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Legendre polynomials method in time-optimal control of linear single - input systems

by

Jerzy Pelczewski

Faculty of Mathematics, University of L6dz,
S.Banacha 22, 90-238 L6dz, Poland
e-mail: jerpel©imul.uni.lodz.pl

Abstract: The paper deals with application of shifted Legendre polynomials in the time-optimal control problem for a linear, time invariant, undisturbed, single-input system. It was assumed that the normality condition of the time-optimal control is satisfied, the state matrix is nonsingular, and all its eigenvalues are real nonpositive. The method of evaluating the approximate switching instants of the bang-bang control is presented. The proposed computational procedure is based on the solution of algebraic matrix equation, which corresponds to the differential state equation, and was obtained according to the properties of Legendre polynomials.

Keywords: Legendre polynomials, time-optimal control, single-input linear systems

1. Introduction

General solution of the time optimal control problem for linear, time-invariant, undisturbed systems is well known. The optimal bang-bang control, the state-vector and the the costate vector must satisfy the canonical equations. Hence, the applied methods are based on the solution of a set of ordinary differential equations, describing the state and costate systems, while one of them is unstable. The initial and final conditions are known only for the state. These facts cause some inconveniences in application of numerical methods.

Another method, which avoids the numerical integration of differential equations was presented by Pelczewski (1988). The solution of the state-equation of the linear system was used, with fundamental matrix found by straightforward computation, according to results obtained in Pelczewski (1987).

A different approach to the optimal control is related to properties of orthogonal polynomials and has been investigated by many authors. Thus, for instance, already in 1965 Kulikowski (Kulikowski, 1965) presented the use of

Chebyshev polynomials for a particular case of the time-optimal control. In the last 15 years the general methods of Hermite, Jacobi, Laguerre, Chebyshev and Legendre polynomials were well established and applied in several optimal control problems.

In this paper the shifted Legendre polynomials method for the considered time-optimal control is presented. It is based on publications of Chang, Wang (1983), Chou, Horng (1985), Horng, Chou (1986), Hwang, Chen (1985), Paraskevopoulos (1985), Perng (1986), Shih, Kung (1986), and Wang, Chen (1983). The approximate values of switching instants were found after having solved the algebraic matrix equations for particular time intervals of the bang-bang control. The obtained effective computational procedure justifies the choice of Legendre polynomials method.

2. Time-optimal control of the single-input system

Let us consider the linear, time invariant, single-input system described by the state-equation

$$x(t) = Ax(t) + Bu(t) \tag{1}$$

where x, u are n-dimensional state vector and scalar control signal, respectively, A, B are constant matrices of appropriate dimensions, t is time.

We accept the following assumptions:

1. The state vector is unbounded
$$x(t) \in R^n \tag{2}$$
2. The control signal satisfies the constraint
$$|u(t)| \le U_{max} \tag{3}$$
3. The matrix A is nonsingular
$$\det A \neq 0 \tag{4}$$
and all its eigenvalues are real nonpositive,
4. The normality condition of the time-optimal control is satisfied
$$\det\,[B|AB|A^2B| \ldots\ldots\ldots |A^{n-1}B] \neq 0 \tag{5}$$

It is known that the time-optimal control, which transfers the system considered from the given initial state $x(t_0)$ at $t = t_0$ to the desired final state x_f, while minimizing the performance index

$$I = \int_0^t dt \tag{6}$$

is of the bang-bang type, with at most n switching intervals. Without loss of generality we put $t_0 = 0$ and obtain the time optimal control $u^*(t)$ of the form:

$$u^*(t) = \begin{cases} 0 & \forall t < 0 \\ \sigma U_{max} & \forall t \in [0, t_1) \\ -\sigma U_{max} & \forall t \in [t_1, t_2) \\ \cdot & \cdot \\ \cdot & \cdot \\ (-1)^{n-1} \sigma U_{max} & \forall t \in (t_{n-1}, t_n) \\ U_c(t) & \forall t \geq t_n \end{cases} \tag{7}$$

where σ denotes the sign of the control signal in the first time interval $[0, t_1)$ and $t_1, t_2, \ldots, t_n$ - the optimal switching instants, such that at $t = t_n$ the state becomes $x(t^*) = x_f$. The control signal $u_c(t)$ corresponds to the desired system's behaviour for $t \geq t_n$.

