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Caratheodory theory for the Bernoulli problem*

by

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Abstract: The variational formulation of the interior Bernoulli free boundary problem is considered. The problem is formulated as follows. Choose an arbitrary bounded simply connected domain $G \subset \mathbb{R}^2$ and smooth positive functions $g : \partial G \to \mathbb{R}$, $Q : G \to \mathbb{R}$. Denote by $\mathscr{C}$ the totality of all connected compact sets $\omega \subset G$, such that the flow domain $\Omega = G \setminus \omega$ is double-connected. The notation $\mathscr{C}^+ \subset \mathscr{C}$ stands for the totality of the set $\omega \in \mathscr{C}$ of positive measure. The cost function $\mathcal{J}(\omega)$ is defined by the equalities

$$\mathcal{J}(\omega) = \int_\Omega \big(|\nabla u|^2 + Q^2 \big)\, dx,$$

$$\Delta u = 0 \;\text{ in }\; \Omega, \;\; u = g \;\text{ on }\; \partial G, \;\; u = 0 \;\text{ on }\; \partial\omega.$$

We prove that, under the natural nondegeneracy assumption, the variational problem $\min_{\omega \in \mathscr{C}^+} \mathcal{J}(\omega)$ has a solution $\omega \in \mathscr{C}^+$. The approach is based on the methods of complex variables theory and the potential theory. The key observation is that every subset of $\mathscr{C}$, separated from ∂G is sequentially compact with respect to the Caratheodory-Hausdorff convergence.

Keywords: Bernoulli problem, shape optimization, potential theory

1. Introduction

This paper is devoted to the application of the theory of holomorphic functions to shape optimization problems. We focus on the Bernoulli's free boundary problems to illustrate our approach. This problem is formulated as follows.

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Assume that $G \subset \mathbb{R}^2$ is a bounded simple connected domain with C^∞ boundary. Fix two functions, $g \in C^\infty(\partial G)$, $Q \in C^\infty(G)$, such that

$$g > c_g > 0 \text{ on } \partial G, \quad Q > c_Q \text{ in } G. \tag{1.1}$$

The problem is to find a compact set $\omega \Subset G$ and a potential $u : G \setminus \omega$ satisfying the following equations and boundary conditions

$$\begin{aligned} \Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial G, \quad & u = 0 \text{ on } \partial\omega, \\ & \partial_n u = Q \text{ on } \partial\omega, \end{aligned} \tag{1.2}$$

where n is the outward normal to $\partial\omega$, the domain $\Omega = G \setminus \omega$. This problem has many physical applications, see Crank (1984), Fasano (1992), or Flucher and Rumpf (1997), and references therein. There is a growing massive literature devoted to the existence theory and the regularity properties of solutions to the Bernoulli problem, see, e.g., Aguilera, Alt and Cafarelli (1986), Alt and Cafarelli (1981), Beurling (1957), Cafarelli and Spruck (1982), Daniljuk (1972), Flucher and Rumpf (1997), Hamilton (1982) or Haslinger et al. (2004). Note that there are two kinds of solutions to the Bernoulli problem. They are named elliptic and hyperbolic solutions. The theory usually deals with the elliptic solutions. The difficult hyperbolic case was considered in Henrot and Onodera (2021).

The Bernoulli problem admits the weak variational formulations. The most general is the the Alt-Cafarelli problem

$$\min J(u), \quad J(u) = \int_G |\nabla u|^2 dx + \int_{u>0} Q^2\, dx.$$

Here, the minimum is taken over the class of functions $u \in W^{1,2}(G)$, satisfying the boundary condition $u = g$ on ∂G. The existence and regularity of solutions to this problem was established in Aguilera, Alt and Cafarelli (1986) and Alt and Cafarelli (1981). In general case, the domain $\Omega = \{u > 0\}$ is multiply connected, see Acker (1980) for examples. Hence, the topology of Ω is not defined. Our goal is to prove the existence results for the problem with fixed topology. We focus on the simplest case of doubly-connected domain Ω. The following definition describes the class of admissible domains.

Definition 1 *We denote by $\mathscr{C}$ the totality of all connected compact sets $\omega \subset G$, such that $\Omega = G \setminus \omega$ is open and connected, that is, Ω is doubly-connected. We also denote by $\mathscr{C}^+ \subset \mathscr{C}$ the totality of all elements of $\mathscr{C}$ with the positive 2D measure $|\omega| =$ meas $\omega > 0$.*

We thus come to the following variational problem.

$$\min_{\omega \in \mathscr{C}^+} \mathcal{J}(\omega), \quad \mathcal{J}(\omega) = \int_\Omega |\nabla u|^2 dx + \int_\Omega Q^2\, dx, \quad \Omega = G \setminus \omega, \tag{1.3}$$

where u is a solution to the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega, \ u = g \text{ on } \partial G, \ u = 0 \text{ on } \partial\omega. \tag{1.4}$$

Since ω is an arbitrary compact connected set, the meaning of the boundary condition at $\partial\omega$ should be refined. Further on, we deal with weak solutions of problem (1.4). In order to give a weak formulation of this problem, we introduce the following notation.

DEFINITION 2 *Denote by $\mathscr{W}(\Omega)$, $\Omega = G \setminus \omega$, the linear space of functions $u \in W^{1,2}(G)$ with the following property: Each function $u \in \mathscr{W}$ vanishes on some neighborhood of ω. We assume that $\mathscr{W}$ is endowed with the norm $W^{1,2}(G)$, which is equivalent to the norm*

$$\|u\|_1^2 = \int_\Omega |\nabla u|^2 dx + \int_{\partial G} |u|^2 \, ds.$$

We also denote by $\mathcal{H}^1$ the completion of $\mathscr{W}$ with respect to the norm $\|\cdot\|_1$.

It is worthy noting that if the condenser capacity of ω (see Section 4 for definitions) equals zero, then $\mathcal{H}^1 = W^{1,2}(\Omega)$. It is clear that every function $u \in \mathcal{H}^1$ has the extension $u^* : G \to \mathbb{R}$ such that

$$\|u^*\|_{W^{1,2}(G)} \le c\|u\|_1.$$

In particular, u has the trace $u\big|_{\partial G} \in W^{1/2}(\partial G)$.

