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Constitutive parameters and their evolution[1]

by

Pauli Pedersen and Daniel A. Tortorelli

Department of Solid Mechanics,
Technical University of Denmark, DK-2800 Lyngby, Denmark

Abstract: The three-dimensional problem of anisotropic evolution is formulated and solved using the tools from optimal design. Basic assumptions are the principle of maximum energy dissipation and a function that relates the damage, as measured by the rate of a constitutive matrix norm, to an effective stress measure. The presentation relies on alternative description for the 21 constitutive parameters and only restricts the constitutive matrix to be semi-positive definite.

Keywords: anisotropic evolution, damage, stiffness and energy

1. Introduction

The present paper is similar to the one by Hammer and Pedersen (1997), with an extension from 2D to 3D problems. However, several differences should be noted. We shall here describe the constitutive matrix directly by its components and not by the engineering moduli. This means that constraints on definiteness must be included, but other formulations will then be more simple. Another difference relates to the strength criterion which is here modelled without an evolving "strength-surface". Instead, a damage rate function is postulated. This function, identified from experimental data, relates a stiffness norm rate to an effective stress measure.

Damage or degradation is by no means a well defined subject, but two different classes of modelling are clear. In the micromechanics approach the evolution of voids, cracks, etc. is directly calculated by, say, a finite element model. The macromechanics, i.e. the continuum approach, takes a more phenomenological view point: it compares directly with experimental values and assumes a smeared out damage. This continuum damage viewpoint is chosen in the present paper.

Damage may influence stiffness as well as strength. Although stiffness decreases, strength may increase as in plasticity models with hardening. Our main

[1]D.A. Tortorelli Guest Professor from University of Illinois at Urbana-Champaign, USA.

concern is the evolution of anisotropy and the total (and local) damage that will be measured by a constitutive matrix norm. We model evolution in time, the strength is indirectly given by the rate of damage as a function of an equivalent stress measure.

Under a fixed load two different results may occur. Either damage continues until collapse or equilibrium is obtained in a partly damaged state. This equilibrium could be obtained by changes in the stiffness field as well as in the strength field. In our modelling the strength field is kept unchanged and then equilibrium is only obtained by change in the stiffness field, i.e. by a stress redistribution. When equilibrium is obtained for a fixed load, the load may be incremented and a new equilibrium or collapse will be found. We shall assume the rate of load to be small compared to the rate of damage.

A local as well as a global damage distribution problem exist and have to be solved. In 3D problems with anisotropic evolution, we have 21 local parameters in the constitutive matrix. The relations between these are determined by the principle of maximum energy dissipation with constraints given by the semi-positive definite nature of the constitutive matrix. Thus, our experience from optimal design can be applied. The global distribution problem can be viewed from the same point of view, but here the given damage rate sets the constraints.

The layout of our presentation is as follows. Various descriptions of constitutive parameters are presented and although not all of them are used in the paper, we find that Section 2 is valuable in itself. These descriptions are given by 4th order tensor, 2nd order matrix, 1st order vector, eigenvalues, spectral decomposition and energy densities.

In Sections 3 and 4 the pointwise stiffness and energy descriptions (densities) are discussed. Norms and volumes are introduced and total stiffness as well as total elastic energy are then subjected to sensitivity analysis. Simplicity and importance are the keywords for these results that deserve to be better known.

The evolution model and the finite element discretization are given in the last sections, which also includes a description of the numerical procedure and examples.

2. Alternative descriptions for constitutive parameters

Traditionally the linear elastic constitutive relations between stress and strain are described by

$$\sigma_{ij} = \textit{Lijklfkl} \text{ or } \{a\text{-}\} = [L]\{f.\}$$
$$\epsilon_{ij} = \textit{Mijkl<7kl} \text{ or } \{E\} = [M]\{<7\} = [LJ\text{-}1\{<7\} \quad (1)$$

with second order tensors <7ij, Eij for stress, strain and fourth order tensors ***Lijkl, Mijkl*** for stiffness, compliance. The contracted notations with vectors {<7}, {f.} for stress, strain use the constitutive matrix [L], which for 3D problems is a quadratic, symmetric, semi-positive definite matrix of order six.

