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Wellposedness and long time behavior for a general class of Moore-Gibson-Thompson equations*

by

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Abstract: We consider the well-posedness and the long time behavior of third order in time linear evolution equations, general and abstract version of the Moore-Gibson-Thompson system. We find sufficient but strong conditions that guarantee the exponential decay of the system and present some illustrative examples. Then, by comparing the behavior of the resolvent of the Moore-Gibson-Thompson system with the one of the resolvent of the wave equation with a frictional interior damping, we furnish weaker conditions that guarantee exponential, polynomial or even logarithmic decay of the solution of the Moore-Gibson-Thompson system in a bounded domain of $\mathbb{R}^n, n \geq 1$.

Keywords: third order systems, wave equation, stabilization

1. Introduction

In this paper, we consider the well-posedness and the long time behavior of third order in time linear (abstract) evolution equations. These kind of equations has found much interest recently due to the large number of applications in nonlinear acoustics, where Fourier's law is replaced by the more realistic Maxwell–Cattaneo's law, which accounts for a finite speed of propagation of acoustic waves (Kaltenbacher, 2015). This leads to three derivatives in time, while classical models are second order in time.

Before going on, let us formulate the Hilbert setting and the basic assumptions. Let H and V be two Hilbert spaces such that V is continuously and densely embedded into H and let V' be the dual of V (with H as pivot space). We

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suppose to be given two sesquilinear and continuous forms a_0 and a_1 on V and such that a_1 is symmetric and coercive, namely

$$a_1(u,v) = \overline{a_1(v,u)}, \forall u, v \in V,$$

and

$$a_1(u,u) \gtrsim \|u\|_V^2, \forall u \in V.$$

Then we introduce the associated (bounded) operators $\mathcal{A}_0$ and $\mathcal{A}_1$ from V into V' defined by

$$\langle \mathcal{A}_i u, u' \rangle_{V-V'} = a_i(u,u'), \forall u, u' \in V, i = 0, 1,$$

where here and below $\langle \cdot, \cdot \rangle$ means the duality pairing between V and V'. Note that $\mathcal{A}_1$ is selfadjoint due to the assumptions on a_1. Note that $\mathcal{A}_1^{-1}\mathcal{A}_0$ is bounded from V into itself, but we suppose that it can be extended into a bounded operator from H into itself. Finally, we suppose also given a bounded operator B from H into itself.

In this setting we consider the third order in time abstract evolution equation set in the Hilbert space H:

$$\begin{cases} u_{ttt} + Bu_{tt} + \mathcal{A}_0 u + \mathcal{A}_1 u_t = 0, \\ u(0) = u_0, u_t(0) = u_1, u_{tt}(0) = u_2, \end{cases} \tag{1}$$

where u_0, u_1, and u_2 are initial data in the appropriate Hilbert spaces, specified later on.

The case of $B = \alpha I$ and $\mathcal{A}_0 = \beta \mathcal{A}_1$, with a real number α and a positive real number β, was analyzed in Kaltenbacher, Lasiecka and Marchand (2011) and Marchand, McDevitt and Triggiani (2012) (see also Kaltenbacher, Lasiecka and Pospieszalska, 2012), where existence is proved using semi-group theory and an exponential stability result is proved under the assumption: $\alpha - \beta > 0$. Note that under this assumption, the optimal exponential decay rate of the solutions is proved in Pellicer and Solà-Morales (2019) by showing that the associated operator is normal in an appropriate inner product. Let us also mention that under the assumption that $\alpha = \beta$, problem (1) is conservative, see Kaltenbacher, Lasiecka and Marchand (2011, Theorem 1.3), while in Conejero, Lizama and Rodenas (2015) it is demonstrated that if $\alpha - \beta < 0$, a chaotic behaviour of the system may occur, as shown for a particular example (namely system (2) in $\mathbb{R}$).

In this paper, our first goal is to show that problem (1) is well-posed with the sole assumptions stated above. This allows for treating concrete examples, where the operators $B, \mathcal{A}_0$, and $\mathcal{A}_1$ have space variable coefficients (see (18) below), in particular, we can consider the standard Moore-Gibson-Thompson system

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta \Delta u - \Delta u_t = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0, \cdot) = u_0, u_t(0, \cdot) = u_1, u_{tt}(0, \cdot) = u_2, & \text{in } \Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain of $\mathbb{R}^n$, β is a positive constant and $\alpha \in L^\infty(\Omega)$, a case mentioned in Liu and Triggiani (2014, p. 306), for which an existence result is proved in Kaltenbacher and Lasiecka (2012). Our approach does not allow for treating the non-autonomous situation when α may depend on the time variable, for such a situation we refer to Kaltenbacher and Lasiecka (2012), Kaltenbacher, Lasiecka and Pospieszalska (2012) and Kaltenbacher and Nikolić (2019) for existence and exponential decay. We further find sufficient conditions, similar to the ones from Kaltenbacher and Lasiecka (2012), Kaltenbacher, Lasiecka and Marchand (2011), Kaltenbacher, Lasiecka and Pospieszalska (2012), Kaltenbacher and Nikolić (2019), and Marchand, McDevitt and Triggiani (2012) that guarantee the exponential decay of the energy. We then illustrate our theory to the system (18), in particular, for problem (2) such a condition reduces to

$$\alpha - \beta \geq \kappa > 0, \text{ a. e. in } \Omega. \tag{3}$$

Since this sufficient condition is quite strong, we concentrate on the degenerate case

$$\alpha - \beta \geq 0, \text{ a. e. in } \Omega,$$

for which, as we will show, exponential, polynomial or even logarithmic decays are available. This is performed by comparing the resolvent of our operator with the one of the wave equation with frictional interior damping

$$\begin{cases} u_{tt} - \Delta u + (\alpha - \beta) u_t = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = u_0 \text{ and } u_t(\cdot, 0) = u_1 & \text{in } \Omega. \end{cases}$$

Indeed, we show that the same behavior of the resolvent of (2) is the square of the behavior of the resolvent of this damped wave equation. This allows for the weakening of (3) into

$$\alpha - \beta \geq \kappa > 0 \qquad \text{a. e. in } \omega_0, \tag{4}$$

where ω_0 is a non empty open subset of Ω. Hence, under some geometrical condition on ω_0 (but weaker than (3)), system (2) is proved to be exponential, polynomial or even logarithmic decaying.

We finally mention some recent papers, concerning the system with a memory damping term, where the exponential decay of the energy is proved provided that the kernel is exponentially decaying, see Lasiecka and Wang (2015, 2016) and Alves et al. (2018); long time behavior of third order nonlinear systems was analysed by Caixeta, Lasiecka and Cavalcanti (2016a,b); and problems set in the whole space were considered in Pellicer and Said-Houari (2019).

The paper is organized as follows: The well-posedness of our problem is proved in Section 2 by using semi-group theory and an appropriate change of unknowns. An illustrative example is also presented. In Section 3, we find sufficient conditions that guarantee the exponential decay of the energy of our abstract system and again illustrate such a result. The link between system (2) and the wave equation with a frictional interior damping is extricated in Section 4, where we show that the decay rate of the wave equation lead to a similar decay for our system (2).

Let us finish this introduction with some notation used in the paper. The inner product (respectively norm) of H will be denoted by $(\cdot,\cdot)$ (respectively $\|\cdot\|$). The inner product (respectively norm) of V will be denoted by $(\cdot,\cdot)_V$ (respectively $\|\cdot\|_V$). The usual norm and semi-norm of $H^s(\Omega)$ ($s \geq 0$) are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, respectively. For $s = 0$ we drop the index s. By $a \lesssim b$, we mean that there exists a constant $C > 0$ independent of a, b, such that $a \leq Cb$.

2. An existence result

In this section we first prove the well-posedness of the system (1), then we give one illustrative example.

2.1. General setting

In order to show that system (1) is well-posed, we introduce the following operator $\mathcal{A}$ on the Hilbert space $\mathcal{H} = V \times V \times H$, endowed with the inner product

$$((u,v,w)^\top, (u',v',w'))_{\mathcal{H}} = a_1(u,u') + a_1(v,v') + (w,w'),$$
$$\forall (u,v,w)^\top, (u',v',w')^\top \in \mathcal{H}.$$

On this space we define the unbounded operator $\mathcal{A}$ by

$$D(\mathcal{A}) = \{(u,v,w)^\top \in V^3 \,|\, \mathcal{A}_0 u + \mathcal{A}_1 v \in H\}, \tag{5}$$

and

$$\mathcal{A}(u,v,w)^\top = (v, w, -(\mathcal{A}_0 u + \mathcal{A}_1 v + Bw)), \quad \forall (u,v,w)^\top \in D(\mathcal{A}). \tag{6}$$

With that definition we see that formally u is a solution of (1) if and only if $U = (u, u_t, u_{tt})$ is a solution of the first order evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{7}$$

where $U_0 = (u_0, u_1, u_2)$.

This formal equivalence is correct as soon as strong solutions are concerned. Namely a strong solution of (7) yields a solution to (1), more precisely we have the following equivalence. Since its proof is immediate we leave it to the reader.