DEFINITION 3 *Let $\omega \in \mathscr{C}$. We say that u is a weak solution to problem* (1.4) *if*

$$u \in \mathcal{H}^1(\Omega), \quad u = g \text{ on } \partial G, \tag{1.5}$$

$$\int_\Omega |\nabla u|^2 \, dx = \min_{\tilde{u} \in \mathscr{J}} \int_\Omega |\nabla \tilde{u}|^2 \, dx, \tag{1.6}$$

where $\mathscr{J} \subset \mathcal{H}^1$ is the set of all functions $\tilde{u} \in \mathcal{H}^1$, satisfying the boundary condition $\tilde{u} = g$ on ∂G. In particular, the integral identity

$$\int_\Omega \nabla u \cdot \nabla \zeta \, dx = 0 \tag{1.7}$$

holds for all $\zeta \in C_0^\infty(\Omega)$.

Note that the affine space $\mathscr{J}$ is convex and weakly closed. The Dirichlet integral in the left hand side of (1.6) defines the strictly convex and coercive functional on $\mathscr{J}$. Therefore, problem (1.6) has a unique solution $u \in \mathscr{J}$. The following theorem is the main result of this paper.

Theorem 1 *Assume that G, g, and Q satisfy the following nondegeneracy condition. There is $\omega_0 \in \mathscr{C}^+$ such that*

$$\mathcal{J}(\omega_0) < \int_G |\nabla U|^2 dx + \int_G Q^2 dx,$$

where U is the solution to the Dirichlet problem

$$\Delta U = 0 \quad in \quad G, \qquad U = g \quad on \quad \partial G.$$

Then the variational problem (1.3) *has a solution $\omega \in \mathscr{C}^+$.*

Note that the nondegeneracy condition is fulfilled if the constant c_Q in (1.1) is sufficiently large. The remainder of the paper is devoted to the proof of Theorem 1.

The paper is organized as follows. In Section 2 we give definitions of various types of convergence of domains of the Euclidean plane. In particular, we introduce the notion of convergence of plane domains in the Caratheodory-Hausdorff sense, which differs from the generally accepted notion of convergence in the Mosco sense. In Section 3 we present basic facts about conformal mappings of multiply connected domains. Section 4 is devoted to the estimates of the capacity of two-dimensional condensers. In Section 5 we consider solutions of the Dirichlet problem for harmonic functions in doubly-connected domains. We study the stability of these solutions regarding perturbations of the domains in the Caratheodory-Hausdorff topology. The last section of the paper is devoted to the proof of the main theorem.

2. Preliminaries

2.1. Notation

In this section, we consider the various kinds of convergence of plane domains. First, we introduce the notations, which will be used throughout the paper.

Further, the notation G stands for a bounded simple-connected domain in space $\mathbb{R}^2$ of points $x = (x_1, x_2)$. We assume that ∂G is a smooth curve of class C^∞. Notice that G is diffeomorphic to the unit disc $B = \{|x| < 1\}$.

We denote by G_h, $h > 0$, the compact set

$$G_h = \{x \in G : \text{ dist } (x, \Gamma) \geq h\}.$$

It follows from the smoothness conditions imposed on ∂G that there is $h_0 > 0$, depending on G only, such that for every $h \in (0, h_0)$, the set

$$\Sigma_h = \partial G_h = \{x \in G : \text{ dist } (x, \partial G) = h\}$$

is a Jordan curve of the class C^∞. Throughout the paper we will assume that $h \in (0, h_0)$. It follows that the curvilinear annulus

$$\mathcal{A}(h) = G \setminus G_h \tag{2.1}$$

is a double-connected domain with C^∞ boundary $\partial G \cup \Sigma_h$. Starting from this point we assume that G and h are fixed. It is convenient to introduce the following definition:

Definition 4 *By $\mathscr{C}_h$ we denote the class of all connected compact sets $\omega \subset G_h$, such that the open sets $\Omega = G \setminus \omega$ are connected. In other words, Ω is a doubly connected domain for every $\omega \in \mathscr{C}_h$.*

Note that the class $\mathscr{C}_h$ includes the degenerate sets of capacity zero and even isolated single points.

2.2. Domain convergence

Hausdorff convergence

Recall the definition of the Hausdorff metrics. If ω and π are compact subsets of $\mathbb{R}^2$, then the Hausdorff distance $d_H(\omega, \pi)$ is defined by the equality

$$d_H(\omega, \pi) = \inf\{\varepsilon > 0 : \omega \subset \pi_\varepsilon \text{ and } \pi \subset \omega_\varepsilon\}.$$

Here

$$\omega_\varepsilon = \{x : \text{ dist } (x, \omega) \equiv \inf_{z \in \omega} |x - z| < \varepsilon\}$$

is the ε-neighborhood of ω, and π_ε is the ε-neighborhood of π. The following two simple lemmas will be used throughout this section.

Lemma 1 *If a compact set $\omega \subset \mathbb{R}^2$ is not connected, then there are two compact sets ω', ω'' with the following properties: $\omega = \omega' \cup \omega''$ and* dist $(\omega', \omega'') > 0$. *This means that there is $\epsilon > 0$ such that $\omega'_\epsilon \cap \omega''_\epsilon = \emptyset$, where ω'_ϵ and ω''_ϵ are ϵ-neighborhoods of ω' and ω''.*

Proof We will consider ω as a compact metric space $\mathbf{O}$, equipped with the standard $\mathbb{R}^2$ metric. A set $E \subset \omega$ is open in $\mathbf{O}$ if there is an open set $\mathcal{G} \subset \mathbf{R}^2$, such that $E = \omega \cap \mathcal{G}$. Conversely, the set $E \subset \omega$ is closed in $\mathbf{O}$ if there is a closed set $\mathcal{F} \subset \mathbb{R}^2$, such that $E = \mathcal{F} \cap \omega$. The set ω is connected if and only if it has the following property, see Dieudonné (1960, Ch. 3, Sec. 19). If subset $\omega' \subset \mathbf{O}$ is closed and open, then $\omega' = \mathbf{O}$ or $\omega' = \emptyset$. If ω is not connected, then there is a nonempty subset $\omega' \subset \omega$, such that ω' is closed and open in $\mathbf{O}$ and $\omega'' = \omega \setminus \omega' \neq \emptyset$.