When the contracted stresses and strains are defined by

$\{\sigma\}^T$:= { cru 0-22 033 v'2cr1 2 V20-13 v'20-23}

$\{\epsilon\}^T$:= { cu c22 c33 v'2c:12 v'2c:13 v'2c:23} (2)

they transform from one Cartesian coordinate system to a rotated one by the orthonormal transformation matrix [T] ([TJ-^{1} = [T]I'), see Pedersen (1995). This implies that the constitutive matrix [L] correspondingly transforms with an orthonormal transformation matrix and most of the difficulties described by Ting (1986) are eliminated. In terms of the tensor components the complete constitutive matrix is

[L] =
[**Lnn** Lu22 Lu33 ; L2222 L2233 ; L3333] v'2 [*L,m* L1u3 ; L22l2 L2213 L2223 ; L3312 L3313 L3323]
symmetric 2 [*Lm,* L1213 L1223 ; L1313 L1323 ; L2323] (3)

with the invariant trace Tr being

Tr = Luu + L2222 + L3333 + 2(L12l2 + L1313 + L2323) (4)

and the invariant squared Frobenius norm F^2 being

F^2 = (Lirn + L 222 + L333) + 2(Lil22 + Li133 + L 233)
+ 4(Lil12 + Lim + Lil23 + L212 + L 213 + L 223 + L31 2
+ L313 + L3 23) (5)
+ 4(Li2l2 + Li313 + L323) + 8(Li213 + Li223 + Li323)

A further contraction of the constitutive matrix [L] to a constitutive vector {L}, defined by

$\{L\}^T$,— {{Luu L2222 L3333}, v'2{Lu22 Lu33 L2233},
2{L1u2 Lul3 Lu23 L22l2 L2213 L2223 L3312 L3313 L3323}, (6)
2{L12l2 L1313 L2323}, 2V2{L1213 L1223 L1323}}

makes it possible to determine an orthonormal matrix [R] ([RJ-^{1} = [Rf) of order 21 that describes rotations between mutual rotated Cartesian coordinate systems x and y simply by

{L}x = [R]{L}$_y$ {L}$_y$ = [Rf{L}x (7)

Note that the Frobenius norm F is the length of the contracted vector {L}

$$F^2 = \{L\}^T\{L\} \tag{8}$$

Along with these different notations for the constitutive relations by tensor, matrix or vector, we will also state the more indirect descriptions. The symmetric, semi-positive definite matrix [L] has six non-negative eigenvalues A_i and corresponding mutual orthogonal eigenvectors $\{Ai\}$ for $i = 1, 2, \ldots, 6$. We choose to order and normalize them by

$$
\begin{aligned}
&0 \;::;\; A1 \;::;\; A2 \;::;\; A3 \;::;\; A4 \;::;\; A5 \;::;\; A5 \\
&\{Ai\}^T\{A_j\} = D_j \\
&\{Ai\}^T[L]\{A_j\} = D_j\, Ai
\end{aligned}
\tag{9}
$$

where D_j is Kronecker delta. Again with general results from linear algebra we can describe the constitutive matrix by the spectral decomposition

$$
[L] = \underset{i=1}{\overset{6}{L}} Ai\{Ai\}\{Ai\}^T \tag{10}
$$

with $\{Ai\}\{Ai\}^T$ being the dyadic product of the i'th eigenvector. From these dyadic products we may construct vectors $\{Ai\}$ of order 21, see (6), and then alternatively write the spectral decomposition as

$$
\{L\} = \underset{i=1}{\overset{6}{L}} Ai\, \{Ai\} \tag{11}
$$

Finally, we see how constitutive parameters can be described in terms of energy densities, very much in parallel to the tools used in homogenization theory, as described in Pedersen (1997). Let us write out the elastic, strain energy density u for linear elasticity

$$
\begin{aligned}
u &= \frac{1}{2} L_{ijkl} E_{ij} E_{kl} = \frac{1}{2}\{E\}^T[L]\{E\} \qquad (12)\\
&= \frac{1}{2}((L_{nn}E_n^2 + L_{2222}E_{22}^2 + L_{3333}E_{33}^2) \\
&+ 2(L_{n22}E_nE_{22} + L_{n33}E_nE_{33} + L_{2233}E_{22}E_{33}) \\
&+ 4(L_{n12}E_nE_{12} + L_{n13}E_nE_{13} + L_{n23}E_{11}E_{23} + L_{2212}E_{22}E_{12} + L_{2213}E_{22}E_{13} \\
&+ L_{2223}E_{22}E_{23} + L_{3312}E_{33}E_{12} + L_{3313}E_{33}E_{13} + L_{3323}E_{33}E_{23}) \\
&+ 4(L_{1212}E_{i2} + L_{1313}E_{i3} + L_{2323}E_{?3}) \\
&+ 8(L_{1213}E_{12}E_{13} + L_{1223}E_{12}E_{23} + L_{1323}E_{13}E_{23}))
\end{aligned}
$$

