check easily by (28) that $val(P_E)$ is finite.

Because the controlled inclusion admits a solution (Theorem 3), then we may consider a minimizing sequence $(x^l,u^l)_{l\in\mathbb{N}}$; i.e., s.t

$$\begin{cases} (x^l_t, u^l_t, \frac{d}{dt}x^l_t) \overset{a.e}{\in} E(t) : x^l_\tau \in \Omega, \forall l \in \mathbb{N} \text{ and} \\ \qquad \lim_{l\to+\infty} \left(\varphi(x^l,u^l)\right) = val(P_E). \end{cases} \tag{30}$$

Thus, for $\epsilon > 0$ sufficiently small, there exist $l_\epsilon \in \mathbb{N}$ such that

$$\varphi(x^l,u^l) \leq val(P_E) + \epsilon \quad \forall l \geq l_\epsilon. \tag{31}$$

Clearly, L^r_U is reflexive (U is reflexive and $r > 1$). Then, $(u^l)_{l\in\mathbb{N}}$ is weakly compact, hence bounded and by condition (7) there exist $\delta > 0,\ r > 1,\ \omega \in L^1_+$, satisfying

$$\left\| \frac{d}{dt} x^l_t \right\| \overset{a.e}{\leq} \omega_t(1 + \left\|x^l_t\right\|) + \delta \left\|u^l_t\right\|^r .$$

By Gronwall's lemma $(x^l)_{l\in\mathbb{N}}$ is bounded in $C_X(J)$ and then, for some $\beta > 0$,

$$\left\| \frac{dx^l}{dt} \right\| \leq \beta\omega_t \text{ for all } l. \tag{32}$$

Thus, $(x^l)_{l\in\mathbb{N}}$ is equicontinuous, hence relatively compact in $C_X(J)$.

By the bound (32), the sequence $(\frac{dx^l}{dt})_{l\in\mathbb{N}}$ is equi-integrable on every open subset $J_0 \subset J$, and then relatively compact for the topology $\sigma(L^1, L^\infty)$ (by Denford-Pettis Theorem). Thus, there exist $N \subset \mathbb{N}$, $\widehat{\eta} \in L^1_X$, $\widehat{x} \in C_X(J)$ and $\widehat{u}$ such that

$$\begin{aligned}(u^l)_{l\in N} &\to \widehat{u} \text{ (weakly) in } L^r_U \text{ and } (\frac{dx^l}{dt})_{l\in N} \to \widehat{\eta} \text{ (weakly) in } L^1_X,\\ (x^l)_{l\in N} &\to \widehat{x} \text{ in } C_X(J) \text{ and then } g(\widehat{x}_\tau, \widehat{x}_T) \le \underline{\lim}_{l\in N}\, g(x^l_\tau, x^l_T). \end{aligned} \quad (33)$$

Further, by the identity $\widehat{x}_t \overset{a.e}{=} c + \int_{[0,t]} \widehat{\eta}_s$ (for some $c \in \Omega$) we get

$$\frac{d}{dt}\widehat{x}_t \overset{a.e}{=} \widehat{\eta}(t). \quad (34)$$

Mazur's theorem applies then to the sequence $(x^l, u^l, \frac{dx^l}{dt})_{l\in N}$ and there exists a sequence $(v^n, w^n, \eta^n)_{n\in\mathbb{N}}$ such that

$$(v^n, w^n, \eta^n) \in conv\left\{(x^l, u^l, \frac{dx^l}{dt}) : l \in N\right\} \;\; \forall n \in \mathbb{N} \quad (35)$$

and

$$(v^n, w^n, \eta^n)_{n\in\mathbb{N}} \to (\widehat{x}, \widehat{u}, \widehat{\eta}) \text{ strongly in } L^1_X \times L^r_Z \times L^1_X.$$

Recall (Ahmed and Teo, 1981, Theorem 1.1.5, p.7) that for each $n \in \mathbb{N}$, (v^n, w^n, η^n) is a finite convex combination of $(x^l, u^l, \frac{dx^l}{dt})_{l\in N}$; i.e., for some $l_1, l_2, .., l_n \in N$,

$$\begin{aligned}\exists \lambda_{l_1}, .., \lambda_{l_n} &\ge 0 \;\; s.t \;\; \lambda_{l_1} + .. + \lambda_{l_n} = 1 \text{ and}\\ (v^n, w^n, \eta^n) &= \sum_{1\le i\le n} \lambda_{l_i}(x^{l_i}, u^{l_i}, \frac{dx^{l_i}}{dt}).\end{aligned}$$

Because $gh(F(t, ., .)) = E(t)$ is convex, then

$$(v^n_t, w^n_t, \eta^n_t) \overset{a.e}{\in} E(t) \text{ for all } n \in \mathbb{N} \quad (36)$$

and without loss of generality we may suppose that for some $J_1 \subset J$ one has

$$(v^n_t, w^n_t, \eta^n_t)_{n\in\mathbb{N}} \to (\widehat{x}_t, \widehat{u}_t, \widehat{\eta}_t) \text{ for all } t \in J - J_1 \text{ and } measure(J_1) = 0.$$