LEMMA 1 *$U = (u, v, w)^\top \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), D(\mathcal{A})$ is a solution of (7) if and only if $u \in C^2([0,\infty), V) \cap C^3([0,\infty), H)$ is a solution of (1), with $v(t) = u_t(t)$ and $w(t) = u_{tt}(t)$ $\mathcal{A}_0 u(t) + \mathcal{A}_1 u_t(t) \in H$ for all $t \in [0,\infty)$.*

Now we are left to proving the existence of a solution to (7), which is obtained using semigroup theory after a change of unknowns. For that purpose, according to the standard decomposition of the solution into $z = u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u$ and u, see Kaltenbacher and Lasiecka (2012, §3.2), Marchand, McDevitt and Triggiani (2012, §2) and Kaltenbacher, Lasiecka and Pospieszalska (2012, §2.1), where z satisfies a wave equation (see Remark 1 below) and u an abstract ODE with exponential decay, we introduce the bounded operator $\mathcal{M}$ from $\mathcal{H}$ into itself, defined by

$$\mathcal{M}(u,v,w)^\top = \begin{pmatrix} I & 0 & 0 \\ \mathcal{A}_1^{-1}\mathcal{A}_0 & I & 0 \\ 0 & \mathcal{A}_1^{-1}\mathcal{A}_0 & I \end{pmatrix} (u,v,w)^\top, \forall (u,v,w)^\top \in \mathcal{H}.$$

This operator is even an isomorphism, since its inverse is the bounded operator given by

$$\mathcal{M}^{-1}(u,v,w)^\top = \begin{pmatrix} I & 0 & 0 \\ -\mathcal{A}_1^{-1}\mathcal{A}_0 & I & 0 \\ (\mathcal{A}_1^{-1}\mathcal{A}_0)^2 & -\mathcal{A}_1^{-1}\mathcal{A}_0 & I \end{pmatrix} (u,v,w)^\top, \forall (u,v,w)^\top \in \mathcal{H}.$$

Now we prove the following Lemma (compare with Section 3 from Kaltenbacher, Lasiecka and Marchand, 2011).

LEMMA 2 *$U = (u, v, w)^\top \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), D(\mathcal{A})$ is a strong solution of (7) if and only if $\tilde{U} = (u, z, y)^\top = \mathcal{M}U \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), D(\tilde{\mathcal{A}}))$ is a strong solution of*

$$\begin{cases} \tilde{U}_t = \tilde{\mathcal{A}}\tilde{U}, \\ \tilde{U}(0) = \tilde{U}_0, \end{cases} \tag{8}$$

where $\tilde{U}_0 = \mathcal{M}U_0 = (u_0, u_1 + \mathcal{A}_1^{-1}\mathcal{A}_0 u_0, u_2 + \mathcal{A}_1^{-1}\mathcal{A}_0 u_1)^\top$,

$$D(\tilde{\mathcal{A}}) = V \times D(\mathcal{A}_1) \times V, \tag{9}$$

and

$$\tilde{\mathcal{A}}(u,z,y)^\top = (z - \mathcal{A}_1^{-1}\mathcal{A}_0 u, y, -\mathcal{A}_1 z - R(u,z,y)^\top), \quad \forall (u,z,y)^\top \in D(\tilde{\mathcal{A}}), \tag{10}$$

when

$$R(u,z,y)^\top = (B - \mathcal{A}_1^{-1}\mathcal{A}_0)\left(y - \mathcal{A}_1^{-1}\mathcal{A}_0 z + (\mathcal{A}_1^{-1}\mathcal{A}_0)^2 u\right).$$

Proof. Let us first show that if $U = (u,v,w)^\top$ is a solution of (7), then $\tilde{U} = (u,z,y)^\top$ is a solution of (8). Indeed, (7) directly implies that $v = u_t, w = v_t = u_{tt}$ and u satisfies (1). Hence, by their definition, $z = u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u$,

$$z_t = u_{tt} + \mathcal{A}_1^{-1}\mathcal{A}_0 u_t = y,$$

and

$$\begin{aligned} y_t &= w_t + \mathcal{A}_1^{-1}\mathcal{A}_0 v_t \\ &= u_{ttt} + \mathcal{A}_1^{-1}\mathcal{A}_0 w \\ &= -Bu_{tt} - \mathcal{A}_0 u - \mathcal{A}_1 u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 w \\ &= (-Bu_{tt} + \mathcal{A}_1^{-1}\mathcal{A}_0)w - \mathcal{A}_0 u - \mathcal{A}_1 v \\ &= -\mathcal{A}_1 z - R(u,z,y)^\top. \end{aligned}$$

This directly leads to (8).

Let us notice that the regularity

$$\tilde{U} = (u,z,y)^\top \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), D(\tilde{\mathcal{A}}))$$

follows from the fact that the operator $\mathcal{A}_1^{-1}\mathcal{A}_0$ is bounded from H into itself as well as from V into itself.

The converse implication is proved in a fully similar manner. ■

Remark 1 From (8), we see that z satisfies

$$z_{tt} + \mathcal{A}_1 z - (B - \mathcal{A}_1^{-1}\mathcal{A}_0)z_t + (B - \mathcal{A}_1^{-1}\mathcal{A}_0)(\mathcal{A}_1^{-1}\mathcal{A}_0 z - (\mathcal{A}_1^{-1}\mathcal{A}_0)^2 u) = 0, \tag{11}$$

which, under the assumption that $B - \mathcal{A}_1^{-1}\mathcal{A}_0$ is a non-negative operator from H into itself, is an weakly damped wave type equation with lower order term $(B - \mathcal{A}_1^{-1}\mathcal{A}_0)(\mathcal{A}_1^{-1}\mathcal{A}_0 z - (\mathcal{A}_1^{-1}\mathcal{A}_0)^2 u)$. Similarly, u is a solution of

$$u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u = z.$$

which is a sort of ODE, since $\mathcal{A}_1^{-1}\mathcal{A}_0$ is bounded from H into itself.

THEOREM 1 *Under the above assumptions, the operator $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$.*

Proof. We first prove that $\tilde{\mathcal{A}}$ generates a C_0-semigroup on $\mathcal{H}$. For that purpose, we notice that $\tilde{\mathcal{A}}$ can be split up into

$$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_d + \mathcal{B},$$

where the operator $\mathcal{B}$, defined by

$$\mathcal{B}(u,z,y)^\top = (z - \mathcal{A}_1^{-1}\mathcal{A}_0 u, 0, -R(u,z,y)^\top), \quad \forall (u,z,y)^\top \in \mathcal{H},$$

is a bounded operator (in $\mathcal{H}$), and the unbounded operator $\tilde{\mathcal{A}}_d$ is defined by

$$\tilde{\mathcal{A}}_d(u,z,y)^\top = (0, y, -\mathcal{A}_1 z), \quad \forall (u,z,y)^\top \in D(\tilde{\mathcal{A}}_d) = D(\tilde{\mathcal{A}}). \tag{12}$$

Therefore, by a standard bounded perturbation theorem (see for instance Pazy, 1983, Theorem 3.1.1), it suffices to demonstrate that $\tilde{\mathcal{A}}_d$ generates a C_0-semigroup on $\mathcal{H}$. This last property holds since $\tilde{\mathcal{A}}_d$ is a maximal dissipative operator, hence, by Lumer-Phillips' theorem, it generates a C_0-semigroup of contraction on $\mathcal{H}$ (it even generates a group).

The dissipativity is mainly direct because for $U = (u,z,y)^\top \in D(\tilde{\mathcal{A}})$, we have

$$\Re(\tilde{\mathcal{A}}_d U, U)_{\mathcal{H}} = \Re(a_1(y,z) - (\mathcal{A}_1 z, y)) = 0.$$

The maximality is also quite direct. Indeed, for $\lambda > 0$ and $F = (f,g,h) \in \mathcal{H}$ fixed, we look for $U = (u,z,y)^\top \in D(\tilde{\mathcal{A}})$ solution of $(\lambda I - \tilde{\mathcal{A}}_d)U = F$, or equivalently

$$\begin{cases} \lambda u = f \text{ in } V, \\ \lambda z - y = g \text{ in } V, \\ \lambda y + \mathcal{A}_1 z = h \text{ in } H. \end{cases} \tag{13}$$

This means that $u = f/\lambda \in V$, $y = \lambda z - g$ and

$$\lambda^2 z + \mathcal{A}_1 z = h + \lambda g \text{ in } H.$$

Since $\lambda^2 I + \mathcal{A}_1$ is an isomorphism from $D(\mathcal{A}_1)$ into H, we find a unique solution $z \in D(\mathcal{A}_1)$ of this problem and hence $y = \lambda z - g$ indeed belongs to V.

Denote by $(\tilde{T}(t))_{t\geq 0}$ the C_0-semigroup generated by $\tilde{\mathcal{A}}$, then we define

$$T(t) = \mathcal{M}^{-1}\tilde{T}(t)\mathcal{M}, \forall t \geq 0,$$

and as $\mathcal{M}$ is an isomorphism from $\mathcal{H}$ into itself, we directly deduce that $(T(t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{H}$. According to Lemma 2, its generator is nothing else than $\mathcal{A}$, the proof is then complete. ■

Corollary 1 *(Existence and uniqueness of the solution) If $U_0 \in \mathcal{H}$, then problem* (7) *admits a unique weak solution $U = (u, v, w)^\top \in C^0([0, \infty), \mathcal{H})$. On the contrary, if $U_0 \in D(\mathcal{A})$, then problem* (7) *admits a unique strong solution $U = (u, v, w)^\top \in C^1([0, \infty), \mathcal{H}) \cap C^0([0, \infty), D(\mathcal{A}))$.*

2.2. An illustrative example

Let $\Omega \subset \mathbb{R}^d, d \geq 1$ be a bounded open set with a Lipschitz boundary Γ. We take $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Now we define the operators $\mathcal{A}_i$, $i = 0$ and 1 and B as follows. For $i = 0, 1$, we suppose to be given scalar functions $b_i \in L^\infty(\Omega)$, and matrix valued functions $M_i \in L^\infty(\Omega; \mathbb{R}^{d\times d})$. Suppose also that we are given a scalar function $\alpha \in L^\infty(\Omega)$, and a vector field function $\mathbf{c} \in L^\infty(\Omega; \mathbb{R}^d)$. Then we define

$$\begin{aligned} a_1(u, v) &= \int_\Omega (M_1 \nabla u \cdot \nabla \bar{v} + b_1 u \bar{v})\, dx, \\ a_0(u, v) &= \int_\Omega (M_0 \nabla u \cdot \nabla \bar{v} + (\mathbf{c} \cdot \nabla u)\bar{v} + b_0 u \bar{v})\, dx, \end{aligned}$$

for all $u, v \in H_0^1(\Omega)$ and

$$Bu = \alpha u, \forall u \in L^2(\Omega). \tag{14}$$

This yields two sesquilinear and continuous forms on $H_0^1(\Omega)$ and a bounded and selfadjoint operator B from $L^2(\Omega)$ into itself.