If ω' is closed in $\mathbf{O}$, then it is closed and compact in $\mathbb{R}^2$. Since ω' is open in $\mathbf{O}$, for every point $z_0 \in \omega'$ there is a circle of radius $\varepsilon > 0$, centered at z_0, such that

$$\omega' \cap \{|z - z_0| < \varepsilon\} \subset \omega'.$$

Since ω' is a compact set, the finite collection of such circles covers ω'. Hence, there is a ε-neighborhood ω'_ε of ω' such that $\omega \cap \omega'_\varepsilon = \omega'$. Therefore, the set

$$\omega'' = \omega \setminus \omega' = \omega \setminus \omega'_\epsilon$$

is compact and the distance between ω' and ω' is greater than ε. This completes the proof of Lemma 1. ■

LEMMA 2 *If a sequence of connected compact sets $\omega_n \subset G$ converges to a compact set ω in the Hausdorff metric, then $\omega \subset \overline{G}$ is connected.*

PROOF Suppose, contrary to our claim, that ω is not connected. By virtue of Lemma 1, there are compact sets ω', ω'', and $\epsilon > 0$ such that

$$\omega = \omega' \cup \omega'', \quad \omega'_\epsilon \cap \omega''_\epsilon = \emptyset,$$

where ω'_ϵ and ω''_ϵ are ϵ-neighborhoods of ω' and ω''. It follows from the definition of the Hausdorff metric that dist $(\omega_n, \omega) \to 0$ as $n \to \infty$. In particular, dist $(\omega_n, \omega) < \epsilon/2$ for all sufficiently large n. We thus get $\omega_{n,\epsilon/2} \subset \omega_\epsilon$. Since $\omega_{\epsilon/2} = \omega'_{\epsilon/2} \cup \omega''_{\epsilon/2}$, we have

$$\omega_n = \omega'_n \cup \omega''_n, \quad \text{where} \quad \omega'_n = \omega_n \cap \omega'_{\epsilon/2}, \; \omega''_n = \omega_n \cap \omega'_{\epsilon/2}.$$

Obviously, the sets ω'_n, ω''_n are nonempty and dist $(\omega'_n \cup \omega''_n) > \epsilon$. Hence, ω_n are disconnected for all sufficiently large n. The contradiction proves the lemma. ■

REMARK 1 *The totality of all compact subsets $\omega \subset G_h$, equipped with the Hausdorff metric, is a compact metric space. In particular, every sequence $\omega_n \in \mathscr{C}_h$, $n \geq 1$, contains a subsequence, which converges in the Hausdorff metric to some compact set $\omega_\infty \subset G_h$. It follows from Lemma 2 that ω_∞ is connected. However, ω_∞ does not belong to the class $\mathscr{C}_h$ in the general case, since $\Omega_\infty = G \setminus \omega_\infty$ may be not connected. We will return to this question at the end of the section.*

Kernels. Caratheodory convergence to kernel

Recall the definition of the kernel of a sequence of domain.

DEFINITION 5 *We say that a domain $A \subset \mathbb{R}^2$ is the kernel of a sequence of domains $A_n \subset \mathbb{R}^2$ with respect to a point $z_0 \in A$ if*

A.1 *$z_0 \in A_n$ for all n.*

A.2 *Every compact set $\omega \subset A$ belongs to domains A_n for all sufficiently large n.*

A.3 *A is a maximal domain, satisfying conditions* (**A.1**)-(**A.2**).

DEFINITION 6 *We say that a sequence of domains $A_n \subset \mathbb{R}^2$, containing a fixed point z_0, converges in the sense of Caratheodory to a domain $A \ni z_0$ if A is a kernel of each subsequence $\{A_k\} \subset \{A_n\}$.*

Caratheodory–Hausdorff convergence

Let us consider the following construction. Recall Definition 4 of the set $\mathscr{C}_h$. Choose an arbitrary sequence $\omega_n \in \mathscr{C}_H$, such that $\omega_n \to \omega_\infty$ in the Hausdorff metric. The limiting set $\omega_\infty \subset G_h$ is connected and compact. In its turn, the open set $G \setminus \omega_\infty$ is a union of a countable set of disjoint, maximal, open, connected components. Moreover, $G \setminus \omega_\infty$ contains the annulus $\mathcal{A}_h$. Therefore, there is the only maximal connected component of $G \setminus \omega_\infty$, which contains $\mathcal{A}_h$. We denote this component by Ω. The other maximal open connected components of $G \setminus \omega_\infty$ we denote by $\boldsymbol{\pi}_n$, $n \geq 1$. We thus get

$$\Omega = G \setminus \omega, \quad \omega = \omega_\infty \bigcup \Big(\bigcup_n \boldsymbol{\pi}_n \Big). \tag{2.2}$$

THEOREM 2 *The set ω, defined by* (2.2), *is compact and connected. In particular, Ω is a doubly connected domain. It is a kernel of the sequence $\Omega_n = G \setminus \omega_n$ with respect to an arbitrary point $z_0 \in \mathcal{A}_h$. The domains Ω_n converge to Ω in the sense of Caratheodory.*

PROOF Let us prove that ω is compact. To this end, it suffices to show that ω contains all its limiting points. Suppose, contrary to our claim, that the limiting point z_0 of ω does not belong to ω. Since $\omega \in G_h$, we have $z_0 \in G_h \subset G$. Hence

$$z_0 \in G \setminus \omega = \Omega.$$

There is a sequence $z_n \in \omega$, $n \geq 1$, such that

$$\omega \ni z_n \to z_0 \in \Omega \quad \text{as} \quad n \to \infty.$$

Only a finite number of the elements z_n belong to ω_∞. In the opposite case, there is a subsequence $\{z_{n_k}\} \subset \{z_n\}$ such that $z_{n_k} \in \omega_\infty$. Since ω_∞ is compact, we have $z_0 \in \omega_\infty \subset G \setminus \Omega$, which is impossible. Hence,

$$z_n \in \omega \setminus \omega_\infty = \bigcup_n \boldsymbol{\pi}_n \quad \text{for all sufficiently large} \quad n.$$

In particular, $z_n \in \boldsymbol{\pi}_{k_n}$ for some sequence k_n. Since Ω is open, there is an open circle $B_r = \{|z - z_0| < r\} \subset \Omega$. Obviously, $z_n \in B_r$ for large n. Hence,

$$z_n \in \boldsymbol{\pi}_{k_n} \cap B \subset \boldsymbol{\pi}_{k_n} \cap \Omega,$$

and the connected sets $\boldsymbol{\pi}_{k_n}$ and Ω have nonempty intersection. From this and the generic properties of connected sets in metric spaces, see Dieudonné (1960, Ch. 3, Sec. 19, **n** 3.19.3), the open set $\boldsymbol{\pi}_{k_n} \cup \Omega \subset G \setminus \omega_\infty$ is connected. This contradicts the fact that Ω is a maximal connected component of $G \setminus \omega_\infty$.