Thus, $(\widehat{x}_t, \widehat{u}_t, \widehat{\eta}_t) \overset{a.e}{\in} E(t)$, since $E(t)$ is closed, and with (34) we obtain

$$(\widehat{x}_t, \widehat{u}_t, \; \frac{d}{dt}\widehat{x}_t) \overset{a.e}{\in} E(t). \quad (37)$$

Having that for all $n \in \mathbb{N}$, $(v^n, w^n) \in conv\left\{(x^l, u^l) : l \in N\right\}$, and because $f(t, ., .)$ and $g(.)$ are convex, then with (31) we get

$$val(P_E) \leq \varphi(v^n, w^n) \leq \sum_{1\leq i\leq n} \lambda_{l_i}(val(P_E) + \epsilon) = val(P_E) + \epsilon.$$

Passing to the limit for $n \to \infty$ leads to

$$val(P_E) \leq \varphi(\widehat{x}, \widehat{u}) \leq \underline{\lim}\,(\varphi(v^n, w^n)) \leq val(P_E) + \epsilon$$

and with (33)-(37) the proof is complete, since ϵ is selected arbitrarily. ∎

6.2. $P_{\mathcal{F}}$-format, duality and system of optimality

The optimal control of the inclusion is handled by the alternative mold of optimization, the so-called $P_{\mathcal{F}}$-format

$$(P_{\mathcal{F}}):\ inf\left\{\varphi\left(z\right)\ :\ 0 \in \mathcal{F}(z)\right\}, \tag{38}$$

which is a unified approach for a large field of optimization problems; see Mokhtar-Kharroubi (1987, 2017).

Let the data $\mathcal{F}$ and φ be proper. Then,

$$\sup_{p\in Y^*}\ \left(\varphi\left(z\right) + s_{\mathcal{F}}\left(z,p\right)\right) = \varphi\left(z\right)\ \text{ if } 0 \in \mathcal{F}(z) \text{ and} + \infty \text{ otherwise.}$$

In this way, the Lagrangian

$$\mathcal{L} : Z \times Y^* \to \overline{\mathbb{R}} : \mathcal{L}\left(z,p\right) := \varphi\left(z\right) + s_{\mathcal{F}}\left(z,p\right), \tag{39}$$

allows for rewriting the primal problem $(P_{\mathcal{F}})$ with a dual one $(D_{\mathcal{F}})$ as

$$\begin{aligned}(P_{\mathcal{F}}) \quad &: \quad \inf_{z\in Z}\sup_{p\in Y^*}\mathcal{L}(z,p)\\ (D_{\mathcal{F}}) \quad &: \quad \sup_{p\in Y^*}\inf_{z\in Z}\mathcal{L}\left(z,p\right).\end{aligned}$$

Then, *weak duality* $val\left(D_{\mathcal{F}}\right) \leq val(P_{\mathcal{F}})$ always holds. But *strong duality* occurs if

$$-\infty < val\left(P_{\mathcal{F}}\right) = val\left(D_{\mathcal{F}}\right) \text{ and } sol(D_{\mathcal{F}}) \neq \varnothing;$$

or,

$$\inf_{z\in Z}\sup_{p\in Y^*}\mathcal{L}(z,p) = \max_{p\in Y^*}\inf_{z\in Z}\ \mathcal{L}\left(z,p\right). \tag{40}$$

The main results to be used later are summarized in

Theorem 5 *(Mokhtar-Kharroubi, 1987, 2017). Assume that the following assumptions are fulfilled.*

(I). $\widehat{\varphi} := val(P_{\mathcal{F}})$ is finite.

(II). φ is convex, closed and u.s.c at some $\widetilde{z} \in int(dom(\mathcal{F}))$.

(III). $gh(\mathcal{F})$ is convex, closed and

$$0 \in int(rg(\mathcal{F})). \tag{41}$$

Then, strong duality (40) holds and for each optimal solution $\widehat{z}$ there exists a dual solution $\widehat{p}$ and $(\widehat{z}, \widehat{p})$ is a saddle point for the Lagrangian (39) on $Z \times Y^*$. In addition, the system of optimality is

$$\begin{cases} 0 \in \partial\varphi(\widehat{z}) + \partial_z s_{\mathcal{F}}(\widehat{z}, \widehat{p}), \\ -\widehat{p} \in N_{\mathcal{F}(\widehat{z})}(\widehat{z}) \quad \text{and } s_{\mathcal{F}}(\widehat{z}, \widehat{p}) = 0. \end{cases} \tag{42}$$

Proof Let us sketch the proof for reader's convenience.
Define in $\mathbb{R} \times Y$ the subsets

$$A_0 := \{(\alpha, w) \ : \ w = 0 \ , \ \alpha < 0\},$$
$$A := \{(\alpha, w) \ : \exists z \in Z, \ \alpha + \widehat{\varphi} \geq \varphi(z) \ \text{ and } w \in \mathcal{F}(z)\}.$$

Clearly, A_0 is convex,$A \cap A_0 = \varnothing$, and it is easy to show that A is convex, since $\mathcal{F}$ and φ are convex, too. Because φ is continuous at some $\widetilde{z}$, then for $\lambda < \varphi(\widetilde{z})$, there exists $\rho > 0$ such that $\varphi(z) < \lambda$ for all z in the ball $B_Z(\widetilde{z}, \rho) := \widetilde{z} + \rho B_Z$.