We further assume that a_1 is symmetric and coercive on $H_0^1(\Omega)$. The symmetry of a_1 is clearly guaranteed if and only if M_1 is symmetric. The coerciveness of a_1 holds if we further assume that M_1 is uniformly positive definite, namely for almost all $x \in \Omega$,

$$M_1(x)\xi \cdot \bar{\xi} \geq m\|\xi\|_2^2, \forall \xi \in \mathbb{C}^d,$$

for some $m > 0$ (independent of x) and if the negative part $b_1^- = \max\{-b_1, 0\}$ of b_1 is small enough (see below). First define

$$B_1 = \sup_{x\in\Omega} b_1^-(x),$$

and let $c_0 > 0$ be the Poincaré constant

$$c_0\|u\|^2_{L^2(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)^d}, \forall u \in H_0^1(\Omega).$$

Since by the above assumption and definition, we have

$$a_1(u, u) \geq (mc_0 - B_1)\|u\|^2_{L^2(\Omega)}, \forall u \in H_0^1(\Omega), \tag{15}$$

then if we assume that

$$B_1 < mc_0, \tag{16}$$

then a_1 will be coercive on $H_0^1(\Omega)$. The assumption (16) means that the negative part b_1^- of b_1 is small enough with respect to M_1 and is easily checked in practice, since c_0 is explicitly known for some domains Ω or different upper bounds are available in the literature, see Kuznetsov and Nazarov (2015) and the references cited there.

It remains to check the assumption that $\mathcal{A}_1^{-1}\mathcal{A}_0$ can be extended into a bounded operator from $L^2(\Omega)$ into itself. The trivial case is to take $a_0 = a_1$, here is a non trivial one.

LEMMA 3 *Assume that the boundary Γ is of class $C^{1,1}$ and that $M_0 \in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d})$, as well as $\mathbf{c} \in W^{1,\infty}(\Omega;\mathbb{R}^d)$, then $\mathcal{A}_1^{-1}\mathcal{A}_0$ can be extended into a bounded operator from $L^2(\Omega)$ into itself.*

Proof. Define the unbounded operator A_1 that is the extension of $\mathcal{A}_1$ from $L^2(\Omega)$ into itself, defined by

$$D(A_1) := \{u \in H_0^1(\Omega) : \exists g_u \in L^2(\Omega) \text{ such that } a_1(u,v) = \int_\Omega g_u \bar{v}\, dx, \forall v \in H_0^1(\Omega)\},$$

and

$$A_1 u = g_u, \forall u \in D(A_1).$$

From our assumptions, it is well known that this operator is positive and selfadjoint. It is then an isomorphism from $D(A_1^s)$ to $D(A_1^{s-1})$, for every real number s.

Now the assumption on the boundary guarantees that

$$D(A_1) = H^2(\Omega) \cap H_0^1(\Omega),$$

see, for instance, Grisvard (1985, Theorem 2.2.2.3).

We now show that the mapping $\mathcal{A}_0$ can be extended into a continuous mapping from $L^2(\Omega)$ into $D(A_1^{-1}) = (H^2(\Omega) \cap H_0^1(\Omega))'$, the dual of $H^2(\Omega) \cap H_0^1(\Omega)$. This holds if we can show that

$$|a_0(u,v) \lesssim \|u\|_{L^2(\Omega)} \|v\|_{H^2(\Omega)}, \tag{17}$$

for any $u \in H_0^1(\Omega)$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Indeed, let $u \in H_0^1(\Omega)$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$, then by Green's formula (allowed by our assumptions on M_0 and $\mathbf{c}$), we get

$$a_0(u,v) = \int_\Omega \left(-u \operatorname{div}(M_0^\top \nabla \bar{v}) - u \operatorname{div}(\mathbf{c}\bar{v}) + b_0 u \bar{v}\right) dx.$$

By Cauchy-Schwarz's inequality we obtain (17).

In conclusion, as the restriction of A_1 to $D(A_1^{\frac{1}{2}}) = H_0^1(\Omega)$ coincides with $\mathcal{A}_1$, the operator $\mathcal{A}_1^{-1}\mathcal{A}_0$ can then be extended from $L^2(\Omega)$ into itself. ■

Altogether, this means that the system

$$\begin{cases} u_{ttt} + \alpha u_{tt} + \operatorname{div}(M_0\nabla u) + (\mathbf{c}\cdot\nabla u) + b_0 u + \operatorname{div}(M_1\nabla u_t) + b_1 u_t = 0, \\ u = 0 \qquad \text{on } \partial\Omega\times(0,\infty), \quad \text{in } \Omega\times(0,\infty), \\ u(0,\cdot) = u_0, u_t(0,\cdot) = u_1, u_{tt}(0,\cdot) = u_2, \quad \text{in } \Omega \end{cases} \tag{18}$$

is well-posed in $H_0^1(\Omega)^2 \times L^2(\Omega)$.

Remark 2 Note that other choices for B are possible, for instance – an integral operator is possible, namely if a scalar kernel $k \in L^\infty(\Omega\times\Omega)$ is given, we may choose

$$Bu(x) = \int_\Omega k(x,y)u(y)\,dy, \forall u \in L^2(\Omega),$$

that is a bounded operator B from $L^2(\Omega)$ into itself.

3. Uniform stability results

In this section, inspired by Kaltenbacher and Lasiecka (2012, §4), Kaltenbacher, Lasiecka and Marchand (2011, §4), Kaltenbacher, Lasiecka and Pospieszalska (2012, §3) and Marchand, McDevitt and Triggiani (2012, §4), we prove that the semi-group $(T(t))_{t\geq 0}$, generated by $\mathcal{A}$, decays exponentially under some additional assumptions. In the whole section we assume that $\mathcal{A}_0$ and B are selfadjoint, that

$$\mathcal{A}_1^{-1}\mathcal{A}_0 = \mathcal{A}_0\mathcal{A}_1^{-1}, \tag{19}$$

$$\mathcal{A}_1^{-1}\mathcal{A}_0 B = B\mathcal{A}_1^{-1}\mathcal{A}_0, \tag{20}$$

and that

$$(Bv, v) \geq 0, \forall v \in H, \tag{21}$$

$$(\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0)v, v) \geq 2\delta\|v\|^2, \forall v \in H, \tag{22}$$

$$((B - \mathcal{A}_1^{-1}\mathcal{A}_0)v, v) \geq 2\delta\|v\|^2, \forall v \in H, \tag{23}$$

$$a_0(u,u) \geq \alpha_0\|u\|_V^2, \forall u \in V, \tag{24}$$

for some positive constants δ and α_0. According to Remark 1, the assumption (23) is certainly needed to obtain the exponential decay of $T(t)$.

Let us first define the following energies

$$\begin{aligned} E(t) &= \frac{1}{2}\Big(a_1(u_t+\mathcal{A}_1^{-1}\mathcal{A}_0u, u_t+\mathcal{A}_1^{-1}\mathcal{A}_0u) \\ &+ \|u_{tt}+\mathcal{A}_1^{-1}\mathcal{A}_0u_t\|^2+(\mathcal{A}_1^{-1}\mathcal{A}_0(B-\mathcal{A}_1^{-1}\mathcal{A}_0)u_t,u_t)\Big), \\ E_0(t) &= \frac{1}{2}((Bu_t,u_t)+a_0(u,u)), \\ E_{\text{tot}}(t) &= E(t)+\delta E_0(t), \forall t\geq 0. \end{aligned}$$

Note that our assumptions guarantee that $\mathcal{A}_1^{-1}\mathcal{A}_0$ is selfadjoint as well as $\mathcal{A}_1^{-1}\mathcal{A}_0B$.

Notice that $E(t)$ and $E_0(t)$ are non negative, both are not equivalent to $\|(u,u_t,u_{tt})\|^2_{\mathcal{H}}$ in general, but under the previous assumptions, the sum is, as shown in the next Lemma, compare with Kaltenbacher and Lasiecka (2012, Remark 4.2), Kaltenbacher, Lasiecka and Marchand (2011, Remark 4.1), Kaltenbacher, Lasiecka and Pospieszalska (2012, Remark 3.2).