If we define a vector of quadratic strains {E} of order 21, see (6),

$$
\begin{aligned}
\{E\} := \{&\{E_{i1}\; E_{?2}\; E_{?3}\}, V2\{E_n E_{22}\;\; E_n E_{33}\;\; E_{22} E_{33}\}, \\
&2\{E_{11} E_{12}\;\; E_n E_{13}\;\; E_{11} E_{23}\;\; E_{22} E_{12}\;\; E_{22} E_{13}\;\; E_{22} E_{23}\;\; E_{33} E_{12}\;\; E_{33} E_{13}\;\; E_{33} E_{23}\}, \\
&2\{E_{i2}\; E_{i3}\; E_{?3}\}, 2V2\{E_{12} E_{13}\;\; E_{12} E_{23}\;\; E_{13} E_{23}\}\}
\end{aligned}
\tag{13}
$$

then the strain energy density u for linear elasticity is given by a scalar product

$$u = \bar{u}\left(\{\hat{\epsilon}\}\right) = \frac{1}{2}\{\hat{\epsilon}\}^T\{L\} \tag{14}$$

Using the above we can express the constitutive parameters in terms of strain energy densities by defining only six elementary strain states and corresponding energies

$$\begin{array}{lll}
\text{UQOOOO} & := & u(\{E\}^T = \{\ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \}) \\
\text{UQQOOO} & := & u(\{E\}^T = \{\ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \}) \\
\text{uoo1000} & := & u(\{E\}^T = \{\ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \}) \\
\text{uooo100} & := & u(\{E\}^T = \{\ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \}) \\
\text{uoooo10} & := & u(\{E\}^T = \{\ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \}) \\
\text{uoooool} & := & u(\{E\}^T = \{\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \})
\end{array} \tag{15}$$

In this way the constitutive parameters are given by

$$\begin{array}{lll}
\text{Luu} & = & \text{2u100000},\ \text{L2222} = \text{2uo10000},\ L3333 = \text{2uoo1000} \\
\text{Lu22} & = & \text{UUOOOO-}\ \text{U100000}\ -\ \text{UQQOOO} \\
\text{Lu33} & = & \text{UQQOO}\ -\ \text{UQOOOO}\ \text{UQQOOO} \\
\text{L2233} & = & \text{uouooo}\ -\ \text{uo10000}\ -\ \text{uoo1000} \\
\text{2L1212} & = & \text{2uooo100},\ \text{2L1313} = \text{2uoooo10},\ \text{2L2323} = \text{2uoooool} \\
\text{2L1213} & = & \text{uooouo}\ -\ \text{uooo100}\ -\ \text{uoooo10} \\
\text{2L1223} & = & \text{Uooo101}\ -\ \text{uooo100}\ -\ \text{uoooool} \\
\text{2L1323} & = & \text{uoooo11}\ -\ \text{uoooo10}\ -\ \text{uoooool} \\
\text{V2Lu12} & = & \text{u100100}\ -\ \text{u100000}\ -\ \text{uooo100} \\
\text{V2L1m} & = & \text{u100010}\ -\ \text{u100000}\ -\ \text{uoooo10} \\
\text{V2L1123} & = & \text{u100001}\ -\ \text{u100000}\ -\ \text{Uooooo1} \\
\text{V2L2212} & = & \text{uo10100}\ -\ \text{uo10000}\ -\ \text{uooo100} \\
\text{V2L2213} & = & \text{u010010}\ -\ \text{uo10000}\ -\ \text{uoooo10} \\
\text{V2L2223} & = & \text{uo10001}\ -\ \text{uo10000}\ -\ \text{uoooo01} \\
\text{V2L3312} & = & \text{uoouoo}\ -\ \text{Uoo1000}\ -\ \text{uooo100} \\
\text{V2L3313} & = & \text{uoo1010}\ -\ \text{uoo1000}\ -\ \text{uoooo10} \\
\text{V2L3323} & = & \text{uoo1001}\ -\ \text{uoo1000}\ -\ \text{uoooool}
\end{array} \tag{16}$$

Note that the definitions in (15) for the shear strains include the factor J:2, say J2E12 = l. Also note that the factors on Lijkl in (16) are chosen to agree with those in the matrix of (3).

3. Local versus global stiffness and energy

The notion of stiffness is very broad and is used for material stiffness, laminate stiffness, finite element stiffness and many others, mostly with different physical dimensions. The constitutive matrix includes what we may classify as material stiffness, valid at a specific point and of dimension N/m^2.

Change in stiffness has physical meaning only when it takes place in a finite volume. We shall therefore introduce the notion of total stiffness Ψ_e for a domain e where we assume homogeneous constitutive parameters (or define mean values). The definition is

$$\Psi_e := V_e \|[L_e]\|_F = V_e F_e \tag{17}$$

where V_e is the volume of domain e and F_e is the Frobenius norm for the constitutive matrix $[L_e]$, see (8). It follows from the definition (17) that the total stiffness has dimension energy (Nm) but, as can be seen, is independent of the actual stress/strain state.