By Robinson-Ursescu's Theorem (Robinson, 1976; Ursescu, 1975) $\mathcal{F}$ is $l.s.c$ on $int(dom(\ \mathcal{F}))$, and then

$$int(\mathcal{F}(B_Z(\widetilde{z}; \rho))) \neq \varnothing.$$

Thus, for $\varepsilon > 0$ and β such that $\lambda < \beta + \ \widehat{\varphi} - \varepsilon$, there holds

$$\varnothing \neq \,]\beta - \varepsilon, \beta + \varepsilon[\, \times int(\mathcal{F}(B_Z(\widetilde{z}; \rho))) \subset A.$$

Hence, A and A_0 can be separated by a nondegenerated hyperplane, or, there exists $(\lambda, p) \in \mathbb{R}_+ \times Y^*$ s.t $(\lambda, p) \neq 0$ and

$$\lambda\varphi(z) + s_{\mathcal{F}}(z, p) \geq \lambda\widehat{\varphi} \text{ for all } z \in Z.$$

We claim that $\lambda > 0$. For otherwise, having $s_{\mathcal{F}}(z, p) \geq 0$ for all $z \in Z$, then $p = 0$ by condition (41). In this way, with $\widehat{p} = \lambda^{-1}p$, we get

$$\varphi(z) + s_{\mathcal{F}}(z, \widehat{p}) \geq \widehat{\varphi} \text{ for all } z \in Z$$

and strong duality holds; or,

$$\inf_{z \in Z} \sup_{p \in Y^*} \mathcal{L}(z, p) = \max_{p \in Y^*} \inf_{z \in Z} \ \mathcal{L}(z, p) = \inf_{z \in Z} \ \mathcal{L}(z, \widehat{p}).$$

Having $\widehat{p} \in sol\ (D_{\mathcal{F}})$, the system of optimality (42) follows from the saddle point relations of the Lagrangian and the condition *(II)* which amounts to

$$int(dom(\mathcal{F})) \cap dom(\varphi) \neq \varnothing. \tag{43}$$

■

PROPOSITION 3 *Under the same conditions, the Fenchel dual is given by*

$$\max_{p\in \mathbf{Y}^*} \max_{z^* \in \mathbf{Z}^*} \left[s_{gh(\mathcal{F})}(-z^*, p) - \varphi^*(z^*)\right]$$

and when $\mathcal{F}$ is a convex process, the Fenchel dual of a $P_{\mathcal{F}}$–format is a $P_{\mathcal{F}^*}$-format

$$(P_{\mathcal{F}^*}) : \max \left\{-\varphi^*(z^*) : 0 \in \mathcal{F}^*(p) + z^* \right\}. \tag{44}$$

PROOF Having that $s_{\mathcal{F}}(.,p)$ is convex and φ is convex and continuous at some $\widetilde{z} \in dom(s_{\mathcal{F}}(.,p))$, we get that the Fenchel's Theorem (see Bot and Csetnek, 2012) applies with φ and $-s_{\mathcal{F}}(.,p)$, and leads to

$$\inf_{z\in \mathbf{Z}} \left[\varphi(z) + s_{\mathcal{F}}(z,p)\right] = \max_{z^*\in \mathbf{Z}^*} \left[s_{gh(\mathcal{F})}(-z^*, p) - \varphi^*(z^*)\right],$$

and so, the dual problem $(D_{\mathcal{F}})$ is reduced to

$$\sup_{p\in \mathbf{Y}^*} \max_{z^* \in \mathbf{Z}^*} \left[s_{gh(\mathcal{F})}(-z^*, p) - \varphi^*(z^*)\right].$$

But by strong duality the p-supremum is a maximum, and the Fenchel dual is

$$\max_{p\in \mathbf{Y}^*} \max_{z^* \in \mathbf{Z}^*} \left[s_{gh(\mathcal{F})}(-z^*, p) - \varphi^*(z^*)\right]. \tag{45}$$

If, in addition, $\mathcal{F}$ is a convex process, and then $gh(\mathcal{F})$ is a cone, we check easily that

$$0 \in \mathcal{F}^*(p) + z^* \text{ iff } s_{gh(\mathcal{F})}(-z^*, p) = 0.$$

This ends the proof. ■

A useful selection result is provided in the following proposition.