Lemma 4 *Let $U=(u,u_t,u_{tt})$ be a strong solution of* (7) *with an initial datum $U_0=(u_0,u_1,u_2)\in D(\mathcal{A})$. Then, the following holds*

$$E_{\text{tot}}(t)\sim\|(u,u_t,u_{tt})\|^2_{\mathcal{H}}, \forall t\geq 0. \tag{25}$$

Proof. Since the estimate

$$E_{\text{tot}}(t)\lesssim\|(u,u_t,u_{tt})\|^2_{\mathcal{H}},$$

is immediate, let us concentrate on the converse estimation. As

$$\|(u,u_t,u_{tt})\|^2_{\mathcal{H}}=a_1(u,u)+a_1(u_t,u_t)+\|u_{tt}\|^2,$$

by the continuity of a_1 and (24), we get

$$\|(u,u_t,u_{tt})\|^2_{\mathcal{H}}\lesssim E_0(t)+a_1(u_t,u_t)+\|u_{tt}\|^2.$$

For the two last terms of this right-hand side, we insert some zero term to get

$$\begin{aligned} \|(u,u_t,u_{tt})\|^2_{\mathcal{H}} &\lesssim E_0(t)+2a_1(u_t+\mathcal{A}_1^{-1}\mathcal{A}_0u,u_t+\mathcal{A}_1^{-1}\mathcal{A}_0u) \\ &+ 2\|u_{tt}+\mathcal{A}_1^{-1}\mathcal{A}_0u_t\|^2 \\ &+ 2a_1(\mathcal{A}_1^{-1}\mathcal{A}_0u,\mathcal{A}_1^{-1}\mathcal{A}_0u)+2\|\mathcal{A}_1^{-1}\mathcal{A}_0u_t\|^2. \end{aligned}$$

By the boundedness properties of $\mathcal{A}_1^{-1}\mathcal{A}_0$, mentioned before, we obtain

$$\begin{aligned} \|(u,u_t,u_{tt})\|^2_{\mathcal{H}} &\lesssim E_0(t)+a_1(u_t+\mathcal{A}_1^{-1}\mathcal{A}_0u,u_t+\mathcal{A}_1^{-1}\mathcal{A}_0u) \\ &+ 2\|u_{tt}+\mathcal{A}_1^{-1}\mathcal{A}_0u_t\|^2+\|u_t\|^2. \end{aligned}$$

Using the assumption (22), we arrive at

$$\|(u, u_t, u_{tt})\|^2_{\mathcal{H}} \lesssim E(t) + E_0(t),$$

as requested. ■

In the first step, we give an explicit expression of the derivative of the energy E (compare with Kaltenbacher and Lasiecka, 2012, Lemma 4.3; Kaltenbacher, Lasiecka and Marchand, 2011, Lemma 4.1; Kaltenbacher, Lasiecka and Pospieszalska, 2012, Lemma 3.1).

LEMMA 5 *Let $U = (u, u_t, u_{tt})$ be a strong solution of (7) with an initial datum $U_0 = (u_0, u_1, u_2) \in D(\mathcal{A})$. Then*

$$E'(t) = -((B - \mathcal{A}_1^{-1}\mathcal{A}_0)u_{tt}, u_{tt}). \tag{26}$$

In particular, the energy E is non increasing.

Proof. Introduce the continuous form

$$\begin{aligned} ((u,v,w)^\top, (u',v',w'))_{\mathcal{H}_0} &= a_1(v + \mathcal{A}_1^{-1}\mathcal{A}_0 u, v' + \mathcal{A}_1^{-1}\mathcal{A}_0 u') \\ &+ (w + \mathcal{A}_1^{-1}\mathcal{A}_0 v, w' + \mathcal{A}_1^{-1}\mathcal{A}_0 v') \\ &+ (\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0)v, v'), \\ &\quad \forall (u,v,w)^\top, (u',v',w')^\top \in \mathcal{H}. \end{aligned} \tag{27}$$

As underlined before, since $\mathcal{A}_1^{-1}\mathcal{A}_0$ and $\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0)$ are selfadjoint, the above form is symmetric.

Now we notice that

$$2E(t) = (U(t), U(t))_{\mathcal{H}_0},$$

hence

$$E'(t) = \Re(U'(t), U(t))_{\mathcal{H}_0} = \Re(\mathcal{A}U(t), U(t))_{\mathcal{H}_0}.$$

To get the conclusion it then remains to show that

$$\Re(\mathcal{A}U, U)_{\mathcal{H}_0} = -((B - \mathcal{A}_1^{-1}\mathcal{A}_0)w, w), \forall U = (u, v, w) \in D(\mathcal{A}). \tag{28}$$

But in view of the definition of $\mathcal{A}$, for $U = (u, v, w) \in D(\mathcal{A})$, we have

$$\begin{aligned} (\mathcal{A}U, U)_{\mathcal{H}_0} &= a_1(w + \mathcal{A}_1^{-1}\mathcal{A}_0 v, v + \mathcal{A}_1^{-1}\mathcal{A}_0 u) \\ &+ (-\mathcal{A}_0 u - \mathcal{A}_1 v - Bw + \mathcal{A}_1^{-1}\mathcal{A}_0 w, w + \mathcal{A}_1^{-1}\mathcal{A}_0 v) \\ &+ (\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0)w, v). \end{aligned}$$

Using the definition of $\mathcal{A}_1$, we get

$$\Re(\mathcal{A}U, U)_{\mathcal{H}_0} =$$
$$\Re\big\{\langle \mathcal{A}_1 w + \mathcal{A}_0 v, v + \mathcal{A}_1^{-1}\mathcal{A}_0 u\rangle - \langle \mathcal{A}_0 u, w + \mathcal{A}_1^{-1}\mathcal{A}_0 v\rangle - \langle \mathcal{A}_1 v, w + \mathcal{A}_1^{-1}\mathcal{A}_0 v\rangle$$
$$+(-Bw + \mathcal{A}_1^{-1}\mathcal{A}_0 w, w + \mathcal{A}_1^{-1}\mathcal{A}_0 v) + (\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0)w, v)\big\}.$$

Using our assumptions, some terms of this right-hand side cancel out to reduce to the right-hand side of (28). ■

In the second step, we need the following identity (compare with Kaltenbacher and Lasiecka, 2012, Lemma 4.4; Kaltenbacher, Lasiecka and Marchand, 2011, identity (30), and Kaltenbacher, Lasiecka and Pospieszalska, 2012, Lemma 3.2).

LEMMA 6 *Let $U = (u, u_t, u_{tt})$ be a strong solution of* (7) *with an initial datum $U_0 = (u_0, u_1, u_2) \in D(\mathcal{A})$, then, the following holds*

$$a_1(u_t, u_t) = \|u_{tt}\|^2 - \frac{d}{dt}E_0(t) - \frac{d}{dt}\left(\Re(u_{tt}, u_t)\right). \tag{29}$$

Proof. Taking the inner product of the first identity of (1) with u_t we directly get

$$(u_{ttt}, u_t) + (Bu_{tt}, u_t) + a_0(u, u_t) + a_1(u_t, u_t) = 0.$$

Taking the real part of this identity, we obtain

$$a_1(u_t, u_t) = -\Re(u_{ttt}, u_t) - \frac{d}{dt}E_0(t).$$

As

$$\frac{d}{dt}(u_{tt}, u_t) = (u_{ttt}, u_t) + (u_{tt}, u_{tt}),$$

the two previous identities directly yield (29). ■

We are ready to state the exponential decay result (compare with Kaltenbacher and Lasiecka, 2012, Lemma 4.5 and §4.3; Kaltenbacher, Lasiecka and Marchant, 2011, steps 3 and 4, pp. 982–983, and Kaltenbacher, Lasiecka and Pospieszalska, 2012, steps 3 and 4, pp. 20–21).

THEOREM 2 *Under the additional assumptions of this section, the semigroup generated by $\mathcal{A}$ is exponentially stable in $\mathcal{H}$, namely there exist two positive constants M and ω such that*

$$\|e^{t\mathcal{A}}U_0\| \leq Me^{-\omega t}\|U_0\|, \forall U_0 \in \mathcal{H}.$$

Proof. Let us first fix $U_0 \in D(\mathcal{A})$ and let $U(t) = (u(t), v(t), w(t)) = e^{t\mathcal{A}}U_0$ be the strong solution of (7) (that satisfies $v = u_t$ and $w = u_{tt}$). For such a solution, using the identities (26) and (29), we have

$$\begin{aligned}\frac{d}{dt}E_{\text{tot}}(t) &= -((B-\mathcal{A}_1^{-1}\mathcal{A}_0)u_{tt}, u_{tt}) \\ &+ \delta\|u_{tt}\|^2 - \delta a_1(u_t, u_t) - \delta\frac{d}{dt}\left(\Re(u_{tt}, u_t)\right).\end{aligned}$$

Hence, by our assumption (23), we get

$$\frac{d}{dt}E_{\text{tot}}(t) \leq -\delta\|u_{tt}\|^2 - \delta a_1(u_t, u_t) - \delta\frac{d}{dt}\left(\Re(u_{tt}, u_t)\right).$$

Integrating this estimate in $t \in (0, T)$ for an arbitray $T > 0$, one gets

$$\begin{aligned}E_{\text{tot}}(T) - E_{\text{tot}}(0) + \delta\int_0^T \left(\|u_{tt}\|^2 + a_1(u_t, u_t)\right)\, dt \leq \\ -\delta\Re(u_{tt}(T), u_t(T)) + \delta\Re(u_{tt}(0), u_t(0)).\end{aligned} \tag{30}$$

For the second term of this right hand side, using Cauchy-Schwarz's inequality and Lemma 4, we get

$$\Re(u_{tt}(0), u_t(0)) \lesssim E_{\text{tot}}(0). \tag{31}$$

On the contrary, for the first term, we write

$$(u_{tt}(T), u_t(T)) = (u_{tt}(T) + \mathcal{A}_1^{-1}\mathcal{A}_0 u_t(T), u_t(T)) - (\mathcal{A}_1^{-1}\mathcal{A}_0 u_t(T), u_t(T)).$$

Using Cauchy-Schwarz's inequality and Young's inequality and the boundedness of $\mathcal{A}_1^{-1}\mathcal{A}_0$ from H into itself, we find

$$(u_{tt}(T), u_t(T)) \lesssim \|u_{tt}(T) + \mathcal{A}_1^{-1}\mathcal{A}_0 u_t(T)\|^2 + \|u_t(T)\|^2;$$

note that here and below the constant involved in $\lesssim$ is independent of T. By the assumption (22) and the definition of $E(t)$, we get

$$(u_{tt}(T), u_t(T)) \lesssim E(T),$$

and since E is non increasing, we arrive at

$$(u_{tt}(T), u_t(T)) \lesssim E(0).$$

This estimate and (31) in (3) directly yield

$$E_{\text{tot}}(T) + \delta\int_0^T \left(\|u_{tt}\|^2 + a_1(u_t, u_t)\right)\, dt \lesssim E_{\text{tot}}(0).$$