The accumulated total stiffness Ψ for an actual structure/continuum is

$$\Psi = \sum_e \Psi_e \tag{18}$$

and the change in total stiffness for the structure/continuum (amount of damage or degradation) $\Delta\Psi$ is thus

$$\Delta\Psi = \sum_e \Delta\Psi_e = \sum_e \left(V \frac{1}{F} \{L\}^T \{\Delta L\} \right)_e \tag{19}$$

For elastic strain energy, the local pointwise density is based on a linear assumption given by (12) or (14). For convenience, we use the description (14) for the mean strain energy density and then in a homogeneous domain e with volume V_e, the elastic strain energy U_e is

$$U_e = V_e \frac{1}{2} \{\hat{\epsilon}\}^T \{L\} \tag{20}$$

Accumulating over all domains gives the total elastic strain energy U

$$U = \sum_e U_e \tag{21}$$

The change in U for a fixed strain field is

$$\Delta U = \sum \Delta U_e = \sum_e \left(V \frac{1}{2} \{\hat{\epsilon}\}^T \{\Delta L\} \right)_e \tag{22}$$

In the present paper we present the formulations in strains, stiffness and elastic strain energy. However, a parallel formulation can be given in stresses, compliance and elastic stress energy.

4. Sensitivity analysis

Change in total elastic strain energy U, change in total stiffness ïïr and change in the status of being semi-positive definite with respect to changes in the constitutive parameters are the information we require to solve the evolution problem described in the next section. We will here derive these results, i.e. perform the sensitivity analysis.

A general result (see Cheng and Pedersen, 1997) makes it possible to perform a localized sensitivity analysis for the total elastic strain energy and we get

$$\frac{dU}{d(L_{ijkl})_e} = -\left(\frac{8U}{8(L_{ijkl})_e}\right)_{\text{fixed strains}} = \_Ve\left(\frac{OUe}{8(L_{ijkl})_e}\right)_{\text{fixed strains}}$$
$$= -\left(\frac{v!}{2}\{E\}^T\frac{8[£]}{8L_{ijkl}}\{E\}\right)_e = -\left(\frac{v!}{2}\{E\}^T\frac{d\{L\}}{dL_{ijkl}}\right)_e \tag{23}$$

The total stiffness ïïr is accumulated from domain stiffnesses ïïr$_e$, see (17) and (18), and hence for local dependence we obtain

$$\frac{\_dw\_}{d(L_{ijkl})_e} = \left(\frac{v}{dL_{ijkl}}\right)_e \; \frac{!!:!}{F}\left(\{L\}^T\frac{d\{L\}}{dL_{ijkl}}\right)_e \tag{24}$$

To determine whether [£] is semi-positive definite, we monitor the values of its lowest eigenvalues $>_1, >_2, \ldots$. Let us assume that $>_1$ is a simple eigenvalue (no multiplicity), then from the sensitivity analysis for eigenvalue problems it follows, see Haftka (1990)

$$\frac{d>1}{d(L_{ijkl})_e} = \left(\{A_1\}^T\frac{d[L]}{dL_{ijkl}}\{A_1\}\right)_e \tag{25}$$

A more complicated problem arises when multiple eigenvalues appear, say $>_1 = >_2$. Then, in principle all combinations of the eigenvectors $\{A\} = c_1\{A_1\} + c_2\{A_2\}$ are eigenvectors and we have by a sub-eigenvalue problem to locate the eigenvectors $\{A_i\}, \{A_2\}$ that do not couple in directional sensitivities, i.e.

$$\{A_i\}^T\frac{d[L]}{dL_{ijkl}}\{Ai\} = 0 \text{ for } i \# j \tag{26}$$

This technique is well known in eigenvalue sensitivity analysis, see Haftka (1990).

An alternative to the direct control of eigenvalues is to control the determinant I[£] I of [£], as suggested in Ringertz (1993). The sensitivity analysis for determinants gives

$$\frac{dI[L]I}{d(L_{ijkl})_e} = \left(I[L]I[L]^T\frac{d[L]}{dL_{ijkl}}\right)_e \tag{27}$$

see Gurtin (1981) or Carlson and Hoger (1986) for details.

We conclude this section by stating that the gradients of total elastic energy, of total stiffness and of lowest eigenvalues or determinant for each constitutive matrix is available in analytical form.