PROPOSITION 4 *Let $r \geq 1$ and $\Gamma : L^1_X \times L^r_U \rightrightarrows L^1_X$ be given by*

$$\Gamma(x,u) := \left\{v : (x_t, u_t, v_t) \overset{a.e}{\in} E(t)\right\}. \tag{46}$$

Then, the following equivalence holds true:

$$(x^*, u^*, v^*) \in N_{gh(\Gamma)}(\widehat{x}, \widehat{u}, \widehat{v}) \ iff \ (x^*_t, u^*_t, v^*_t) \overset{a.e}{\in} N_{E(t)}(\widehat{x}_t, \widehat{u}_t, \widehat{v}_t). \tag{47}$$

PROOF Let $(x, u, v) \in L^1_X \times L^r_U \times L^1_X$. Define

$$\varpi(x_t, u_t, v_t) := \langle x^*_t, x_t\rangle + \langle u^*_t, u_t\rangle + \langle v^*_t, v_t\rangle.$$

If $(x^*_t, u^*_t, v^*_t) \overset{a.e}{\in} N_{E(t)}(\widehat{x}_t, \widehat{u}_t, \widehat{v}_t)$, then

$$\varpi(\widehat{x}_t, \widehat{u}_t, \widehat{v}_t) \overset{a.e}{=} \sigma_{E(t)}(x^*_t, u^*_t, v^*_t) \tag{48}$$

where $\sigma_{E(t)}$ is the upper support function of $E(t)$, and since

$$\varpi(x_t, u_t, v_t) \overset{a.e}{\leq} \varpi(\widehat{x}_t, \widehat{u}_t, \widehat{v}_t) \text{ for all } (x, u, v) \in gh(\Gamma),$$

by summing up on J we get

$$(x^*, u^*, v^*) \in N_{gh(\Gamma)}(\widehat{x}, \widehat{u}, \widehat{v}). \tag{49}$$

Conversely, if (49) holds, then

$$\int \varpi(\widehat{x}_t, \widehat{u}_t, \widehat{v}_t) = \sup_{(x,u,v)\in gh(\Gamma)} \left(\int \varpi(x_t, u_t, v_t)\right)$$

and by the usual consequence of the Measurable Selection Theorem,

$$\sup_{(x,u,v)\in gh(\Gamma)} \left(\int \varpi(x_t, u_t, v_t)\right) = \int \left(\sup_{(a,b,c)\in E(t)} (\varpi(a, b, c))\right),$$

i.e.

$$\int \varpi(\widehat{x}_t, \widehat{u}_t, \widehat{v}_t) = \int \sigma_{E(t)}(x^*_t, u^*_t, v^*_t). \tag{50}$$

Thus, if (48) does not hold, then there will exist $J_1 \subsetneq J : measure\,(J_1) > 0$ and

$$\begin{array}{rcl} \varpi(\widehat{x}_t, \widehat{u}_t, \widehat{v}_t) & = & \sigma_{E(t)}(x^*_t, u^*_t, v^*_t) \text{ if } t \in J - J_1, \\ \varpi(\widehat{x}_t, \widehat{u}_t, \widehat{v}_t) & < & \sigma_{E(t)}(x^*_t, u^*_t, v^*_t) \text{ if } t \in J_1. \end{array}$$

By summing up on J, the contradiction with (50) ends the proof. ■

Next we provide more explicit conditions of optimality.

THEOREM 6 *Assume for the optimal control problem (27) that:*

(I). Conditions (6)-(7) hold for the multimap E.

(II). The objective satisfies the conditions (28)-(29).

Then, there exist an optimal pair $(\widehat{x}, \widehat{u})$ *and an adjoint state* $\widehat{p} \in A^{\infty}_{X^*}$ *satisfying*

$$\left\{ \begin{array}{c} (\frac{d}{dt}\widehat{p}_t \ , 0, -\widehat{p}_t) \overset{a.e}{\in} N_{E(t)}(\widehat{x}_t, \widehat{u}_t, \frac{d}{dt}\widehat{x}_t) + \left\{\partial_{(a,b)} f(t, \widehat{x}_t, \widehat{u}_t) \times \{0\}\right\} \\ and \\ (-\widehat{p}_\tau \ , \widehat{p}_T) \in \partial g(\widehat{x}_\tau, \widehat{x}_T) + \{N_\Omega \, (\widehat{x}_\tau) \times \{0\}\} \end{array} \right. \tag{51}$$

Proof Let $\Gamma : L^1_X \times L^r_U \rightrightarrows L^1_X$ be given by

$$\Gamma(x, u) := \left\{ v : (x_t, u_t, v_t) \overset{a.e}{\in} E(t) \right\}.$$

The Lagrangien $\mathcal{L} : A^1_X \ \times L^r_U \times L^{\infty}_{X^*} \times X^* \ \to \overline{\mathbb{R}}$ is then

$$\mathcal{L}(x, u, p, \alpha) = \varphi(x, u) + s_\Gamma(x, u, p) - \langle \tfrac{dx}{dt}, p \rangle + s_\Omega(\alpha) - \langle x_\tau, \alpha \rangle$$

where

$$s_\Gamma(x, u, p) = \inf \left\{ \textstyle\int \langle y_t, p_t \rangle : y \in \Gamma(x, u) \right\}. \tag{52}$$