Now our task is to prove that the compact set ω, defined by (2.2), is connected. We will consider ω as the metric space $\mathbf{O}$, equipped with the standard $\mathbb{R}^2$ metric. If ω is not connected, then it follows from Lemma 1 that there are two nonempty compact sets ω', ω'', and $\epsilon > 0$ with the following properties.

$$\omega = \omega' \cup \omega'', \quad \omega'_\epsilon \cap \omega''_\epsilon = \emptyset, \quad \omega', \omega'' \neq \emptyset, \tag{2.3}$$

where ω'_ϵ and ω''_ϵ are open ϵ- neighborhoods of ω' and ω''. Note that ω' and ω'' are open and closed in $\mathbf{O}$. It follows that

$$\omega = \omega'_{\epsilon/2} \cup \omega''_{\epsilon/2}, \text{ dist } (\omega'_{\epsilon/2}, \omega''_{\epsilon/2}) > \epsilon > 0, \ \omega' = \omega'_{\epsilon/2} \cap \omega, \ \ \omega'' = \omega \cap \omega''_{\epsilon/2}. \tag{2.4}$$

Let us prove that one of the sets $\omega'_{\varepsilon/2} \cap \omega_\infty$ and $\omega''_{\epsilon/2} \cap \omega_\infty$ is empty. We have

$$\begin{aligned}\big(\omega''_{\epsilon/2} \cap \omega_\infty\big) \cup \big(\omega''_{\epsilon/2} \cap \omega_\infty\big) = \big(\omega''_{\epsilon/2} \cup \omega''_{\epsilon/2}\big) \cap \omega_\infty \\ = \omega \cap \omega_\infty = \omega_\infty.\end{aligned}$$

On the other hand, relations (2.4) imply

$$\text{dist } \big(\omega'_{\epsilon/2} \cap \omega_\infty, \ , \ \omega''_{\epsilon/2} \cap \omega_\infty\big) > \epsilon > 0.$$

If both sets $\omega'_{\varepsilon/2} \cap \omega_\infty$ and $\omega''_{\epsilon/2} \cap \omega_\infty$ are not empty, then ω_∞ consists of two nonempty components, and the distance between them is positive. This contradicts the connectedness of ω_∞. Hence, one of these components, say $\omega'_{\epsilon/2} \cap \omega_\infty$, is empty. From this and (2.4) we conclude that

$$\begin{gathered}\omega' = \omega'_{\varepsilon/2} \cap \omega = \omega'_{\varepsilon/2} \cap \Big(\omega_\infty \bigcup \Big(\bigcup_n \boldsymbol{\pi}_n\Big)\Big) = \\ \Big(\omega'_{\varepsilon/2} \cap \omega_\infty\ \Big) \bigcup \Big(\omega'_{\varepsilon/2} \cap \Big(\bigcup_n \boldsymbol{\pi}_n\Big)\Big) = \Big(\omega'_{\varepsilon/2} \cap \Big(\bigcup_n \boldsymbol{\pi}_n\Big)\Big).\end{gathered}$$

Note that the sets $\omega'_{\varepsilon/2}$ and $\cup_n \boldsymbol{\pi}_n$ are open in $\mathbb{R}^2$. Hence, ω' is an open subset of $\mathbb{R}^2$. On the other hand, ω' is closed in the metric space $\mathbf{O}$. This means that $\omega' = \omega \cap \mathcal{F}$ for some closed set $\mathcal{F} \subset \mathbb{R}^2$. Since ω is a compact subset of $\mathbb{R}^2$, it follows that ω' is closed in $\mathbb{R}^2$. Therefore, the bounded set ω' is open and closed in $\mathbb{R}^2$. Hence, it is empty, which contradicts (2.3). Hence, ω is connected.

It remains to prove that the domain $\Omega = G \setminus \omega$ is a kernel of the sequence $\Omega_n = G \setminus \omega_n$ with respect to an arbitrary point $z_0 \in \mathcal{A}(h)$. We begin with the observation that $\mathcal{A}(h) \subset \Omega$ and $\mathcal{A}(h) \subset \Omega_n$, since $\omega_n \in \mathscr{C}_h$. Choose an arbitrary compact set $K \Subset \Omega$. Note that $\Omega \cap \omega_\infty$ is the empty set. Hence, dist $(K, \omega_\infty) > \epsilon$ for some $\epsilon > 0$. Since dist $(\omega_n, \omega_\infty) \to 0$ as $n \to \infty$, the inequalities dist $(K, \omega_n) > \epsilon/2 > 0$ hold for all sufficiently large n. This means that

$$K \subset G \setminus \omega_n = \Omega_n \ \text{ for all large } \ n.$$

Now we prove that Ω is the maximal domain with this property. Let $\Omega' \supset \Omega$ be a domain (connected, open subset of G) such that every compact set $K \subset \Omega'$ belongs to domains Ω_n for all large n. Our task is to show that $\Omega' = \Omega$. We begin with the observation that the intersection of Ω' and ω_∞ is empty. Indeed, if $z_0 \in \omega_\infty \cap \Omega'$, then there is a compact neighborhood $B_\rho = \{|z - z_0 \leq \rho\} \in \Omega'$, $\rho > 0$, since Ω' is open. By the assumption,

$$B_\rho \subset \Omega_n = G \setminus \omega_n \quad \text{for all large} \quad n. \tag{2.5}$$

Since ω_n converge to ω_∞ in the Hausdorff metric, there is a sequence

$$\omega_n \ni z_n \to z_0 \in \omega_\infty \ \text{ as } \ n \to \infty.$$

It follows that $z_n \in B_\rho$ for all large n, which contradicts (2.5). Hence,

$$\Omega' \cap \omega_\infty = \emptyset \ \text{ and } \ \Omega' \subset G \setminus \omega_\infty.$$

From this, we conclude that $\Omega' \supset \Omega$ is an open connected component of the set $G \setminus \omega_\infty$. Since Ω is a maximal open connected component with this property, we get $\Omega' = \Omega$. Hence, Ω is a kernel of the sequence Ω_n.