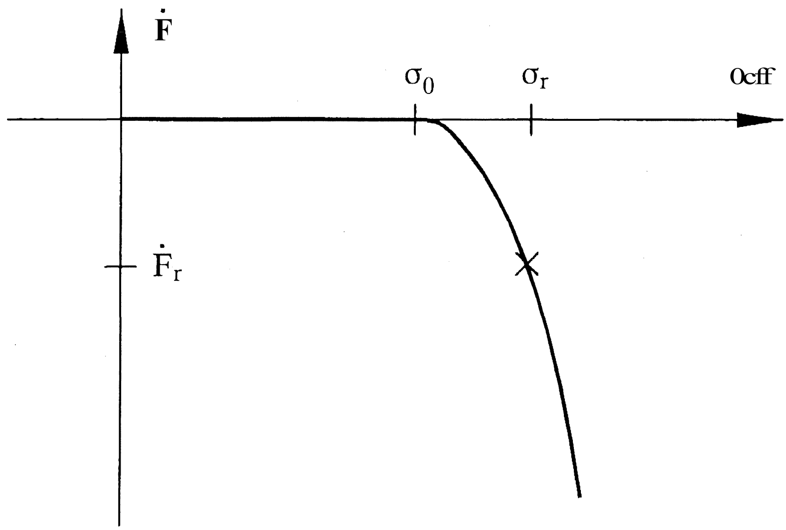

Figure 1. Illustration of the damage rate P as a function of effective stress.

5. Stiffness evolution

In a quasi-static formulation without influence from the inertia forces, we will describe the solution procedure in a finite element formulation. A global energy criterion controls the stiffness evolution of anisotropy and a local stress criterion controls the local rate of total stiffness evolution.

Given a fixed load case by the nodal load vector {A}, the equilibrium at time t is given by

$$[tSl\{tD\} = \{A\} \tag{28}$$

where [tS] is the secant stiffness matrix and {tD} is the resulting nodal displacement vector. From {tD} we find in each element e the strains $\{tE_e\}$ or $\{ti:_e\}$ and the stresses {to-} with the constitutive matrix $[tL_e]$

$$\{_t\sigma_e\} = [_tL_e]\{_t\epsilon_e\} \tag{29}$$

The effective stress in element e is $\sigma_{e,eff}$ and is defined from the stress vector by the function f

$$_t\sigma_{eff} = f(\{tO\text{-}_e\}) \tag{30}$$

This effective stress determines the rate of damage, as measured by P, i.e. the rate of the Frobenius norm of the constitutive matrix, see (8). In Fig. 1 we have illustrated such a dependence, but any experimentally obtained function can be applied. The function in Fig. 1 is described by

$$\dot{F} = 0 \text{ for Ueff :S uo}$$
$$F = \text{g(ueff)} = \text{((ueff - uo)/(ur - uo)tFr} \quad (31)$$

where u_0 is the stress limit where damage begins and the parameters u_0, U_r, n and Fr (Fr < 0) are obtained from experiments. The rate of the Frobenius norm, expressed in the constitutive vector {L} is

$$\dot{F} = \{L\}^T\{\dot{L}\}/F \approx \{L\}^T\{\Delta L\}/(F\Delta t) \quad (32)$$

where flt is the specified finite time increment.

Based on the knowledge from equations (28)-(32) we now want to find the equilibrium at time $t+flt$

$$[_{t+\Delta t}S]\{_{t+\Delta t}D\} = \{A\} \quad (33)$$

The element stiffnesses at time $t+flt$, i.e. $[_{t+6t}S_e]$ will be a function of $\{_{t+6t}L\} = \{_tL\} + \{tflL\}$; although this is notationally complicated, the calculation is straightforward.

The constitutive increments $\{flL_e\}$ for all the elements must satisfy the conditions of a semi-positive definite constitutive matrix $[_{t+6t}L]$ and the evolution function i.e. equations (31) and (32). Still, there will normally be several solutions that satisfy the equilibrium i.e. equation (33). We choose the solution that maximizes the energy dissipation in the time step flt, i.e. we look for the feasible increments $\{flL_e\}$ that

$$\text{Maximize } flU \quad (34)$$

where flU is the increment in the total elastic energy.

6. Numerical procedure and example

The problem of stiffness evolution, as described in section five, can be solved by different numerical procedures, but in view of highly nonlinear nature they must all be based on iteration. Two solution procedures are possible, mathematical programming or optimality criteria, as described in detail in the literature on optimal design. In the methods of mathematical programming we use information at the actual iteration step while in optimality criteria methods we use information related to the unknown solution.