The condition $0 \in int\,(rg(\mathcal{F}))$ holds true (by Theorem 3) then, a dual solution $(\widehat{p}, \widehat{\alpha})$ exists (by Theorem 5). Because a primal solusion $(\widehat{x}, \widehat{u})$ exists (by Theorem 4), then

$$((\widehat{x}, \widehat{u}), (\widehat{p}, \widehat{\alpha})) \text{ is a saddle point of the Lagrangian;}$$

or, for all $(x, u) \in A^1_X \times L^r_U$ and all $(p, \alpha) \in L^{\infty}_{X^*} \times X^*$ there holds

$$\mathcal{L}(\widehat{x}, \widehat{u}, p, \alpha) \leq \mathcal{L}(\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\alpha}) \leq \mathcal{L}(x, u, \widehat{p}, \widehat{\alpha}). \tag{53}$$

The first inequality of (53) ensures that for all $(p, \alpha) \in L^{\infty}_{X^*} \times X^*$

$$\begin{array}{l} s_\Gamma(\widehat{x}, \widehat{u}, p) - \langle \frac{d\widehat{x}}{dt}, p \rangle + s_\Omega(\alpha) - \langle \widehat{x}_\tau, \alpha \rangle \leq \\ s_\Gamma(\widehat{x}, \widehat{u}, \widehat{p}) - \langle \frac{d\widehat{x}}{dt}, \widehat{p} \rangle + s_\Omega(\widehat{\alpha}) - \langle \widehat{x}_\tau, \widehat{\alpha} \rangle. \end{array}$$

Hence,

$$-\widehat{p} \in N_{\Gamma(\widehat{x}, \widehat{u})}(\frac{d\widehat{x}}{dt}) \text{ and } -\widehat{\alpha} \in N_\Omega \, (\widehat{x}_\tau) \, . \tag{54}$$

From the second inequality of (53), for all $(x, u) \in A^1_X \times L^r_U$ and $\lambda > 0$,

$$\lambda^{-1} \left(\mathcal{L}(\widehat{x} + \lambda x, \widehat{u} + \lambda u \ , \widehat{p}, \widehat{\alpha}) - \mathcal{L}(\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\alpha}) \right) \geq 0. \tag{55}$$

Let $z := (x, u)$ and $\widehat{z} := (\widehat{x}, \widehat{u})$. Upon writing $\omega(z)$ in place of $s_\Gamma(z, p)$ and passing to the limit in (55) for $\lambda \to 0_+$, we obtain

$$\left[\begin{array}{c} \omega^o(\widehat{z}; z) + g^o((\widehat{x}_\tau, \widehat{x}_T) \, ; (x_\tau, x_T)) + \\ + \psi^o(\widehat{z}; z) - \langle \frac{dx}{dt}, \widehat{p} \rangle - \langle \widehat{\alpha}, x_\tau \rangle \end{array} \right] \geq 0 \tag{56}$$

where $\omega^o(\widehat{z};.)$, $g^o((\widehat{x}_\tau, \widehat{x}_T)\ ;.)$ and $\psi^o(\widehat{z}\ ;.)$ are the directional derivatives of the convex functions $\omega(.)$, $g(.,.)$ and $\psi(.)$.

Because these derivatives are the upper support functions of the subdifferentials, which are convex and weak* compact, then (56) amounts to

$$\inf_{\substack{x\in A^1_X \\ u\in L^r_U}} \max_{\substack{(\eta,\xi)\in\partial\psi(\widehat{x},\widehat{u}) \\ (\widehat{\eta},\widehat{\xi})\in\partial\omega(\widehat{x},\widehat{u}) \\ (\xi_\tau,\xi_T)\in\partial g(\widehat{x}_\tau,\widehat{x}_T)}} \begin{bmatrix} \langle \eta+\widehat{\eta}, x\rangle + \langle \xi+\widehat{\xi}, u\rangle - \langle \frac{d}{dt}x, \widehat{p}\rangle + \\ + \langle \xi_\tau - \widehat{\alpha}, x_\tau\rangle + \langle \xi_T, x_T\rangle\rangle \end{bmatrix} \geq 0. \quad (57)$$

In this way, by the so-called lop-sided minmax theorem of Aubin (Aubin, 1972, Theorem 7, p. 319)

$$\exists (w_\tau, w_T) \in \partial g(\widehat{x}_\tau, \widehat{x}_T) \subset X^* \times X^*,$$
$$\exists (\widetilde{x}^*, \widetilde{u}^*) \in \partial\psi(\widehat{x}, \widehat{u}) \subset L^\infty_{X^*} \times L^{r^*}_{U^*}\ , (\tfrac{1}{r} + \tfrac{1}{r^*} = 1)$$

and

$$\exists (\widetilde{y}^*, \widetilde{v}^*) \in \partial\omega(\widehat{x}, \widehat{u}) = \partial_{(x,u)} s_\Gamma(\widehat{x}, \widehat{u}, \widehat{p}) \subset L^\infty_{X^*} \times L^{r^*}_{U^*},$$

such that for all $(x,u) \in A^1_X \times\ L^r_U$ there holds

$$\langle \widetilde{x}^* + \widetilde{y}^*, x\rangle + \langle \widetilde{u}^* + \widetilde{v}^*, u\rangle - \langle \frac{d}{dt}x, \widehat{p}\rangle + \langle w_\tau - \widehat{\alpha}, x_\tau\rangle\rangle + \langle w_T, x_T\rangle \geq 0.$$