The switching instants can be found by computation, minimizing - as a function of $t_1, t_2, \ldots, t_n$ - the euclidean norm

$$N(t_n) = \| x_f - x(t_n) \| = \sqrt{\sum_{i=1}^{n} [x_{fi} - x_i(t_n)]^2} \tag{8}$$

representing in R^n the distance between $x(t_n)$ and x_f.

In this procedure the consecutive integrations of the state equation are needed. Another method - like the one in Pelczewski (1988) - is based on evaluation of the state vector's transients, after the straithforward computation of system's fundamental matrix e^{At}, according to Pelczewski (1987).

A different approach is presented in this paper. In order to find the vector $x(t)$ on the time interval $[t_0, t_n)$ we will apply the Legendre polynomials method and solve the algebraic matrix equation, obtained from the state equation. That will enable us to find the approximate values of the switching instants.

According to the properties of the shifted Legendre polynomials the control signal $u(t)$ and the state vector $x(t)$ can be approximately expressed on the time interval $[0, t_1]$ in the following form

$$u(t) = F s_m(t), \qquad \forall t \in [0, t_1] \tag{9}$$

$$x(t) = D s_m(t), \qquad \forall t \in [0, t_1] \tag{10}$$

where F, D are constant matrices of dimensions $1 \times m$ and $n \times m$ respectively

$$F = [f_{10} \quad f_{11} \quad \cdot \quad \cdot \quad \cdot \quad f_{1(m-1)}] \tag{11}$$

$$D = \begin{bmatrix} d_{10} & d_{11} & \cdot & \cdot & \cdot & \cdot & d_{1(m-1)} \\ d_{20} & d_{21} & \cdot & \cdot & \cdot & \cdot & d_{2(m-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{n0} & d_{n1} & \cdot & \cdot & \cdot & \cdot & d_{n(m-1)} \end{bmatrix} \tag{12}$$

and $s_m(t)$ is a vector of dimension m x 1 - like (49) in the Appendix:

$$s_m(t) = [\,s_o(t)s_1(t)s_2(t)\ldots\ldots s_{m-1}(t)\,]^T$$

whose components $s_j(t), j = 0, 1, 2, \ldots, m-1$ are shifted Legendre polynomials on the time interval $[0, t_1]$ - like in the formulae (39), (40) and (41) in the Appendix. It must be underlined that the approximation of expressions in (9) and (10) depends on the number m of vector's $s_m(t)$ components and that the accuracy increases with m.

Substituting u(t) and x(t) from (9) and (10) into the state equation (1), we obtain

$$D = V(0) + ADH_m + BFH_m \tag{13}$$

where V(0) denotes the following n x m matrix

$$V(0) = \begin{bmatrix} x_1(0) & 0 & . & . & 0 \\ x_2(0) & 0 & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ x_n(0) & 0 & . & . & 0 \end{bmatrix} \tag{14}$$

and H_m is the m x m operational matrix of vector's $s_m(t)$ integration - like (51) in the Appendix.

In the case where the control signal u(t) on the time interval $[0, t_1]$ is known, we find the elements $f_{ij}, i = 1, 2, \ldots n, j = 0, 1, 2, \ldots, m-1$ of the matrix F in (9), according to (44) of the Appendix

$$f_{ij} = \frac{2j+1}{t_1}\int_0^{t_1} u(t)s_j(t)dt \tag{15}$$

$j = 0, 1, 2, \ldots, m-1$

Introducing the known matrix F into (13) and finding the unknown matrix D, we obtain from (10) the approximate solution x(t) of the state equation (1), corresponding - on the time interval $[0, t_1]$ - to the given control signal u(t).