Using Lemma 4, we arrive at

$$\|(u(T), u_t(T), u_{tt}(T))\|_{\mathcal{H}}^2 + \int_0^T \left(\|u_{tt}\|^2 + a_1(u_t, u_t)\right) dt \lesssim \\ \|(u(0), u_t(0), u_{tt}(0))\|_{\mathcal{H}}^2. \tag{32}$$

It now remains to estimate $\int_0^T \|u\|_V^2 \, dt$. For that purpose, we take the inner product in H of the first identity of (1) with u to get

$$(u_{ttt} + Bu_{tt} + \mathcal{A}_0 u + \mathcal{A}_1 u_t, u) = 0.$$

Taking the real part of this identity, we find that

$$a_0(u,u) + \frac{1}{2}\frac{d}{dt}a_1(u,u) = -\Re(u_{ttt} + Bu_{tt}, u).$$

As

$$\Re(u_{ttt}, u) = -\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \Re\frac{d}{dt}(u_{tt}, u),$$

and

$$(Bu_{tt}, u) = \frac{d}{dt}(Bu_t, u) - (Bu_t, u_t),$$

we get

$$a_0(u,u) + \frac{1}{2}\frac{d}{dt}a_1(u,u) = (Bu_t, u_t) \\ + \frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 - \Re(u_{tt}, u) - \Re(Bu_t, u)\right).$$

By integrating this estimate between 0 and $T > 0$, we find

$$\int_0^T a_0(u,u)\, dt + \frac{1}{2}a_1(u(T), u(T)) - \frac{1}{2}a_1(u(0), u(0)) \le \int_0^T (Bu_t, u_t)\, dt \\ + \frac{1}{2}\|u_t(T)\|^2 + |(u_{tt}(T), u(T))| + |(Bu_t(T), u(T))| \\ + \frac{1}{2}\|u_t(0)\|^2 + |(u_{tt}(0), u(0))| + |(Bu_t(0), u(0))|.$$

Using Cauchy-Schwarz's inequality, the boundedness of B, the continuous embedding of V into H and the coerciveness of a_1, we obtain

$$\int_0^T a_0(u,u)\, dt \lesssim \int_0^T a_1(u_t, u_t)\, dt \\ + \|(u(T), u_t(T), u_{tt}(T))\|_{\mathcal{H}}^2 + \|(u(0), u_t(0), u_{tt}(0))\|_{\mathcal{H}}^2.$$

With the help of (3), we get

$$\int_0^T a_0(u,u)\,dt \lesssim \|(u(0),u_t(0),u_{tt}(0))\|_{\mathcal{H}}^2.$$

This estimate and again (3) lead to

$$\int_0^T \left(\|u_{tt}\|^2 + a_1(u_t,u_t) + a_0(u,u)\right)\,dt \lesssim \|(u(0),u_t(0),u_{tt}(0))\|_{\mathcal{H}}^2,$$

and by Lemma 4, we finally obtain

$$\int_0^T \|(u(t),u_t(t),u_{tt}(t))\|_{\mathcal{H}}^2\,dt \lesssim \|(u(0),u_t(0),u_{tt}(0))\|_{\mathcal{H}}^2.$$

Since this estimate is valid for all $T > 0$ and as $(u(t),u_t(t),u_{tt}(t)) = e^{tA}U_0$, we get

$$\int_0^\infty \|e^{tA}U_0\|_{\mathcal{H}}^2\,dt \lesssim \|U_0\|_{\mathcal{H}}^2.$$

As $D(\mathcal{A})$ is dense in $\mathcal{H}$, this estimate remains valid for all $U_0 \in \mathcal{H}$ and by Datko (1970, Corollary) (see also Pazy, 1983, Theorem 4.1.4) we conclude that e^{tA} is exponentially stable. ■

Let us end up with some examples.

Example 1 In the setting of Subsection 2.2, assuming that $M_0 = \beta M_1$, $b_0 = \beta b_1 + r$, $\mathbf{c} = \mathbf{0}$ for some real number r and a positive real number β, the operator $\mathcal{A}_0$ is selfadjoint. Then, due to (15), (24) will be valid if

$$r > \beta(B_1 - mc_0).$$

Now, denoting by $\mathbb{I}$ the identity operator, as

$$\mathcal{A}_0 = \beta\mathcal{A}_1 + r\mathbb{I},$$

we deduce that

$$\mathcal{A}_1^{-1}\mathcal{A}_0 = \mathcal{A}_0\mathcal{A}_1^{-1} = \beta\mathbb{I} + r\mathcal{A}_1^{-1},$$

and hence (19) holds.

If $r \neq 0$, we take $B = \alpha\mathbb{I}$ with a constant α which guarantees that (20) holds. On the contrary, if $r = 0$, we can take $B = \alpha\mathbb{I}$ with $\alpha \in L^\infty(\Omega)$ and (20) remains valid.

In both cases, (21) holds if $\alpha \geq 0$. Hence, it remains to examine (22) and (23). If $r = 0$, as $\mathcal{A}_1^{-1}\mathcal{A}_0 = \beta\mathbb{I}$ with $\beta > 0$, (22) and (23) are equivalent and as

$$B - \mathcal{A}_1^{-1}\mathcal{A}_0 = (\alpha - \beta)\mathbb{I},$$

they hold if and only if

$$\alpha - \beta \geq 2\delta, \text{ a. e. in } \Omega. \tag{33}$$

On the contrary, if $r \neq 0$, then

$$B - \mathcal{A}_1^{-1}\mathcal{A}_0 = (\alpha - \beta)\mathbb{I} - r\mathcal{A}_1^{-1},$$

while

$$\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0) = \beta(\alpha - \beta)\mathbb{I} + (\alpha - 2\beta)r\mathcal{A}_1^{-1} - r^2\mathcal{A}_1^{-2}.$$

As there exists a positive constant C_1 such that

$$\|\mathcal{A}_1^{-1}v\|_{L^2(\Omega)} \leq C_1\|v\|_{L^2(\Omega)}, \forall v \in L^2(\Omega),$$

by Cauchy-Schwarz's inequality we deduce that

$$\begin{aligned} ((B - \mathcal{A}_1^{-1}\mathcal{A}_0)v, v)_{L^2(\Omega)} &= (\alpha - \beta)\|v\|^2_{L^2(\Omega)} - r(\mathcal{A}_1^{-1}v, v)_{L^2(\Omega)} \\ &\geq (\alpha - \beta - |r|C_1)\|v\|^2_{L^2(\Omega)}. \end{aligned}$$

This means that (23) holds if

$$\alpha - \beta - |r|C_1 > 0.$$

Similarly, we have

$$(\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0)v, v)_{L^2(\Omega)} \geq (\beta(\alpha - \beta) - |\alpha - 2\beta||r|C_1 - r^2C_1^2)\|v\|^2_{L^2(\Omega)}.$$

Consequently, (22) holds if

$$\beta(\alpha - \beta) - |\alpha - 2\beta||r|C_1 - r^2C_1^2 > 0.$$

4. A degenerate case

In this section we present examples, where the assumptions (22) and (23) fail and for which exponential, polynomial or logarithmic rate is reached.

In the setting of Subsection 2.2, we assume that Ω is connected and we choose M_1 equal to the identity matrix $= \mathbb{I}_{d\times d}$, $M_0 = \beta\mathbb{I}_{d\times d}$, where β is a positive constant and B in the form (14) with $\alpha \in L^\infty(\Omega)$ such that

$$\alpha \geq \beta \text{ a. e. in } \Omega. \tag{34}$$

In other words, we study Moore-Gibson-Thompson system (2). We further suppose that there exist an non empty open subset ω_0 of Ω and a positive constant κ such that

$$\alpha - \beta \geq \kappa \text{ a. e. in } \omega_0. \tag{35}$$

In this case, all assumptions of Section 3 hold except for (22) and (23), since

$$\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0) = \beta(B - \mathcal{A}_1^{-1}\mathcal{A}_0) = \beta(\alpha - \beta)\mathbb{I},$$

that could be zero on $\Omega \setminus \omega_0$. Hence, the sole case of interest here is the case when ω_0 is different from Ω and $\alpha = \beta$ on a non empty open set of Ω.

According to Remark 1 and commonly found papers about weaker (polynomial or logarithmic) decay rate of the wave equation (see below), we can expect a weaker decay rate for system (2). In this case our stability result is based on a spectral analysis and a resolvent estimate, obtained by a comparison with the resolvent of the wave equation with an interior damping in ω_0.

We then first analyze the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$. Note that the domain of $\mathcal{A}$ is not compactly embedded into $\mathcal{H}$, hence, if it exists, the resolvent of $\mathcal{A}$ is not compact. This renders the analysis more complex and forces us to use a compact perturbation argument (described below).