Clearly, the inequality is actually an equality for all $(x,u) \in A^1_X \times\ L^r_U$

$$\langle \widetilde{x}^* + \widetilde{y}^*, x\rangle + \langle \widetilde{u}^* + \widetilde{v}^*, u\rangle - \langle \frac{d}{dt}x, \widehat{p}\rangle + \langle w_\tau - \widehat{\alpha}, x_\tau\rangle\rangle + \langle w_T, x_T\rangle = 0. \quad (58)$$

On the other hand, it is well known (see Aubin and Clarke, 1979, Theorem 2) that for every

$$(\widetilde{x}^*, \widetilde{u}^*) \in \partial\psi(\widehat{x}, \widehat{u}) \text{ and } (\widetilde{y}^*, \widetilde{v}^*) \in \partial\omega(\widehat{x}, \widehat{u}),$$

there exist (x^*, u^*) and (y^*, v^*) in $L^\infty_{X^*} \times L^{r^*}_{U^*}$ selectors respectively, of $\partial\psi(\widehat{x}, \widehat{u})$ and $\partial\omega(\widehat{x}, \widehat{u})$, satisfying for all $(x,u) \in L^1_X \times L^r_U$ the equalities

$$\langle \widetilde{x}^*, x\rangle + \langle \widetilde{u}^*, u\rangle = \int (\langle x^*_t, x_t\rangle + \langle u^*_t, u_t\rangle),$$
$$\langle y^*, x\rangle + \langle v^*, u\rangle := \int (\langle y^*_t, x_t\rangle + \langle v^*_t, u_t\rangle).$$

By the routine abuse of notation, let the subgradients be denoted by their corresponding selectors. Then, (58) amounts to

$$\langle x^* + y^*, x\rangle + \langle u^* + v^*, u\rangle - \langle \frac{d}{dt}x, \widehat{p}\rangle + \langle w_\tau - \widehat{\alpha}, x_\tau\rangle\rangle + \langle w_T, x_T\rangle = 0. \quad (59)$$

Thus, $(x,u) := (0,u)$, with u selected arbitrarily in L^r_U, leads to

$$u^*_t + v^*_t \overset{a.e}{=} 0. \quad (60)$$

Define $\theta_t := \int_{[t,T]} (x_s^* + y_s^*)$; then, $\theta \in A_{X^*}^{\infty}$ and for all $x \in A_X^1$

$$\langle x^* + y^*, x\rangle = \langle \theta, \frac{dx}{dt}\rangle + \langle \theta_\tau, x_\tau\rangle.$$

By the identity $x_T = x_\tau + \int \frac{d}{dt} x_t$ we get

$$\langle \theta - \widehat{p} + w_T, \frac{d}{dt}x\rangle + \langle w_\tau - \widehat{\alpha} + \theta_\tau + w_T, x_\tau\rangle = 0.$$

Hence,

$$\widehat{p} = \ \theta + w_T \in A_{X^*}^{\infty} \text{ and } (\widehat{p}_\tau,\ \widehat{p}_T) = (\widehat{\alpha} - w_\tau\ , w_T).$$

In this way,

$$\frac{d}{dt}\widehat{p}_t + x_t^* + y_t^* \overset{a.e}{=} 0 \tag{61}$$

and

$$(\widehat{p}_\tau,\ \widehat{p}_T) \in \partial g(\widehat{x}_\tau, \widehat{x}_T) + \{N_\Omega\,(\widehat{x}_\tau) \times \{0\}\}\,. \tag{62}$$

But, in view of (60), one disposes of

$$-\widehat{p} \in N_{\Gamma(\widehat{x},\widehat{u})}(\frac{d\widehat{x}}{dt}) \text{ and } (y^*, -v^*) \in \partial s_\Gamma(\widehat{x}, \widehat{u}, \widehat{p}), \tag{63}$$

which amounts (by Proposition 1 (i)) to

$$(y^*,\ -v^*, -\ \widehat{p}) \in N_{gh(\Gamma)}(\widehat{x}, \widehat{u}, \frac{d\widehat{x}}{dt})$$

and, then, is equivalent (by Proposition 4) to

$$(y_t^*,\ -v_t^*, -\ \widehat{p}_t) \overset{a.e}{\in} N_{E(t)}(\widehat{x}_t,\ \widehat{u}_t,\ \frac{d}{dt}\widehat{x}_t). \tag{64}$$