Finally, note that every subsequence ω_{n_k} converges to ω_∞ in the Hausdorff metric, and the compact set, ω given by (2.2), is independent of the choice of such a subsequence. Hence, Ω is the kernel of each subsequence Ω_{n_k}. This means that Ω_n converges to Ω in the sense of Caratheodory. This completes the proof of Theorem 2 ■

DEFINITION 7 *Let a sequence $\omega_n \in \mathscr{C}_h$, $n \geq 1$, converge to a compact set $\omega_\infty \subset G_h$ in the Hausdorff metric and the compact set ω be defined by (2.2). Consider the domains $\Omega_n = G \setminus \omega_n$ and $\Omega = G \setminus \omega$. We say that $\Omega_n \to \Omega$ ($\omega_n \to \omega$) in the sense of Caratheodory-Hausdorff.*

REMARK 2 *Every sequence $\omega_n \in \mathscr{C}_h$ contains a subsequence, still denoted by ω_n, which converges to some ω_∞. Let the set ω be defined by (2.2). Then $\omega_n \to \omega$ in the sense of Caratheodory-Hausdorff. It follows from Theorem 2 and Definition 4 that $\omega \in \mathscr{C}_h$. Therefore, the space $\mathscr{C}_h$, equipped with the Caratheodory–Hausdorff convergence, is sequentially compact.*

Mosco and Caratheodory-Hausdorff convergence

The other type of convergence of domains in Euclidian spaces is the Mosco convergence. It is widely used in the shape optimization theory, see Bucar and Trebesch (1998) and Švetàk (1993). The following example demonstrates the essential difference between the Mosco and the Caratheodory-Hausdorff convergence.

Example 1 *Assume that* $G = \{|x| < R\}$ *and* $G_h = \{|x| \le R - h\}$. *Let us consider the sequence of arcs*

$$\omega_n = \{x: \; x_1 + ix_2 = re^{i\phi}, \quad 1/n \le \phi \le 2\pi\}, \quad 0 < r < R - h.$$

The sets ω_n *can be viewed as circular atolls surrounding a circular lagoon with the entrance of the width* $1/n$. *Obviously,* $\omega_n \in \mathcal{C}_h$. *It is clear that* ω_n *converge in the Hausdorff metric to the circle* $\omega_\infty = \{|x| = r\}$. *The open set* $G \setminus \omega_\infty$ *consists of two connected components: the annulus* $\Omega = \{r < |x| < R\}$ *and the disc* $B_r = \{|x| < r\}$. *It follows from Definition 7 that the domains* $\Omega_n = G \setminus \omega_n$ *converge to* Ω *in the Caratheodory-Hausdorff sense. On the other hand, the Mosco limit of the sequence* Ω_n *equals* $\Omega \cup B_r$, *and consists of two disjoint components.*

3. Conformal mappings

Conformal mapping of doubly-connected domains

Recall that a bounded domain $\Omega \subset \mathbb{R}^2$ is *doubly connected* if the boundary of Ω consists of two disjoint compact connected sets. If G is a simply connected domain and $\omega \Subset G$ is a compact set, then $\Omega = G \setminus \omega$ is a doubly connected domain if and only if Ω and ω are connected. The following result is a particular case of the general Hilbert Theorem on conformal mappings of multiply connected domains, see Golusin (1969, Ch. 5, §1). Hereinafter we will use the complex notation

$$z = x_1 + ix_2, \quad \zeta = \zeta_1 + i\zeta_2.$$

We also denote by D_μ, $0 \le \mu < 1$, the open annulus

$$D_\mu = \big\{ \, \zeta \in \mathbb{C}: \, \mu < |\zeta| < 1 \, \big\} \tag{3.1}$$

in the complex plane of the variable ζ.

Theorem 3 *Let* Ω *be a doubly connected domain. Then there are* $\mu \in [0, 1)$ *and a conformal one-to-one onto mapping* w *of* Ω *onto the annulus* D_μ. *The* *quantity $1/\mu$ is named the conformal modulus of Ω. The conformal mapping w is uniquely defined by the normalization condition*

$$\zeta_0 = w(z_0), \quad w'(z_0) > 0 \quad \textit{with fixed} \quad z_0 \in \Omega \quad \textit{and} \quad \zeta_0 \in D_\mu.$$

If $\partial\Omega$ consists of Jordan arcs, then $w \in C(\overline{\Omega})$. If $\partial\Omega \in C^\infty$, then $w \in C^\infty(\overline{\Omega})$.

Convergence of conformal mappings

The next general fact is the generalization of the Caratheodory Theorem for multiply connected domains, Golusin (1969, Ch. 5, §5, Thm. 2).

THEOREM 4 *Let domains $A_n \subset \mathbb{C}$ converge to a kernel $A \subset \mathbb{C}$ with respect to a point z_0 in the sense of Caratheodory, and $B_n \subset \mathbb{C}$ converge to a kernel $B \subset \mathbb{C}$ with respect to a point ζ_0 in the sense of Caratheodory. Let $w_n : A_n \to B_n \subset C$ be a conformal bijection such that $w_n(z_0) = \zeta_0$, $w_n'(z_0) > 0$. Then, w_n converge uniformly with all derivatives on every compact subset of A to the conformal mapping $w : A \to \mathbb{B}$, satisfying the normalization condition $w(z_0) = \zeta_0$, $w'(z_0) > 0$.*

The following Proposition is the straightforward consequence of Theorem 3 and Definition 7. Let us consider a sequence of compact sets $\omega_n \in \mathscr{C}_h$, domains $\Omega_n = G \setminus \omega_n$, and conformal mappings w_n, $n \geq 1$, with the following properties.