Lemma 7 *Under the previous assumptions,*

$$\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \Re\lambda \geq 0\} \subset \rho(\mathcal{A}). \tag{36}$$

Proof. Let $\lambda \in \mathbb{C}_+$ and $F = (f, g, h)^\top \in \mathcal{H}$. We look for $U = (u, v, w)^\top \in D(\mathcal{A})$ such that

$$\lambda U - \mathcal{A}U = F, \tag{37}$$

or equivalently

$$\lambda u - v = f, \tag{38}$$
$$\lambda v - w = g, \tag{39}$$
$$(\lambda + \alpha)w - \Delta(\beta u + v) = h \tag{40}$$

Assume that a solution U exists. Then the two first identities yield

$$v = \lambda u - f, \tag{41}$$
$$w = \lambda v - g = \lambda^2 u - \lambda f - g, \tag{42}$$

and by plugging (41) and (42) into (40), we find

$$(\lambda+\alpha)\lambda^2 u-(\beta+\lambda)\Delta u=h+(\lambda+\alpha)(\lambda f+g)-\Delta f \text{ in } \mathcal{D}'(\Omega), \tag{43}$$

where $\mathcal{D}'(\Omega)$ is the space of Schwartz distributions, the dual of the space $\mathcal{D}(\Omega)$ made of smooth and compactly supported functions in Ω, see Schwartz (1966) or Adams (1975, p.19). This equivalently means that

$$a_\lambda(u,v)=F_\lambda(v), \forall v\in\mathcal{D}(\Omega), \tag{44}$$

where for all $u,v\in H_0^1(\Omega)$

$$a_\lambda(u,v)=\int_\Omega \left((\lambda+\alpha)\lambda^2 u\bar{v}+(\beta+\lambda)\nabla u\cdot\nabla\bar{v}\right)\,dx,$$

$$F_\lambda(v)=\int_\Omega \left((h+(\lambda+\alpha)(\lambda f+g))\,\bar{v}+\nabla f\cdot\nabla\bar{v}\right)\,dx.$$

Since a_λ is continuous on $H_0^1(\Omega)\times H_0^1(\Omega)$ and F_λ is continuous on $H_0^1(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, the identity (44) remains valid for the test-functions in $H_0^1(\Omega)$, namely

$$a_\lambda(u,v)=F_\lambda(v), \forall v\in H_0^1(\Omega). \tag{45}$$

Let us now show that this problem has a unique solution $u\in H_0^1(\Omega)$. For that purpose, we distinguish two cases:

1. If $\lambda=0$, we see that

$$a_0(u,v)=\beta\int_\Omega \nabla u\cdot\nabla\bar{v}\,dx,$$

$$F_0(v)=\int_\Omega \left((h+\alpha g)\,\bar{v}+\nabla f\cdot\nabla\bar{v}\right)\,dx.$$

Since a_0 is a continuous sesquilinear and coercive form on $H_0^1(\Omega)$, problem (45) (with $\lambda=0$) has a unique solution $u\in H_0^1(\Omega)$. Upon defining $v=-f$ and $w=-g$ (see (41) and (42)), we easily see that the triple $U=(u,v,w)^\top$ belongs to $D(\mathcal{A})$ and is a solution of (37) (with $\lambda=0$). Hence, 0 belongs to $\rho(\mathcal{A})$.

2. If $\lambda\neq 0$, our argument is more complex and is based on a compact perturbation argument. Namely, introduce the sesquilinear and continuous form b_λ on $H_0^1(\Omega)$, defined by

$$b_\lambda(u,v)=(\beta+\lambda)\int_\Omega \nabla u\cdot\nabla\bar{v}\,dx, \forall u,v\in H_0^1(\Omega).$$

Introduce further the operators A_λ and B_λ by

$$A_\lambda: H_0^1(\Omega)\to H^{-1}(\Omega): u\to A_\lambda u,$$
$$B_\lambda: H_0^1(\Omega)\to H^{-1}(\Omega): u\to B_\lambda u,$$

with

$$\langle A_\lambda u, v\rangle = a_\lambda(u,v), \quad \langle B_\lambda u, v\rangle = b_\lambda(u,v), \forall u, v \in H_0^1(\Omega).$$

Since $\Re(\beta+\lambda) \geq \beta$, the form b_λ is coercive, in the sense that

$$\Re b_\lambda(u,u) \geq \beta \int_\Omega |\nabla u|^2 \, dx \gtrsim \|u\|_{1,\Omega}^2, \forall u \in H_0^1(\Omega),$$

due to Poincaré inequality. Hence, by Lax-Milgram lemma, the operator B_λ is an isomorphism from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Since $A_\lambda - B_\lambda = (\lambda+\alpha)\lambda^2\mathbb{I}$ is a compact operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$, A_λ is then a Fredholm operator of index zero from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Hence, it is an isomorphism if and only if it is injective.

So, let $u \in \ker A_\lambda$, then it is a solution of (45) with $F_\lambda = 0$, namely

$$a_\lambda(u,v) = 0, \forall v \in H_0^1(\Omega). \tag{46}$$

But then, upon defining (compare with (41) and (42))

$$v = \lambda u, w = \lambda v,$$

we easily see that the triple $U = (u, v, w)^\top$ belongs to $D(\mathcal{A})$ and is a solution of

$$\lambda U - \mathcal{A}U = 0.$$

Now we take advantage of the identity (28) that here implies

$$\Re\lambda (U,U)_{\mathcal{H}_0} = -\int_\Omega (\alpha-\beta)|w|^2 \, dx.$$

But, according to its definition (27) and the assumption (34), we have

$$(U,U)_{\mathcal{H}_0} \geq 0, \text{ and } \int_\Omega (\alpha-\beta)|w|^2 \, dx \geq 0,$$

and therefore the previous identity implies that

$$\int_\Omega (\alpha-\beta)|w|^2 \, dx = 0.$$

By (34) and (35), we conclude that

$$w = 0 \text{ on } \omega_0.$$

As $\lambda \neq 0$, we deduce that

$$u = 0 \text{ on } \omega_0. \tag{47}$$

Now, since $u \in H_0^1(\Omega)$, the solution of (46), it satisfies

$$(\lambda+\alpha)\lambda^2 u - (\beta+\lambda)\Delta u = 0 \text{ in } \mathcal{D}'(\Omega).$$

Since the operator Δ is elliptic and u is zero on ω_0 by Calderon uniqueness theorem (see, for instance, Rousseau and Lebeau, 2012, Theorem 4.2), $u = 0$ on the whole Ω. This obviously implies that $v = w = 0$ and hence $U = (0,0,0)^\top$ and the injectivity of A_λ is proved.

In conclusion, A_λ is an isomorphism, which guarantees that problem (45) has a unique solution $u \in H_0^1(\Omega)$. As before, defining v by (41) and w by (42), we easily see that the triple $U = (u,v,w)^\top$ belongs to $D(\mathcal{A})$ and is a solution of (37). The proof is then complete. ∎

The main ingredient to obtain the resolvent estimate is to use the decay rate (exponential, polynomial or less) of the semigroup generated by the wave equation in Ω with Dirichlet boundary condition and with a frictional interior damping in ω_0:

$$\begin{cases} u_{tt} - \Delta u + (\alpha-\beta)u_t = 0 & \text{in } \Omega\times(0,+\infty), \\ u = 0 & \text{on } \Gamma\times(0,+\infty), \\ u(\cdot,0) = u_0 \text{ and } u_t(\cdot,0) = u_1 & \text{in } \Omega, \end{cases} \tag{48}$$

where α and β are the functions introduced before. More precisely, let us introduce the Hilbert space $\mathcal{H}_w = H_0^1(\Omega)\times L^2(\Omega)$ with norm

$$\|(u,v)^\top\|^2_{\mathcal{H}_w} = |u|^2_{1,\Omega} + \|v\|^2_\Omega, \ \ \forall (u,v)^\top \in \mathcal{H}_w,$$

and the operator $\mathcal{A}_w$ defined by

$$D(\mathcal{A}_w) = \{(u,v)^\top \in H_0^1(\Omega)\times H_0^1(\Omega) \,|\, \Delta u \in L^2(\Omega)\}, \tag{49}$$

and

$$\mathcal{A}_w(u,v)^\top = (v, \Delta u - (\alpha-\beta)v)), \quad \forall (u,v)^\top \in D(\mathcal{A}_w). \tag{50}$$

It is well-known that $\mathcal{A}_w$ generates a C_0-semigroup of contractions $(T_w(t))_{t\geq 0}$, see Arendt et al. (2001, p. 232). Hence, it is straightforward that its resolvent set $\rho(\mathcal{A}_w)$ contains the open right half-plane $\{\lambda\in\mathbb{C} : \Re\lambda > 0\}$ and that

$$\|(\lambda\mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)} \leq \frac{1}{\Re\lambda}, \ \ \forall\lambda\in\mathbb{C} : \Re\lambda > 0. \tag{51}$$

Since ω_0 is open and non empty, by Calderon uniqueness theorem (see above), one can show that the imaginary axis is included into $\rho(\mathcal{A}_w)$ and therefore $\mathbb{C}_+ \subset \rho(\mathcal{A}_w)$. The decay rate of the solution to system (1) is based on the following bound on the resolvent of $\mathcal{A}_\omega$ on the imaginary axis

$$\|(i\xi\mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)} \lesssim M(|\xi|), \ \ \forall\xi\in\mathbb{R}, \tag{52}$$

where M is a continuous, positive, and non decreasing function from $[0,\infty)$ into itself.

Before going on, recall that for any $\lambda \in \mathbb{C}_+$ and an arbitrary $F_1 = (f_1, g_1) \in \mathcal{H}_w$, $(u_1, v_1)^\top = (\lambda\mathbb{I} - \mathcal{A}_w)^{-1}F_1 \in D(\mathcal{A}_w)$ satisfies

$$\begin{aligned} v_1 &= \lambda u_1 - f_1, & (53)\\ (\lambda+\alpha-\beta)\lambda u_1 - \Delta u_1 &= g_1 + (\lambda+\alpha-\beta) f_1, & (54) \end{aligned}$$

and the estimate

$$|u_1|_{1,\Omega} + \|v_1\|_\Omega \le \|(\lambda\mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)}(|f_1|_{1,\Omega} + \|g_1\|_\Omega). \quad (55)$$

Due to (53), this implies

$$|u_1|_{1,\Omega} + |\lambda|\|u_1\|_\Omega \le \max\{1, \|(\lambda\mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)}\}(|f_1|_{1,\Omega} + \|g_1\|_\Omega). \quad (56)$$

Now we are ready to prove the following result.