Finally, since $(x^*, -v^*) = (x^*, u^*) \in \partial\psi(\widehat{x}, \widehat{u})$ defines a selector

$$(x_t^*, -v_t^*) \overset{a.e}{\in} \partial f_{(a,b)}(t, \widehat{x}_t, \widehat{u}_t)$$

then, with (61)-(62), the conditions (51) follow. The proof is complete. ∎

6.3. Convex process of evolution and the respective Fenchel dual

Assume that the state x_τ is fixed. Then, define

$$\mathbf{X} := A^1_X \times L^r_U, \mathbf{U} := L^r_U \ , \ \mathbf{Y} := L^1_X$$

and

$$\mathcal{F} : \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y} : \mathcal{F}(x,u) := \Gamma(x,u) - \left\{ \frac{dx}{dt} \right\}$$

where

$$\Gamma : L^1_X \times L^r_U \rightrightarrows L^1_X : \Gamma(x,u) := \left\{ v : (x_t, u_t, v_t) \overset{a.e}{\in} E(t) \right\}.$$

Proposition 5 *Assume that E is cone valued, u.s.c satisfying (6); or,*

$$int(E(t)) \neq \varnothing \text{ for all } t \in J.$$

Let $(E(t))^+$ be the polar cone of $E(t)$. Then, the conjugate(s)

(i). Γ^* *is a strict convex process from* $L^\infty_{X^*}$ *to* $L^\infty_{X^*} \times L^{r^*}_{U^*}$ $(\frac{1}{r} + \frac{1}{r^*} = 1)$:

$$\Gamma^*(p) = \left\{ (x^*, u^*) : (p_t \ , -x^*_t, -u^*_t) \overset{a.e}{\in} \ (E(t)^+ \right\}.$$

(ii). $\mathcal{F}^*$ *is a convex process from* $A^\infty_{X^*}$ *to* $L^\infty_{X^*} \times L^{r^*}_{U^*}$*, given by*

$$dom(\mathcal{F}^* \ ) = \{ p \in A^\infty_{X^*} : p_T = 0 \} \text{ and}$$

$$\mathcal{F}^*(p) = \left\{ (x^*, u^*) : (p_t, \tfrac{d}{dt} p_t - x^*_t, -u^*_t) \overset{a.e}{\in} (E(t))^+ \right\}.$$

Proof *(i).* Let $(x^*, u^*) \in L^\infty_{X^*} \times L^{r^*}_{U^*}$ and $p \in L^\infty_{X^*}$ be such that

$$(p_t \ , -x^*_t, -u^*_t) \overset{a.e}{\in} (E(t))^+.$$

Having $(x_t, u_t, v_t) \overset{a.e}{\in} E(t)$ for all $(x,u,v) \in gh(\Gamma)$ we obtain that

$$-\langle x^*_t, x_t \rangle - \langle u^*_t, u_t \rangle + \langle p_t, v_t \rangle \geq 0 \text{ for all } t \in J \tag{65}$$

and by summing up on J in (65), we get

$$-\langle, x^* x \rangle - \langle u^*, u \rangle + \langle p, v \rangle \geq 0;$$

hence,

$$(x^*, u^*) \in \Gamma^*(p).$$

Conversely, let $(p, -x^*, -u^*) \in gh(\Gamma^*)$; then, for every

$$(x,u,v) \in L^1_X \times L^s_X \times L^1_X \text{ s.t } (x_t,u_t,v_t) \overset{a.e}{\in} E(t)$$

one has

$$\int (\langle p_t, v_t \rangle) - \langle x_t^*, x_t \rangle - \langle u_t^*, u_t \rangle) \geq 0. \tag{66}$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be defined as

$$\eta_t = \langle p_t, v_t \rangle - \langle x_t^*, x_t \rangle - \langle u_t^*, u_t \rangle.$$

Let $t \in]t_0, T[$ be a Lebesgue point of η such that $int(E(t)) \neq \varnothing$. Such a point exists, since $int(E(t)) \neq \varnothing$ *for all* $t \in J$ and the set of Lebesgue points is of full measure.

Let $(a,b,c) \in int(E(t))$ and $\rho > 0$ be such that $(a,b,c) + \rho\widetilde{B} \subset E(t)$, where $\widetilde{B}$ is the open unit ball of $X \times U \times X$. Then, $]t-\eta, t+\eta[\subset J_0$ for some $\eta > 0$ and, since E is $u.s.c.$, the subset

$$J_0 := E^{-1}((a,b,c) + \rho B)$$

is open.

For $0 < h \leq \eta$ define $(\widehat{x}, \widehat{u}, \widehat{v}) \in L^1_X \times L^r_Z \times L^1_X$ as

$$(\widehat{x}_s, \widehat{u}_s, \widehat{v}_s) := (-a, -b, c) \text{ if } s \in]t-h, t+h[\text{ and } 0 \text{ otherwise.}$$

Clearly, $(\widehat{x}, \widehat{u}, \widehat{v}) \in gh\,(\Gamma)$, since $(-a,-b,c) + \rho\widetilde{B} \subset E(t)$. Thus,

$$\lim_{h \to 0} \frac{1}{2h} \int_{t-h}^{t+h} (\langle p_s, \widehat{v}_s \rangle - \langle x_s^*, \widehat{x}_s \rangle - \langle u_s^*, \widehat{u}_s \rangle) \geq 0.$$

That is,

$$\langle p_t, c \rangle - \langle x_t^*, a \rangle - \langle u_t^*, b \rangle \geq 0.$$

Because $E(t) = cl(int(E(t)))$ and (a,b,c) is selected arbitrarily, then

$$(p_t, -x_t^*, -u_t^*) \in (E(t))^+,$$

which ends the proof of *(i)*.