B.1 The compact sets ω_n and domains $\Omega_n = G \setminus \omega_n$ converge to a set $\omega \in \mathscr{C}_h$ and domain $\Omega = G \setminus \omega$ in the sense of Caratheodory-Hausdorff.

B.2 For every $n \geq 1$, the holomorphic function $w_n : \Omega_n \to \mathbb{C}$ is a conformal bijection, which takes the domain Ω_n onto the annulus D_{μ_n}.

B.3 The conformal modulus $1/\mu_n$ satisfies the conditions

$$0 < \mu^- \leq \mu_n \leq \mu^+ < 1, \quad \mu_n \to \mu \ \text{ as } \ n \to \infty.$$

PROPOSITION 1 *Let conditions* **B.1**-**B.3** *be satisfied. We also assume that w_n satisfy the normalization conditions*

$$w_n(z_0) = \zeta_0 \quad \textit{for fixed} \quad z_0 \in \mathcal{A}_h, \ \zeta_0 \in D_{\mu^+}. \tag{3.2}$$

Then, w_n converge uniformly with all derivatives on every compact subset of Ω to the conformal mapping $w : \Omega \to D_\mu$ satisfying normalization condition (3.2).

PROOF Fix an arbitrary $z_0 \in \mathcal{A}(h)$ and $\zeta_o \in D_{\mu^+}$. It follows from Definition 7 of the Caratheodory-Hausdorff convergence that Ω_n converge to the kernel Ω with respect to z_0 in the sense of Caratheodory. Obviously, the domains D_{μ_n} converge to the kernel D_μ with respect to ζ_0 in the sense of Caratheodory. The application of Theorem 4 completes the proof. ∎

The mappings of boundaries

Choose an arbitrary $\omega \in \mathscr{C}_h$ and denote by $\zeta = w(z)$ the conformal mapping of doubly-connected domain $\Omega = G \setminus \omega$ onto the annulus D_μ. Assume that the conformal modulus $1/\mu$ and w satisfy the conditions

$$0 < \mu^- \leq \mu \leq \mu^+ < 1, \tag{3.3}$$

$$w(z_0) = \zeta_0, \quad w'(z_0) > 0 \ \text{ for fixed } \ z_0 \in \mathcal{A}(h), \ \zeta_0 \in D_{\mu^+}. \tag{3.4}$$

Since ∂G belongs to the class C^∞, it admits the natural parametrization

$$z = Z(s), \quad 0 \leq s \leq L, \quad Z \in C^\infty[0, L], \quad |Z'(s)| = 1.$$

In its turn, the boundary $\mathbb{S}^1$ of the unit circle $B = \{|\zeta| < 1\}$ admits the parametrization

$$\zeta = e^{i\vartheta}, \quad 0 \leq \vartheta \leq 2\pi.$$

The conformal mapping w establishes one-to-one correspondence between parameters s and ϑ. This leads to the following definition.

Definition 8 *We say that the mapping $\varphi : [0, L] \to [0, 2\pi]$ is associated with the conformal mapping w if*

$$e^{i\varphi(s)} = w\big(Z(s)\big), \quad s \in [0, L]. \tag{3.5}$$

The following proposition constitutes the smoothness properties of φ.

Proposition 2 *For every integer $r > 0$, there is a constant $c(r)$, depending only on r, μ^-, h, and ∂G such that*

$$\|\varphi\|_{C^r[0,L]} \leq c(r). \tag{3.6}$$

Proof The mapping w^{-1} is a holomorphic diffeomorphism of D_μ onto Ω. For every $\lambda \in (\mu, 1)$, the circle $|\zeta| = \lambda$ is a compact subset of D_μ. Hence, $\Gamma_\lambda = w^{-1}\{|\zeta| = \lambda\}$ is an analytic Jordan curve. The curves Γ_λ and ∂G belong to the class C_∞ and form the boundary of the doubly-connected domain $\Omega_\lambda = w^{-1}(D_\lambda)$. It follows from this and Theorem 3 that w belongs to the class $C^\infty(\overline{\Omega}_\lambda)$ for every $\lambda \in (\mu, 1)$. Introduce the function

$$\Psi(z) = \ln |w(z)|, \quad z \in \Omega.$$

It is infinitely continuously differentiable in the neighborhood of ∂G and satisfies the relations

$$\ln \mu < \Psi < 0, \quad \Delta\Psi = 0 \ \text{ in } \ \Omega, \quad \Psi = 0 \ \text{ on } \ \partial G. \tag{3.7}$$

Note that the curvilinear C^∞ annulus $\mathcal{A}(h)$ belongs to the domain Ω. It follows from this, (3.7), and estimates of solutions to elliptic equations near the boundary that for every integer $r \geq 0$,

$$\|\Psi\|_{C^r(\mathcal{A}(h/2))} \leq c(r) \quad \text{and} \quad \|\partial_\nu \Psi\|_{C^r(\partial G)} \leq c(r). \tag{3.8}$$

Here, ∂_ν is the outward normal derivative, the constant $c(r)$ depends only on r, μ^-,h, and ∂G. Next, note that $w(z) = |w(z)|e^{i\vartheta(z)}$ and the function $\ln w = \ln|w(z) + i\vartheta(z)$ is locally holomorphic in Ω. From this and the Cauchy-Riemann equations we get

$$\partial_s^r \varphi(s) = \partial_s^r \vartheta(Z(s)) = \partial_s^{r-1}\big(\partial_\nu \Psi(Z(s))\big) \quad \text{for} \quad r \geq 1.$$

Combining this identity with the second inequality in (3.8) we obtain the desired estimate (3.6). ∎

Our next task is to obtain the similar estimate for the inverse mapping $\varphi^{-1} : [0, 2\pi] \to [0, L]$. This question is less trivial and we prove the following particular result.