Theorem 3 *There exists a positive constant C such that*

$$\|(\lambda\mathbb{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le C \max\left\{1, \|(\lambda\mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)}, \frac{\|(\lambda\mathbb{I} - \mathcal{A}_w)^{-1}\|^2_{\mathcal{L}(\mathcal{H}_w)}}{1+|\lambda|^2}\right\},$$
$$\forall \lambda \in \mathbb{C}_+. \quad (57)$$

Proof. 1. For $\lambda \in \mathcal{K} = \{\mu \in \mathbb{C}_+ \,|\, |\mu| \le 1\}$, the estimate (3) is direct using Lemma 7 and since the resolvent operator

$$\lambda \to (\lambda\mathbb{I} - \mathcal{A})^{-1}$$

is holomorphic on $\rho(\mathcal{A})$ (see Arendt et al., 2001, Corollary B.3), hence continuous on $\mathcal{K}$. Therefore

$$\|(\lambda\mathbb{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \lesssim 1, \forall \lambda \in \mathcal{K},$$

where from now on the positive constant, hidden in $\lesssim$, is independent of λ.

2. Let $\lambda \in \mathbb{C}_+$ satisfy $|\lambda| \ge 1$ and $F = (f, g, h)^\top \in \mathcal{H}$ and let $U = (u, v, w)^\top \in D(\mathcal{A})$ be the unique solution of (37). We have seen before that v is given by (41), hence the condition from $D(\mathcal{A})$, $\mathcal{A}_0 u + \mathcal{A}_1 v \in L^2(\Omega)$ takes the form

$$\Delta\left((\beta+\lambda)u - f\right) \in L^2(\Omega).$$

This suggests the introduction of the new unknown

$$u_1 = (\lambda+\beta)u - f \quad (58)$$

that belongs to $H_0^1(\Omega)$ with $\Delta u_1 \in L^2(\Omega)$. Recalling that u satisfies (43), we see that u_1 satisfies

$$(\lambda+\alpha)\lambda^2 u - \Delta u_1 = h + (\lambda+\alpha)(\lambda f + g) \text{ in } L^2(\Omega).$$

As $\lambda+\beta \neq 0$, replacing u by $\frac{1}{\lambda+\beta}(u_1+f)$, we find

$$\frac{\lambda+\alpha}{\lambda+\beta}\lambda^2 u_1 - \Delta u_1 = h + (\lambda+\alpha)g + \frac{\lambda\beta(\lambda+\alpha)}{\lambda+\beta} f \text{ in } L^2(\Omega). \tag{59}$$

By setting

$$\begin{aligned} g_1 &= h + \beta g + \frac{\lambda\beta^2}{\lambda+\beta} f + \frac{\lambda\beta(\alpha-\beta)}{\lambda+\beta} u_1, \\ f_1 &= g + \frac{\lambda\beta}{\lambda+\beta} f, \end{aligned}$$

we see that u_1 is a solution of (54). Hence, upon setting $v_1 = \lambda u_1 - f_1$, we find a pair $(u_1, v_1)^\top \in D(\mathcal{A}_w)$, satisfying (53)-(54). Consequently, the estimate (56) holds for u_1, namely

$$|u_1|_{1,\Omega} + |\lambda| \|u_1\|_\Omega \leq C_\lambda (|f_1|_{1,\Omega} + \|g_1\|_\Omega),$$

where for shortness we have set

$$C_\lambda = \max\{1, \|(\lambda \mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)}\}.$$

Using (41) and (58), we see that

$$\frac{\lambda}{\lambda+\beta} u_1 = v + \frac{\beta}{\lambda+\beta} f,$$

hence

$$g_1 = \beta(\alpha-\beta)v + h + \beta g + \frac{\beta^2(\lambda+\alpha-\beta)}{\lambda+\beta} f.$$

Using this expression of g_1 and the definition f_1, we get

$$\begin{aligned} &|u_1|_{1,\Omega} + |\lambda| \|u_1\|_\Omega \leq \\ &C_\lambda \Big(|g|_{1,\Omega} + \frac{|\lambda|\beta}{|\lambda+\beta|} |f|_{1,\Omega} + \|h\|_\Omega + \beta \|g\|_\Omega + \frac{\beta^2}{|\lambda+\beta|} \|(\lambda+\alpha-\beta) f\|_\Omega \\ &+ \beta \|(\beta-\alpha)v\|_\Omega \Big). \end{aligned}$$

As

$$\frac{|\lambda|}{|\lambda+\beta|} \leq 1, \forall \lambda \in \mathbb{C}_+, \tag{60}$$

we find that

$$|u_1|_{1,\Omega} + |\lambda|\|u_1\|_\Omega \leq \\ C_\lambda(|g|_{1,\Omega} + \beta|f|_{1,\Omega} + \|h\|_\Omega + \beta\|g\|_\Omega + \beta(\beta+K)\|f\|_\Omega + \beta\|(\beta-\alpha)v\|_\Omega), \quad (61)$$

where $K = \max_\Omega(\alpha - \beta)$.

Now we exploit the dissipativeness relation (28) that here implies

$$-\Re((\lambda U - \mathcal{A}U, U)_{\mathcal{H}_0} = \int_\Omega (\alpha-\beta)|w|^2\, dx.$$

Using Cauchy-Schwarz's inequality and the fact that $(U,U)_{\mathcal{H}_0} \lesssim (U,U)_{\mathcal{H}}$, we find

$$\int_\Omega (\alpha-\beta)|w|^2\, dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

By Young's inequality this estimate implies that

$$\|(\alpha-\beta)w\|_\Omega \lesssim \|\sqrt{\alpha-\beta}w\|_\Omega \lesssim \varepsilon^{-1}\|F\|_{\mathcal{H}} + \varepsilon\|U\|_{\mathcal{H}}, \quad (62)$$

for all $\varepsilon > 0$. As (42) yields $\lambda v = w + g$, we get

$$|\lambda|\|(\alpha-\beta)v\|_\Omega \leq \|(\alpha-\beta)w\|_\Omega + \|(\alpha-\beta)g\|_\Omega,$$

and therefore, by (62)

$$|\lambda|\|(\alpha-\beta)v\|_\Omega \lesssim (1+\varepsilon^{-1})\|F\|_{\mathcal{H}} + \varepsilon\|U\|_{\mathcal{H}},$$

for all $\varepsilon > 0$. As we assumed here that $|\lambda| \geq 1$, this estimate in (4) leads to

$$|u_1|_{1,\Omega} + |\lambda|\|u_1\|_\Omega \lesssim C_\lambda\left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right), \quad (63)$$

for all $\varepsilon > 0$.

Now we come back to u, v and w. First, using (58), we have (recalling that β is constant)

$$|\lambda+\beta||u|_{1,\Omega} \leq |u_1|_{1,\Omega} + |f|_{1,\Omega},$$

which, by (63), yields

$$|\lambda+\beta||u|_{1,\Omega} \lesssim C_\lambda\left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right),$$

for all $\varepsilon > 0$. Recalling (60), we deduce that

$$|\lambda||u|_{1,\Omega} \lesssim C_\lambda\left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right), \quad (64)$$

for all $\varepsilon > 0$. By (41), we directly obtain

$$|v|_{1,\Omega} \lesssim C_\lambda \left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right), \tag{65}$$

for all $\varepsilon > 0$.

It remains to estimate the L^2-norm of w. For that purpose, we notice that $u_1 = \beta u + v$, hence

$$\|v\|_\Omega \le \beta\|u\|_\Omega + \|u_1\|_\Omega.$$

Therefore, using (63) and (64) (with Poincaré inequality), we get

$$|\lambda|\|v\|_\Omega \le \beta|\lambda|\|u\|_\Omega + |\lambda|\|u_1\|_\Omega \lesssim C_\lambda \left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right),$$

for all $\varepsilon > 0$. By (42), we get that

$$\|w\|_\Omega \lesssim C_\lambda \left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right), \tag{66}$$

for all $\varepsilon > 0$.

In conclusion, using this estimate and (64)-(66), there exists a positive constant C (independent of λ and ε) such that

$$\|U\|_{\mathcal{H}} = \|(u,v,w)\|_{\mathcal{H}} \le CC_\lambda \left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right),$$

for all $\varepsilon > 0$. Hence, choosing $\varepsilon = \frac{|\lambda|}{2CC_\lambda}$, we conclude that

$$\frac{1}{2}\|U\|_{\mathcal{H}} \le CC_\lambda(1 + 2CC_\lambda|\lambda|^{-2})\|F\|_{\mathcal{H}}, \tag{67}$$

which proves (3) for $|\lambda| \ge 1$. ■

Corollary 2 *Under the previous setting, if we suppose additionally that* (52) *holds for a continuous, positive, and non decreasing function M from* $[0,\infty)$ *into itself, then the semigroup $T(t) = e^{t\mathcal{A}}$, generated by $\mathcal{A}$, is bounded and the bound of the resolvent of $\mathcal{A}$*

$$\|(i\xi\mathbb{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \lesssim \max\{M(|\xi|), \frac{M(|\xi|)^2}{1+|\xi|^2}\}, \forall \xi \in \mathbb{R} \tag{68}$$

holds.