It was shown by Pelczewski (1990), that the solution D of the algebraic matrix equation (13) exists, if and only if there is a nonsingular matrix T, satisfying the equation

$$\begin{bmatrix} A^{-1} & 0 \\ 0 & H_m \end{bmatrix} = T \begin{bmatrix} A^{-1} & C \\ O & H_m \end{bmatrix} T^{-1} \tag{16}$$

This solution is unique if and only if the matrices A^{-1} and $-H_m$ have no eigenvalues in common.

C in (16) denotes the following n x m matrix

$$C = A^{-1}V(0) + A^{-1}BFH_m \tag{17}$$

Assuming that the above conditions of existence and uniqueness are satisfied, we write the equation (13) in the form

$$D - ADHm = K \tag{18}$$

where

$$K = V(0) + BFHm \tag{19}$$

The solution of equation (18) according to Lancaster, Tismenetsky (1985), is

$$vecD = W^{-1} vecK \tag{20}$$

Here *vecD* and *vecK* are following nm x 1 vectors

$$vecD = \begin{bmatrix} d_0 \\ d_1 \\ \cdot \\ \cdot \\ \cdot \\ d_{m-1} \end{bmatrix}, \tag{21}$$

$$vecK = \begin{bmatrix} k_0 \\ k_1 \\ \cdot \\ \cdot \\ \cdot \\ k_{m-1} \end{bmatrix} \tag{22}$$

where $d_0, d_1, \ldots, d_{m-1}$ are columns of the matrix D and $k_0, k_1 \ldots, k_{m-1}$ - columns of the matrix K

$$D = [d_0 \quad d_1 \quad . \quad . \quad . \quad d_{m-1}] \tag{23}$$

$$K = [k_0 \quad k_1 \quad . \quad . \quad . \quad k_{m-1}] \tag{24}$$

By W in (20) the nm x nm matrix is denoted of the form

$$W = [I_{nm} \quad - \quad H@A] \tag{25}$$

where I_{nm} is the unit matrix of dimension nmxnm and@ denotes the Kronecker product.

In the case of bang-bang control the input signal *u(t)* in consecutive time-intervals is: uŪmax, -O"Ūmax, OŪmax, ..., $(-1)^{n-1}$OŪmax. Hence we obtain from (15)

$$u(t) = OU_{max} \{:\} (F)u = [O'_{max} U \quad 0 \quad . \quad . \quad . \quad 0] \tag{26}$$

$$u(t) = \text{-CJUmax } \{c\} \ (F) - u = \left[\text{-C!}\bar{U}max \quad O \quad \bullet \quad \bullet \quad \bullet \quad 0\right] \tag{27}$$

With

$$B = \begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ b_{n1} \end{bmatrix} \tag{28}$$

and Hm from (51) in the Appendix, we find for the time intervals in which $u(t) = CJUmax = canst$

$$B(F)a - \bar{H}m = \frac{t1}{2C!Umax} \begin{bmatrix} bn & bn & 0 & \cdot & \cdot & \cdot & 0 \\ b21 & b21 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ bnl & bnl & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \tag{29}$$

hence

$$(K)u = \begin{bmatrix} (x1(0) + \text{₺}C!Umaxbn) & \text{₺}C!\bar{U}maxbil & 0 & \cdot & \cdot & \cdot & 0 \\ (x2(0) + \text{¥}0'\bar{U}maxb21) & \text{¥}0'Umaxb21 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ (xn(0) + \text{¥}C!Umaxbn1) & \text{₺}C!Umaxbnl & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \tag{30}$$