(ii). Observe that for $\Lambda : A^1_X \to L^1_X$, given by

$$dom(\Lambda) := \left\{ x \in A^1_X : x_\tau = 0 \right\} \text{ and } \Lambda x = \frac{dx}{dt},$$

and then, Λ^* is from $A^\infty_{X^*}$ to $L^\infty_{X^*}$ such that

$$dom(\Lambda^*) = \{p \in A^\infty_{X^*} : p_T = 0\} \text{ and } \Lambda^* p = -\frac{dp}{dt}.$$

Let $F_1 := \Gamma$ and $F_2 := (-\Lambda, 0)$. Then, $\mathcal{F} = F_1 + F_2$ and (by Proposition 5)

$$dom(F_1) - dom(F_2) = A^1_X \times L^r_X.$$

Thus, in view of Aubin and Ekeland (1984, Corollary 16, p. 141), we get

$$\mathcal{F}^* = F_1^* + F_2^* = \Gamma^* - (\Lambda^*, 0)$$

and then,

$$(-x^*, -u^*) \in \mathcal{F}^*(p) \text{ iff } (-x^*, -u^*) \in \Gamma^*(p) - (\Lambda^*, 0)(p),$$

or,

$$(-x^* + \Lambda^* p, -u^*) \in \Gamma^*(p).$$

In this way, $dom(\mathcal{F}^*\ ) = \{p \in A^\infty_{X^*} : p_T = 0\}$ and

$$(-x^*, -u^*) \in \mathcal{F}^*(p) \text{ iff } (p_t,\ -x_t^* + \frac{d}{dt}p_t,\ -u_t^*) \in (E(t))^+.$$

■

Proposition 6 *Assume that E is cone-valued, u.s.c satisfying (6); or,*

$$int(E(t)) \neq \varnothing \text{ for all } t \in J.$$

Then, the Fenchel dual of the optimal control problem is given by

$$\begin{cases} max\,(-\varphi^*(x^*, u^*)) \\ (x^*, u^*, p) \in L^\infty_{X^*} \times L^{r^*}_{U^*} \times A^\infty_{X^*} \ s.t \\ (p_t, -x_t^* + \frac{d}{dt}p_t, -u_t^*) \overset{a.e}{\in} (E(t))^+ \ and\ p_T = 0. \end{cases}$$

Proof Theorem 5 and Proposition 5 yield the result. ■

7. On a class of controlled integro-differential inclusions

The controlled integro-differential inclusions, given by

$$(x_t + \int_0^t k(s,t)x_s\ , u_t, \frac{d}{dt}x_t) \overset{a.e}{\in} \widehat{E}(t) \quad ; \quad x_0 \in \Omega, \tag{67}$$

can be handled by the theory presented here. Indeed, let the integral operator

$T : L^1_X \to A^1_X$ be s.t.,

$$\begin{array}{l}(Tx)_t := x_t + \int_0^t k(s,t)x_s \text{ where} \\ \qquad k : J \times J \to L^1_{\mathcal{L}(X)}.\end{array} \tag{68}$$

Then, T is bounded and T^* is from $A^\infty_{X^*}$ to $L^\infty_{X^*}$; s.t.,

$$\begin{array}{rcl}(T^*p)_t & = & p_t + \displaystyle\int_t^T k^*(t,s)p_s \quad \text{where} \\ k^* & : & J \times J \to L^\infty_{\mathcal{L}(X^*)}.\end{array}$$

Clearly, the inclusion (67) reduces to

$$((Tx)_t\,, u_t, v_t) \overset{a.e}{\in} E(t).$$

Under the same conditions (on the multimap E) one may define

$$\Gamma : L^1_X \times L^r_U \rightrightarrows L^1_X, \quad \text{as} \quad \Gamma(x,u) := \{v : ((Tx)_t, u_t, v_t) \in E(t)\}\,, \tag{69}$$

then, we get the reduction

$$0 \in \mathcal{F}(x,u) := \left\{\Gamma(x,u) - \frac{dx}{dt}\right\} \times \{\Omega - x_\tau\}\,.$$

In this way, the presented theory here applies and the main results hold true for the inclusions (67). Namely,

(i). The results of existence of solution and the well posedness, or, Theorems 3 and 4 and Proposition 5 remain valid.

(2i). The existence of the optimal pair, or, Theorem 6 holds true.

However, some facts must be reworked carefully. To avoid overloading the paper we omit the details. But, we point out the main fact that the adjoint T^* of the operator (68) will appear in Proposition 9 for the normal cone of $E(t)$ at $((Tx)_t, u_t, v_t)$ and in Theorem 10 for the selector of $\partial_x s_\Gamma(Tx,u)$.

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