In other words the problem to be solved is: at time $t+flt$ find nodal displacement vector $\{_{t+6t}D_e\}$, and all the element constitutive matrices $[_{t+6t}L_e]$, that satisfy equilibrium (33) and give maximum energy dissipation flU subject to the evolution rate (as shown in Fig. 1) and definiteness constraints. To solve this problem we use the finite element method to first solve the local problem, i.e. to find the element material stiffnesses that maximize the compliance for a given displacement (i.e. strain), we then update the displacement to satisfy equilibrium.

Using the penalty method to enforce the positive definiteness constraint on [L] (Ringertz, 1993), the local problem is expressed, at the centroid of each element, as

$$\min\left(\frac{1}{2\Delta t}\{E\}^T(\{L\} - \{{}^tL\}) + \mu \ln \det [L]\right)$$

$$\text{s.t.}\ \frac{\{L\}^T}{F\Delta t}(\{L\} - \{{}^tL\}) = g(\sigma_{eff}) \tag{35}$$

where we used equations (23) to note that maximizer of the compliance increment ΔU is equivalent to the minimizer of ½{EF({L } - {tL }). The constraint in the above follows from equation (32) and we choose σ_{eff} proportional to the octahedral shear stress, i.e. proportional to the von Mises stress. An implicit time integration algorithm is used and as such all quantities are evaluated at time t + Δt unless otherwise noted.

Stationarity of the above gives

$$\begin{aligned} \frac{1}{2\Delta t}\{E\} + \mu\{L^{-1}\} &= \lambda\left(2\frac{\{L\} - \{{}^tL\}}{F\Delta t} - \frac{\{L\}\{L\}^T\{{}^tL\}}{F^3\Delta t} - \{\nabla g\}\right) \\ \frac{\{L\}^T}{F\Delta t}(\{L\} - \{{}^tL\}) &= g(\sigma_{eff}) \end{aligned} \tag{36}$$

where $\{\nabla g\}$ is the gradient of g with respect to {L } and where $\{L^{-1}\}$ is the vector representation of $[L]^{-1}$. The above nonlinear equations are solved by Newton's method to determine {L } and λ for the given strain {E}. After {L } and λ are evaluated, a direct differentiation sensitivity analysis is used to evaluate the derivative $d\{L\}/d\{E\}$. This derivative is required to compute the element tangent matrix. Details of this efficient sensitivity analysis appear in Michelaris et al. (1994) in the context of incremental plasticity analysis.

Newton's method is also used on the global level to compute the nodal displacement vector. The internal force vector, i.e. residual {R}, for each element is computed from the Gaussian quadrature

$$\{R(\{D\})\} = \sum_I [B]^T[L][B]\{D\}Jw. \tag{37}$$

where [B] the strain-displacement matrix, {D} the element nodal displacement vector, J the determinant of the isoparametric Jacobian and w Gaussian weighting factor, are evaluated at the Gaussian point I location, unless otherwise noted. The derivative of the above is the, not symmetric, element tangent stiffness matrix, and for this calculation we require the derivative $d\{L\}/d\{D\} = (d\{L\}/d\{E\})[B]$.