For the time intervals in which $u(t) = \text{-CJUmax} = canst$ we have

$$B(F) - a - \bar{H}m = -\frac{t1}{2C!Umax} \begin{bmatrix} bn & bn & 0 & \cdot & \cdot & \cdot & 0 \\ b21 & b21 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ bnl & bnl & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \tag{31}$$

hence

$$(K) - u = \begin{bmatrix} (x1(0) - \text{¥}C!Umaxb11) & -\text{₺}C!Umaxbll & 0 & \cdot & \cdot & \cdot & 0 \\ (x2(0) - \text{¥}C!\bar{U}maxb21) & -\text{¥}C!Umaxbll & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (xn(0) - \text{¥}C!Umaxbn1) & -\text{¥}C!Umaxbnl & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \tag{32}$$

We observe that in matrices (30) and (32) only the elements of the first and the second columns are not equal to zero. From (22) we obtain

$$\operatorname{vec}(K)_\sigma = \begin{bmatrix} (k_0)_\sigma \\ (k_1)_\sigma \\ \mathbf{0} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{0} \end{bmatrix} \tag{33}$$

$$\operatorname{vec}(K)_{-\sigma} = \begin{bmatrix} (k_0)_{-\sigma} \\ (k_1)_{-\sigma} \\ \mathbf{0} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{0} \end{bmatrix} \tag{34}$$

and with

$$\left.\begin{array}{l} \operatorname{vec} D = W^{-1} \operatorname{vec}(K)_\sigma \ \text{ for } u(t) = \sigma U_{max} = \text{const} \\ \operatorname{vec} D = W^{-1} \operatorname{vec}(K)_{-\sigma} \ \text{ for } u(t) = -\sigma U_{max} = \text{const} \end{array}\right\} \tag{35}$$

we form the matrix D, according to (23), and obtain from (10) the state vector $x(t)$.

For a given set of switching instants we can evaluate the matrix D for consecutive time intervals and find at their end the state vector. Thus, for instance, for the *g-th* time interval, putting $x(0) = x(t_{g-1})$ and $t_{1g} = t_g - t_{g-1}$ we obtain the matrix D_g. According to (41) in the Appendix, at $t = t_g$ the values of all vector's $s_m(t)$ components are $s_j(t_g) = 1, j = 0,1,2,\ldots,m-1$. Hence with matrix D_g, corresponding to the *g-th* time interval, we find at its end the state vector

$$x(t_g) = \begin{bmatrix} \sum_{j=0}^{m-1} (d_{1j})_g \\ \sum_{j=0}^{m-1} (d_{2j})_g \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=0}^{m-1} (d_{nj})_g \end{bmatrix} \tag{36}$$

Assuming the set of values $t_1, t_2, \ldots, t_n$ of switching instants, we can find consecutively $x(t_1), x(t_2), \ldots, x(t_n)$ and the norm $N(t_n)$ from (8). Minimizing it as a function of $t_1, t_2, \ldots, t_n$, we obtain for our time-optimal control problem the approximate solution, i.e., such a set of switching instants, which corresponds to the final state $x(t_n)$, belonging to the imposed neighbourhood of x_f.

3. Computational procedure

The above results enable us to formulate the computational procedure for the considered time-optimal control problem. For a given system described by the state equation (1) we know the initial condition x(O), the target xf, and the value of $e(t_n)$, imposed according to the needed approximation of the final solution. It means that the computation will be finished at $t = t_n$ such that

$$N(t_n) \leq e(t_n) \tag{37}$$

Taking into consideration the desired accuracy of results, we choose the number m of vector's $s_m(t)$ components. Next, we must evaluate CJ- the sign of the control signal u(t) in the first time interval. In some cases it can be done directly for the given initial and final conditions. But generally, applying the Legendre polynomials method we compute, for the given x(O) and assumed final time t_1, the transients of x(t), caused by the control signal u(t) = CJUmax = canst, for both CJ= 1 and CJ= -1. For these transients we find the norms N(t) as functions of current time

$$N(t) = llx1 - x(t)II = \sqrt{\sum_{i=1}^{n} [XJi - xi(t)^2]} \tag{38}$$

and comparing them we fix the value of(]", corresponding to minimum N(t).