Proposition 3 *Let compact sets ω_n, ω, domains $\Omega_n = G \setminus \omega_n$, $\Omega = G \setminus \omega$, and conformal mappings $w_n : \Omega_n \to D_{\mu_n}$, $w : \Omega \to D_\mu$ satisfy conditions* **B.1**-**B.3** *and meet all requirements of Proposition 1. Then, for every integer $r > 0$, there is a constant $c(r)$, independent of n, such that*

$$\|\varphi_n\|_{C^r[0,L]} + \|\varphi\|_{C^r[0,L]} \leq c(r), \tag{3.9}$$

$$\|\varphi_n^{-1}\|_{C^r[0,2\pi]} + \|\varphi^{-1}\|_{C^r[0,2\pi]} \leq c(r). \tag{3.10}$$

Moreover, we have

$$\|\varphi_n - \varphi\|_{C^r[0,L]} \to 0, \quad \|\varphi_n^{-1} - \varphi^{-1}\|_{C^r[0,2\pi]} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.11}$$

Proof Without loss of generality we may assume that w_n and w satisfy the normalization conditions

$$w_n(z_0) = w(z_0) = \zeta_0, \; w_n'(z_0) > 0, \; w'(z_0) > 0 \;\; \text{for fixed} \;\; z_0 \in \mathcal{A}(h), \; \zeta_0 \in D_{\mu^+}.$$

Note that estimate (3.9) is the straightforward consequence of Proposition 2. In order to prove the estimate (3.10), we introduce the harmonic functions

$$\Psi_n(z) = \ln|w_n(z)|, \quad z \in \Omega_n, \quad \Psi(z) = \ln|w(z)|, \quad z \in \Omega.$$

Arguing as in the proof of Proposition 2 we conclude that they are infinitely differentiable in the neighborhood of ∂G and satisfy the relations

$$\begin{gathered} \ln\mu^- < \Psi_n < 0, \quad \Delta\Psi_n = 0 \;\text{ in }\; \Omega_n, \quad \Psi_n = 0 \;\text{ on }\; \partial G, \\ \ln\mu^- < \Psi < 0, \quad \Delta\Psi = 0 \;\text{ in }\; \Omega, \quad \Psi = 0 \;\text{ on }\; \partial G. \end{gathered} \tag{3.12}$$

Note that Ψ_n, Ψ take the maximum on ∂G, which, along with the strong maximum principle, yields

$$\inf_{\partial G} \partial_\nu \Psi_n > 0, \quad \inf_{\partial G} \partial_\nu \Psi > 0, \tag{3.13}$$

where ∂_ν is the outward normal derivative. Let us prove that there is $\beta > 0$, independent of n, such that

$$\partial_\nu \Psi_n \geq \beta > 0, \quad \partial_\nu \Psi \geq \beta > 0 \ \text{ on } \ \partial G. \tag{3.14}$$

Suppose, contrary to our claim, that there exists a subsequence of the sequence Ψ_n, still denoted by Ψ_n, and a sequence $z_n \in \partial G$ such that

$$\partial_\nu \Psi_n(z_n) \to 0, \quad z_n \to z^* \in \partial G \ \text{ as } \ n \to \infty. \tag{3.15}$$

It follows from the estimates of solutions to elliptic equations near the boundary that for every integer $r \geq 0$,

$$\|\Psi_n\|_{C^r(\overline{\mathcal{A}(h/2)})} \leq c(r), \tag{3.16}$$

where $c(r)$ is independent of n. On the other hand, Proposition 1 implies that $w_n \to w$ uniformly on every compact subset of $\mathcal{A}(h)$. From this and (3.16) we conclude that

$$\Psi_n \to \Psi \ \text{ in } \ C^r(\overline{\mathcal{A}(h/2)}),$$
$$\partial_\nu \Psi_n \to \partial_\nu \Psi \ \text{ in } \ C^r(\partial G) \ \text{ as } \ n \to \infty.$$

These relations, along with the identities

$$\partial_s^r \varphi_n(s) = \partial_s^{r-1}\big(\partial_\nu \Psi_n(Z(s))\big), \quad \partial_s^r \varphi(s) = \partial_s^{r-1}\big(\partial_\nu \Psi(Z(s))\big)$$

imply the convergence of the sequence φ_n

$$\varphi_n \to \varphi \ \text{ in } \ C^r[0, L] \ \text{ as } \ n \to \infty. \tag{3.17}$$

Obviously, we have

$$\partial_\nu \Psi_n(z_n) \to \partial_\nu \Psi(z^*) = 0,$$

which contradicts (3.13). Hence, the normal derivatives $\partial_\nu \Psi_n$ and $\partial_\nu \Psi$ admit estimate (3.14) with the constant β independent of n. From this and the identities

$$\partial_s \varphi_n(s) = \partial_\nu \Psi_n(Z(s)), \quad \partial_s \varphi(s) = \partial_\nu \Psi(Z(s))$$

we get the inequalities

$$\partial_s \varphi_n(s) \geq \beta > 0, \quad \partial_s \varphi(s) \geq \beta > 0. \tag{3.18}$$

Therefore, the desired estimates (3.10) follow from estimate (3.9) and the inverse function theorem. It remains to note that the first relation in (3.11) follows from (3.17). The second follows from the first and the inverse function theorem. ■

4. Condenser capacity

Recall that a condenser in $\mathbb{R}^2$ is a pair (G, ω) such that $G \subset \mathbb{R}^2$ is an arbitrary domain and $\omega \Subset G$ is an arbitrary compact subset of G. The capacity of the condenser (G, ω) is defined by the equality

$$\text{cap }(G, \omega) = \min \int_G |\nabla \varphi|^2 \, dx, \tag{4.1}$$

where the minimum is taken over the set of all Lipschitz functions $\varphi : G \to \mathbb{R}$, satisfying the conditions

$$\varphi = 0 \text{ on } \partial G, \quad \varphi \geq 1 \text{ on } \omega.$$

There is another definition of the condenser capacity. The Green condenser capacity $C_g(G, \omega)$ is defined by the equality (Landkof, 1972, Ch. 2, §4)

$$C_g(G, \omega) = \mathbf{W}^{-1}, \quad \mathbf{W} = \inf \int_{G \times G} \mathbf{g}(x, y) \, d\boldsymbol{\mu}(x) \, d\boldsymbol{\mu}(y). \tag{4.2}$$

Here $\mathbf{g}$ is the Green function of G, and the infimum is taken over the set of all nonnegative probability measures $\boldsymbol{\mu}$ supported on ω. The following relation establishes the connection between two capacities

$$\text{cap }(G, \omega) = 2\pi C_g(G, \omega).$$

Recall the basic properties of the condenser capacity.