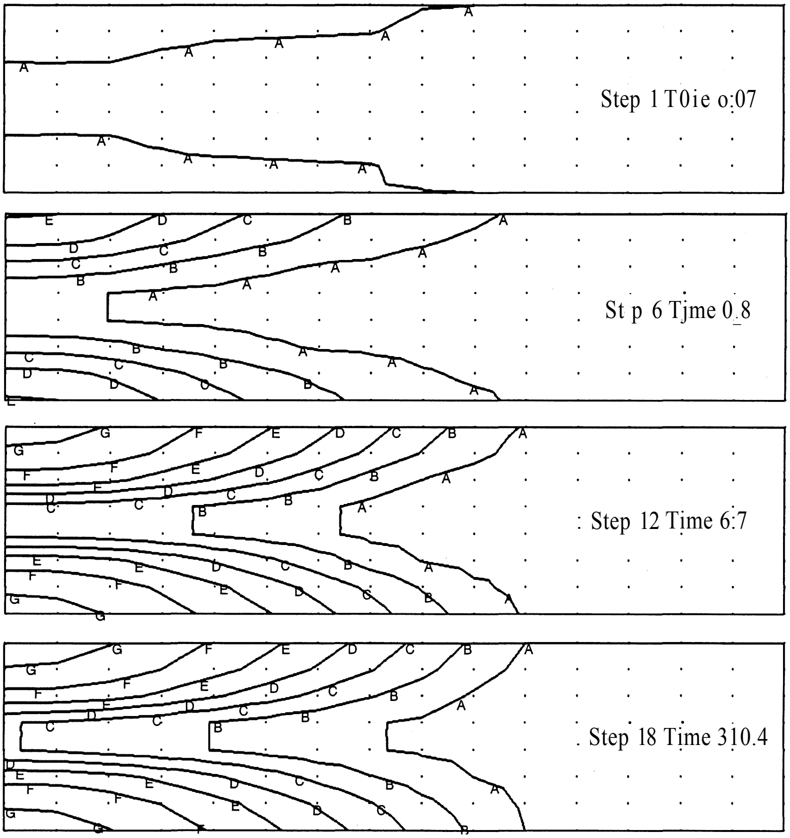

Figure 2. Evolution of the Frobenius norm F of the constitutive matrix. In units (x 10^3) the levels are: A = 224, B = 205, C = 186, D = 170, E = 148, F = 129, G = 110.

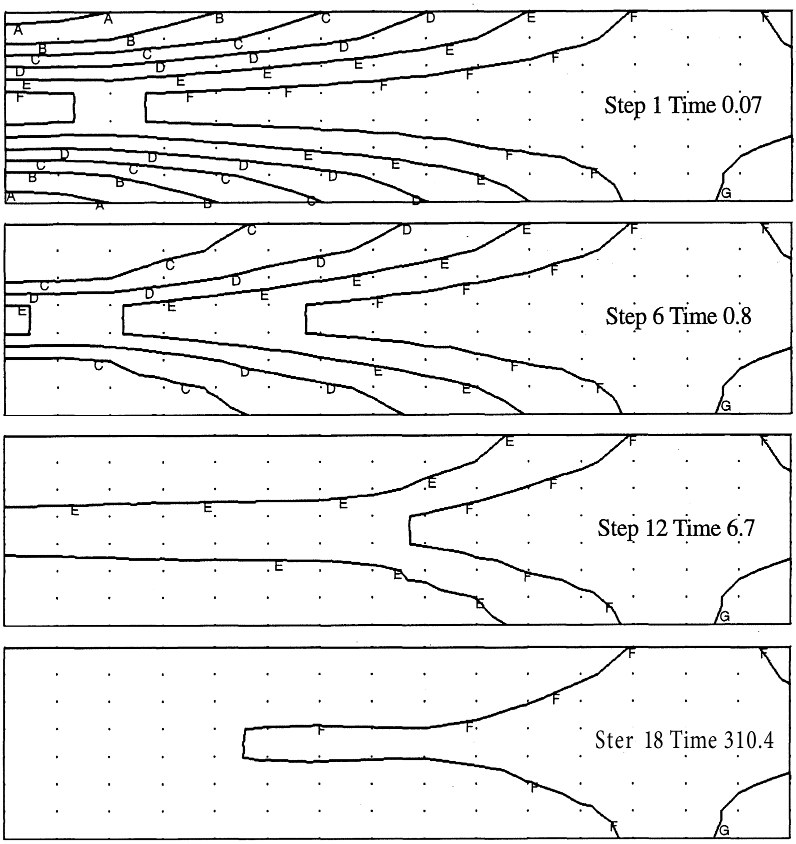

Figure 3. History of the effective a_{eff}, the levels are: $A = 260$, $B = 221$, $C = 182$, $D = 143$, $E = 104$, $F = 65.1$, $G = 26.1$.

The element residual and tangent stiffness matrix are assembled to form the global residual and tangent stiffness matrix in the usual manner. Thereafter, a Newton iteration is performed to update the value of the nodal displacement vector until convergence is obtained at which occasion the time is incremented and the procedure begins anew. It is emphasized that the evaluations of {L} and d{L}/d{E} are performed locally in each element whereas the evaluation of the nodal displacement vector is performed at the global level, i.e. as per usual.

To exemplify this model a cantilever beam is analyzed in two-dimensions. The 30 x 7 beam, discretized by a 15 x 7 mesh, is fixed on the left and subject to a 100 unit vertical load at the top right corner. The plane stress model uses an initially isotropic material with a Young's modulus of 100,000 and a Poisson's ratio of 0.3. The parameters for the function g of equation (31) are as follows: aa = 100, Q' = 110, n = 2 and Fr = -500 and the penalty parameter value (35) is μ = 10. The analysis is performed for 310 time units at which time steady-state condition is obtained.

The main result of the analysis appears in Figs. 2 and 3. Fig. 2 displays contours of F at various instants in time. As seen in the figure, the stiffness matrix norm deteriorates from the outer edge where the bending stress is the greatest. The effective stress relaxes at the outer edges as time progresses as witnessed by the contour plot of Fig. 3.

The evolution of anisotropy is shown in Figs. 4 and 5. Fig. 4 displays contours of the ratio £2222/£1111 at various instants in time (initially for the isotropic material we had £2222 = £1111 everywhere). With dominating En strains £1111 is degradating faster. Fig. 5 displays contours of the ratio £1212/£1111 also at the various instants in time.