After these preliminary preparations we proceed as follows:

1. We choose the set of switching times ii, $t_2, \ldots, t_n$ for the first computation.
2. In the first time interval t E [O, t_1) we put $t_{19} = t_{11} = t_1$ and find (for the above fixed value of Q) the matrix K_1 from (30) or (32). Next we form vecK_1 - like in (33) or (34).
3. We compute for the first time interval the matrix W_1 from (25), and its inverse.
4. From (35) we find vecD_1 and form - like in (23) - the matrix D_1. For $t = t_1$ we obtain from (36) the state vector $x(t_1)$ at the end of the first time interval.
5. In the second time interval we change CJ to -CJ, put x(O) = $x(t_1)$ and $t_{19} = t_1 2 = t_2 - t_1$. Proceeding analogously like in 2° to 4° above, we find K_2, vecK2, W_2, w_2^{-1}, vecD_2, D_2 and $x(t_2)$.
6. We apply the same procedure for the next, consecutive time intervals, finding x(t3),x(t4), ... ,$x(t_n)$ and the norm $N(t_n)$ at the end of the last interval.
7. If $N(t_n)$ satisfies the condition (37), the set of switching instants chosen in 1° corresponds (with accepted approximation) to the solution of the considered time-optimal control problem. In the opposite case we minimize the norm $N(t_n)$ as the function of ti, $t_2, \ldots, t_n$. Proceeding like in 2° to 6° we arrive at the result, satisfying the condition (37). The applied numerical procedure must enable us to find the approximate solution in the desired neighbourhood of the global minimum of the norm $N(t_n)$-

REMARK 3.1 *The obtainable accuracy of the final result depends on the choice of the number* m *of shifted Legendre polynomials which are the components of the vector* $s_m(t)$. *In the case where for a given* m *we cannot satisfy the condition (31), the value of* m *must be increased.*

4. Numerical example

For the system described by (1) with

$$A = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & -4 & 3 & 3 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}; \qquad B = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

the time-optimal control, transfering the state from $x(0) = [20 \quad -10 \quad 40 \quad -30]^T$ to $x_f = 0$, has to be found. The constraint of the form (3) imposed on the control vector is $\bar{u}_{max} = 8$. The final results must satisfy the accuracy given by $e(t1) = 0.2$.

After preliminary computations the value $\sigma = +1$ and the number m = 12 of vector's $s_m(t)$ components were fixed. The initial set of switching times:

$$t1 = 1; \; t2 = 1.5; \; t3 = 1.9, \; t4 = tn = 2.1$$

was chosen for the first computation.

The procedure based on the Legendre polynomials method was applied and the following results were found:

$$x1(t4) = -1.0254; \; x2(t4) = -2.2558; \; x3(t4) = -0.66443;$$

$$x4(t4) = -2.7120;$$

$$N(t_4) = 3.73317 > e(t4) = 0.2.$$

The condition (37) is not satified, hence the norm $N(t_4)$ was minimized as a function of switching times. The final results are: $t_1 = 1.0367902$; $t_2 = 1.6205391$; $t_3 = 1.9458606$; $t_4 = 2.0457417$; $X1(t4) = -0.027411$; $X2(t4) = -0.092158$; $X3(t4) = 0.13555$; $X4(t4) = -0.11118$; $N(t4) = 0.199949$.

5. Concluding remarks

The method presented, based on properties of Legendre polynomials, enables us to find the approximate solution of the time-optimal control problem for a linear single input system. An effective computational procedure is applied in the case, where the external disturbances do not affect the system, the conditions (2), (3), (4), (5), (16) are satisfied and the matrices $A^{-1}, -H_m$ have no eigenvalues in common. By consecutive algebraic operations the distance between the target state and the state at the end of the last time interval is minimized - as a function of switching times.

The accuracy of results depends on the choice of m - the number of shifted Legendre polynomials used in the approximate expressions for the control signal and for the state vector.

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Appendix

The properties of Legendre polynomials are given according to Horng, Chou (186), Hwang, Chen (1985), Perng (1986), Shih, Kung (1986), Szego (1975), Wang, Chang (1983).