Proof. To prove the first statement, recall (see for instance Arendt et al., 2001) that the spectral bound of the operator $\mathcal{A}$ is defined by

$$s(\mathcal{A}) = \sup\,\{\Re\lambda \ : \ \lambda \in Sp(\mathcal{A})\},$$

while

$$s_0(\mathcal{A}) := \inf\ \{x > s(\mathcal{A})\ :\ \exists C_x > 0\ :\ ||(\lambda\mathbb{I} - \mathcal{A})^{-1}|| \leq C_x \text{ whenever } \Re\lambda > x\}.$$

By Theorem 5.2.1 in Arendt et al. (2001), we know that

$$\begin{aligned}\omega(T) &:= \inf\left\{\omega \in \mathbb{R}\ :\ \exists M_\omega > 0 \text{ such that } ||T(t)||_{\mathcal{L}(\mathcal{H})} \leq M_\omega e^{\omega t}, \forall t \geq 0\right\}\\ &= s_0(\mathcal{A}).\end{aligned} \tag{69}$$

First, owing to Lemma 7, $s(\mathcal{A}) \leq 0$. Secondly, by combining the estimates (51) and (3), we directly get

$$\|(\lambda\mathbb{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \lesssim \max\{1, \frac{1}{(\Re\lambda)^2}\}, \forall\lambda \in \mathbb{C} : \Re\lambda > 0.$$

This proves that

$$s_0(\mathcal{A}) \leq 0,$$

and by (4), we deduce that $T(t)$ is bounded.

Finally, the bound (68) is a direct consequence of (3) and (52), since $M(x) \geq M(0) > 0$. ■

This result, combined with the frequency domain approach yields the following decay rates of the semi-group generated by $\mathcal{A}$. We start with the exponential decay.

Corollary 3 *Assume that $\mathcal{A}_w$ generates a C_0-semigroup of contractions $(T_w(t))_{t\geq 0}$ that is exponentially stable, namely*

$$\|T_w(t)U\|_{\mathcal{H}_w} \leq Me^{-\omega t}\|U\|_{\mathcal{H}_w}, \forall U \in \mathcal{H}_w,$$

for some positive constants M and ω. Then, the semi-group $(e^{t\mathcal{A}})_{t\geq 0}$ is exponentially stable in $\mathcal{H}$.

Proof. By a well-known result, due to Huang and Prüss (see Prüss, 1984, and Huang, 1985) , $(e^{t\mathcal{A}_w})_{t\geq 0}$ is exponentially stable in $\mathcal{H}_w$ if and only if $i\mathbb{R} \cap \sigma(\mathcal{A}_w) = \emptyset$ and (52) with $M(x) = 1$ holds. Hence, by Corollary 2, (68) holds with $M(x) = 1$, and again applying the Huang/Prüss theorem to $\mathcal{A}$, we conclude. ■

For polynomial decays, we replace the Huang/Prüss theorem by the Borichev-Tomilov theorem (Borichev and Tomilov, 2010, Theorem 2.4) to obtain the next result.

Corollary 4 *Assume that the semi-group* $(e^{t\mathcal{A}_w})_{t\geq 0}$ *is polynomially stable in* $\mathcal{H}_w$*, namely there exists a positive real number* ℓ *such that*

$$||e^{t\mathcal{A}_w}U_0||_{\mathcal{H}_w} \lesssim t^{-\frac{1}{\ell}}||U_0||_{\mathcal{D}(\mathcal{A}_w)}, \quad \forall U_0 \in \mathcal{D}(\mathcal{A}_w), \quad \forall t > 1. \tag{70}$$

Then the semi-group $(e^{t\mathcal{A}})_{t\geq 0}$ *is polynomially stable in* $\mathcal{H}$*, i.e.,*

$$||e^{t\mathcal{A}}U||_{\mathcal{H}} \lesssim t^{-\frac{1}{2\ell-2}}||U||_{\mathcal{D}(\mathcal{A})}, \quad \forall U \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1,$$

if $\ell > 2$*, while if* $\ell \leq 2$*, one has*

$$||e^{t\mathcal{A}}U||_{\mathcal{H}} \lesssim t^{-\frac{1}{\ell}}||U||_{\mathcal{D}(\mathcal{A})}, \quad \forall U \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1.$$

Proof. As the semi-group generated by $\mathcal{A}_w$ is bounded and $i\mathbb{R} \cap \sigma(\mathcal{A}_w) = \emptyset$, by Borichev and Tomilov (2010, Theorem 2.4), (70) holds if and only if (52) holds with $M(x) = 1 + x^\ell$. As before, the conclusion follows with the help of Corollary 2, and again applying Borichev and Tomilov (2010, Theorem 2.4) to $\mathcal{A}$ by noticing that

$$\max\{1 + |\xi|^\ell, 1 + |\xi|^{2\ell-2}\} = \begin{cases} 1 + |\xi|^{2\ell-2}, \text{ if } \ell > 2, \\ 1 + |\xi|^\ell, \ \ell \leq 2. \end{cases}$$

■

For lower decay, the equivalence between the semi-group decay rate and the asymptotic behavior of the resolvent on the imaginary axis is not guaranteed, but by taking advantage of a result due to Batty and Duyckaerts (2008, Theorem 1.5) and our Corollary 2, we get as before the following corollary.

Corollary 5 *Assume that* (52) *holds with a continuous, positive, and non decreasing function* M *from* $[0, \infty)$ *into itself. Then, the semi-group* $(e^{t\mathcal{A}})_{t\geq 0}$ *has the following asymptotic decay in* $\mathcal{H}$*:*

$$||e^{t\mathcal{A}}U||_{\mathcal{H}} \lesssim \frac{1}{\tilde{M}_{\log}^{-1}\left(\frac{t}{C}\right)}||U||_{\mathcal{D}(\mathcal{A})}, \quad \forall U \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1,$$

for some positive constant C*, and* $\tilde{M}_{\log}$ *is defined by*

$$\tilde{M}_{\log}(x) = \tilde{M}(x)\left(\log(1 + \tilde{M}(x)) + \log(1 + x)\right), \forall x \geq 0,$$

with $\tilde{M}(x) = \max\{M(x), \frac{M(x)^2}{1+x^2}\}$.

Let us finish this section with some illustrative examples for which exponential, polynomial or logarithmic decay is available. Of course, due to our previous results, it suffices to mention the result for the wave equation.

Example 2 There are many cases for which the frictional damping in ω_0 is sufficient to guarantee the exponential stability of the wave equation (48). Let us mention a few of them.

1) By Rauch and Taylor (1974, Theorem 2) (see also Bardos, Lebeau and Rauch, 1988, and Lebeau, 1996), (48) is exponentially stable if the boundary of Ω is of class C^∞ and ω_0 satisfies the Geometric Control Condition (GCC). Recall that the GCC can be formulated as follows: For a subset ω of Ω, we shall say that ω satisfies the Geometric Control Condition if there exists $T > 0$ such that every geodesic traveling at speed one issued from Ω at time $t = 0$ intersects ω before time T.

2) From Lions (1988, Lemme VII.2.4) (see also Zuazua, 1990, Theorem 1.1 and Remark 1.2, or Haraux,1989, Exemple 3) (48) is exponentially stable if the boundary of Ω is of class C^2 and ω_0 is a neighborhood of $\bar{\Gamma}(x^0)$, for some $x^0 \in \mathbb{R}^d$, where

$$\Gamma(x^0) = \{x \in \partial\Omega \mid (x - x^0) \cdot \nu(x) > 0\},$$

$\nu(x)$ being the unit outward normal vector at $x \in \partial\Omega$.

3) From Haraux (1989, Exemple 1), (48) is exponentially stable if $d = 1$, $\Omega = (0, \ell)$ for some positive real number ℓ, and ω_0 is a non empty open subset of Ω.

4) In Liu (1997, Remark 4.3), further examples of pairs (Ω, ω_0) such that (48) is exponentially stable are given.

Example 3 Let us now mention some examples, for which the frictional damping in ω_0 guarantees the polynomial stability of the wave equation (48).

1) If Ω is the unit square and ω_0 contains a vertical strip of Ω, then the polynomial decay rate is demonstrated in Liu and Rao (2005) and Stahn (2017). Namely, in Liu and Rao (2005), assuming that

$$(a, b) \times (0, 1) \subset \omega_0,$$

for some $0 \le a < b < 1$, it is shown that (70) holds with $\ell = 2$. On the contrary if

$$(0, c) \times (0, 1) \subset \omega_0,$$

for $0 < c < 1$, it is shown in Stahn (2017) that (70) holds with $\ell = 3/2$.

2) If Ω is a partially rectangular domain and ω_0 contains the non-rectangular part of Ω, then it is proved in Burq and Hitrik (2007) (combined with Borichev

and Tomilov, 2010, Theorem 2.4) that (70) holds with $\ell = 2$.

3) Examples of domains Ω and ω_0 leading to (70) holding for some $\ell > 0$ can be found in Phung (2007).

Example 4 It was shown in Lebeau (1996) that if $\alpha - \beta$ is smooth and not identically equal to zero and if the boundary of Ω is smooth or convex, then (52) holds with $M(x) = e^{Cx}$, for some positive constant C. This yields

$$||e^{t\mathcal{A}}U||_{\mathcal{H}} \lesssim \frac{1}{\log t}||U||_{\mathcal{D}(\mathcal{A})}, \quad \forall U \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1,$$

since $\tilde{M}_{\log}^{-1}(t) \sim \log t$, for t large (see Batty and Duyckaerts, 2008, Example 1.6).

Remark 3 In the abstract setting from Section 2, let us assume that $\mathcal{A}_0 = \beta\mathcal{A}_1$ and

$$B = CC^* + \beta\mathbb{I}.$$

with C being a bounded operator from H into itself. Consider the wave type equation

$$\begin{cases} u_{tt} + \mathcal{A}_1 u + CC^* u_t = 0 & t > 0, \\ u(\cdot, 0) = u_0 \text{ and } u_t(\cdot, 0) = u_1. \end{cases} \tag{71}$$

This system generates a C_0-semigroup of contractions $(T_{aw}(t))_{t\geq 0}$ in $V \times H$ equipped with the inner product

$$((u, v)^\top, (u', v')^\top = a_1(u, u') + (v, v')_H.$$

If we assume that Lemma 7 holds, then, as before, one can prove that Theorem 3 holds.

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