7. Conclusion

In the present paper we have shown many alternative descriptions of the constitutive tensor and especially the contracted vector notation has shown useful.

Anisotropy will evolve during damage and thus in general for a 3D-problem, we have to model the evolution of 21 constitutive parameters. It is suggested that this model is based on a principle of maximum energy dissipation with constraints on the semi-positiveness of the constitutive matrix.

As an alternative to a strength model we describe the rate of stiffness change as a function of an effective stress measure. Experimental backup for this modelling is not given, but this should be attempted.

The modelling is applied to a finite element formulation and a numerical procedure for solution of the inherent optimization problem is described. A numerical example is presented.

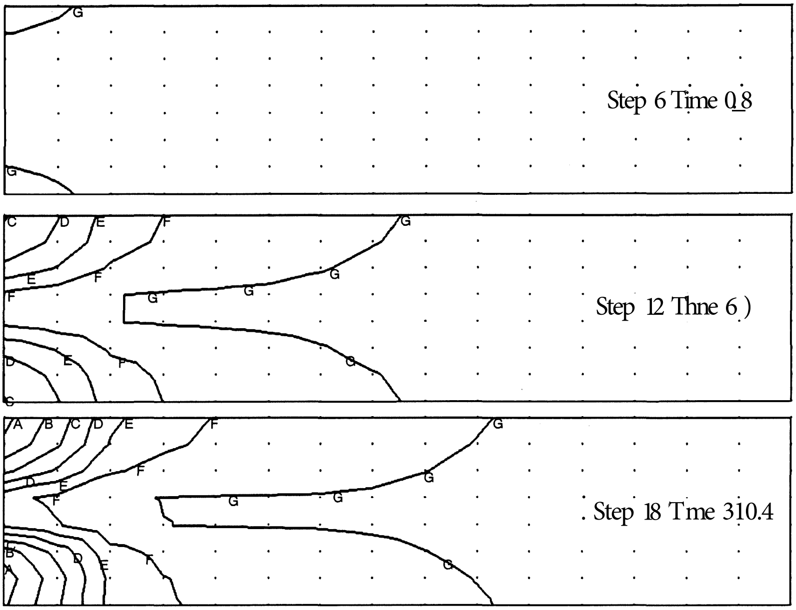

Figure 4. Evolution of anisotropy, exemplified by the ratio $£_{2222}/£_{1111}$.The stronger degradation of $£_{1111}$ shows clearly. The levels are: $A = 14$, $B = 12$, $C = 10$, $D = 8$, $E = 6$, $F = 4$, $G = 2$.

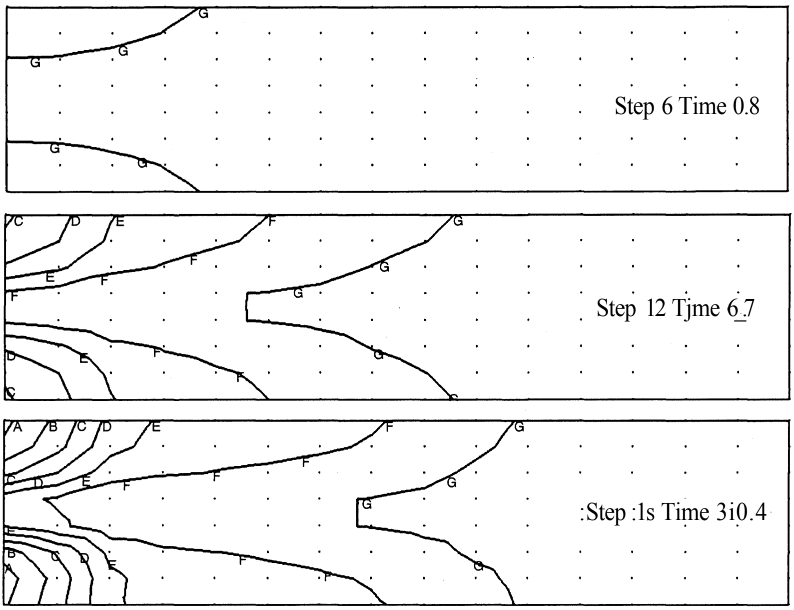

Figure 5. Evolution of anisotropy, exemplified by the ratio L_{1212}/L_{11n}. The stronger degradation of L_{nn} is again dominant. The levels are: $A = 7$, $B = 6$, $C = 4.9$, $D = 3.9$, $E = 2.8$, $F = 1.8$, $G = 0.8$.

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