The shifted Legendre polynomials on the time interval $[0, t_f]$ are defined as

$$s_j(t) = \sum_{h=0}^{j} \frac{(-1)^h(2j-h)!}{h!^2[(j-h)!]}\left(\frac{t}{t_f}\right)^{j-h} \tag{39}$$

For $j = 0, 1, 2, \ldots$ we obtain from (39)

$$\left.\begin{aligned}
so(t) &= 1 \\
s1(t) &= 2(,t) - 1 \\
s2(t) &= 6(-k)2 - 6(-k) + 1 \\
s_3(t) &= 20(,t)^3 - 30(,t)^2 + 12(,t) - 1 \\
s_4(t) &= 70(,t)^4 - 140(,t)^3 + 90(,t)^2 - 20(,t) + 1 \\
S5(t) &= 252(,t)^5 - 630(,t)^4 + 560(,t)^3 - 210C:)^2 + 30C:) - 1
\end{aligned}\right\} \tag{40}$$

etc.

$$\left.\begin{aligned}
s_j(0) &= (-1)^j \\
s_j(t_f) &= 1
\end{aligned}\right\} \tag{41}$$

The shifted Legendre polynomials are orthogonal on the interval $[0, t_f]$

$$I_o^{tt} sj(t)s_p(t)dt = \{ \begin{matrix} 0 \text{ for } p \neq j \\ \frac{t_f}{2j+1} \text{ for } p = j \end{matrix} \tag{42}$$

The scalar function y(t) on the interval $[0, t_1]$ can be represented by the following sum of shifted Legendre polynomials

$$y(t) = \sum_{j=0}^{\infty} CjSj(t), \ t \in [0, t_1] \tag{43}$$

where

$$G = \frac{2j \pm 1}{t_1} \int_o^{tJ} y(t)sj(t)dt \tag{44}$$

Considering only the first m Legendre polynomials $s_0(t), s_1(t), \ldots, s_{m-1}(t)$ we obtain the approximate expression for $y(t)$

$$y(t) \quad \sum_{j=0}^{m-1} c_j s_j \tag{45}$$

The vector function $w(t)$ can be represented approximatively as follows

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \cdot \\ \cdot \\ \cdot \\ w_n(t) \end{bmatrix} \cong \begin{bmatrix} \sum_{j=0}^{m-1} c_{1j} s_j(t) \\ \sum_{j=0}^{m-1} c_{2j} s_j(t) \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=0}^{m-1} c_{nj} s_j(t) \end{bmatrix} \tag{46}$$

or

$$w(t) \cong C_m s_m(t) \tag{47}$$

where C_m is a following n x m matrix, whose elements in particular rows were found from (44) for the corresponding components of $w(t)$

$$C_m = \begin{bmatrix} c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{1(m-1)} \\ c_{20} & c_{21} & \cdot & \cdot & \cdot & c_{2(m-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n0} & c_{n1} & \cdot & \cdot & \cdot & c_{n(m-1)} \end{bmatrix} \tag{48}$$

and $s_m(t)$ is a vector of dimension m x 1, whose components are shifted Legendre polynomials: $s_j(t), j = 0, 1, \ldots, m - 1$

$$s_m = [s_0(t), s_1(t), \ldots\ldots, s_{m-1}(t)]^T \tag{49}$$

The integral of $s_m(t)$

$$\int_0^t s_m(t)dt = \begin{bmatrix} \int_0^t s_0(\tau)d\tau \\ \int_0^t s_1(\tau)d\tau \\ \cdot \\ \cdot \\ \cdot \\ \int_0^t s_{m-1}(\tau)d\tau \end{bmatrix} = H_m s_m(t) \tag{50}$$

where $\bar{H}_m$ denotes the operational m x m matrix of vector's $s_m(t)$ integration

$$H_m = \frac{t_J}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2m-3} & 0 & \frac{1}{2m-3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{2m-1} & 0 \end{bmatrix} \qquad (51)$$