Monotonicity. Let ω' be a compact set, such that $\omega \subset \omega' \Subset G$, then

$$\text{cap }(G, \omega) \leq \text{cap }(G, \omega').$$

Conformal invariance. Let G' be a simply connected domain with regular boundary, $w : G \to G'$ be a conformal mapping, and $\omega' = w(\omega)$. Then

$$\text{cap }(G, \omega) = \text{cap }(G', \omega').$$

Symmetrization. Recall that the Steiner symmetrization ω_{sym} of a compact set $\omega \subset \mathbb{R}^2$ with respect to the real axis is defined by the equality

$$\omega_{sym} = \Big\{ z = x_1 + ix_2 : x_1 \in \Pi\omega, \quad |x_2| \leq \frac{1}{2} \text{ meas } B_{x_1} \Big\}. \tag{4.3}$$

Here, $\Pi\omega$ is the projection of ω onto real axis, B_{x_1} is the intersection of ω with the vertical line $\Re z = x_1$. If $B = \{|z| < 1\}$ is a unit circle and $\omega \Subset B$, then

$$\text{cap }(B, \omega_{sym}) \leq \text{cap }(B, \omega). \tag{4.4}$$

Capacity and conformal modulus. Let $\omega \Subset G$ be a connected compact set and $\zeta = w(z)$ be a conformal mapping of doubly connected domain $G \setminus \omega$ onto the annulus $\mu < |\zeta| < 1$. Then (see Landkof, 1972, Ch. 2, §4)

$$\text{cap }(G,\omega) = 2\pi C_g(G,\omega) = 2\pi\Big(\ln\frac{1}{\mu}\Big)^{-1}. \tag{4.5}$$

The following proposition is the main result of this section.

PROPOSITION 4 *Let $G \subset \mathbb{R}^2$ be a bounded domain with C^∞ boundary and $|G| = \text{meas } G$. Fix an arbitrary $\theta \in (0, |G|)$. Then there are $\lambda \in (0,1)$ and a continuous function $\Phi : (0,\lambda) \to \mathbb{R}^+$, depending on G and θ, with the following properties:*

$$\Phi(\epsilon) \to \infty \quad as \quad \epsilon \to 0, \tag{4.6}$$

and the inequality

$$\text{cap }(G,\omega) \geq \Phi(\epsilon) \tag{4.7}$$

holds for every connected compact set $\omega \Subset G$ such that

$$|\omega| \in [\,\theta,\ |G|\,], \quad \epsilon \equiv \text{dist }(\omega, \partial G) \leq \lambda.$$

The following corollary is a straightforward consequence of Proposition 4.

COROLLARY 1 *Let $G \subset \mathbb{R}^2$ be a bounded domain with C^∞ boundary. Furthermore, assume that a sequence of connected compact sets $\omega_n \Subset G$, $n \geq 1$, satisfies the conditions*

$$0 < \theta \leq |\omega_n|,\ \text{dist }(\partial G, \omega_n) \to 0 \quad as \quad n \to \infty,\ \Omega_n = G \backslash \omega_n \quad is\ doubly\text{-}connected.$$

Then

$$\text{cap }(G,\omega_n) \to \infty \quad as \quad n \to \infty.$$

Let us turn to the proof of Proposition 4. We split the proof into the sequence of Lemmas

LEMMA 3 *Let $B = \{|z| < 1\}$ be a unit circle and $\omega(h) \Subset B$ be a compact segment*

$$\omega(h) = \{z = x_1 + i0 : \quad 0 \leq x_1 \leq h\}, \quad 1/2 < h < 1.$$

Then

$$\text{cap }(B, \omega(h)) = \pi \ln\Big(\frac{4(1+h)}{1-h}\Big) + o(1), \tag{4.8}$$

where $o(1) \to 0$ as $h \to 1$.

Proof We construct the conformal mapping $B\backslash\omega(h) \to D_\mu$ and employ relation (4.5). Our considerations are based on the scheme proposed in Landkof (1972, Ch. 2). Introduce the auxiliary conformal mapping

$$\zeta = \frac{(1-z)^2}{2(1+z)^2-(1-z)^2}. \tag{4.9}$$

It takes diffeomorphically the domain $B \setminus \omega(h)$ onto the domain $\mathbb{C} \setminus (I \cup I_\beta)$. Here, the segments I and I_β are defined by the relations

$$I = \{z = x_1 + i0 : x_1 \in [-1, 0]\}, \quad I_\beta = \{z = x_1 + i0 : x_1 \in [\beta, 1]\}, \\ \beta = \frac{(1-h)^2}{2(1+h)^2-(1-h)^2}. \tag{4.10}$$

By virtue of Akhiezer's formula (Akhiezer, 1990, Ch. 8 §49), the implicit relation

$$\zeta = \Big(2\,\mathrm{sn}^2 \big(\frac{K' \ln\sigma}{\pi} \big) - 1 \Big)^{-1}$$

determines the conformal mapping $\sigma(\zeta)$ of the domain $\mathbb{C} \setminus (I \cup I_\beta)$ onto the annulus

$$D_\mu = \{\mu < |\sigma| < 1\}, \quad \mu = e^{-\pi\frac{K}{K'}}.$$

Here, sn is the Jacobi elliptic function with period $4K + i4K'$, see Whittaker and Watson (1996). The elliptic integrals K and K' are given by the equalities

$$K = \int_0^1 (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}\,dt, \quad K' = \int_0^1 (1-t^2)^{-1/2}(1-{k'}^2t^2)^{-1/2}\,dt,$$

where the elliptic modulus $k \in (0, 1)$,

$$k^2 = \frac{2\beta}{1+\beta}, \quad k'^2 = 1 - k^2. \tag{4.11}$$

The composite mapping $\sigma(\zeta(z))$ determines the conformal mapping of the doubly connected domain $G \setminus \omega(h)$ onto the annulus D_μ. From this and relation (4.5) we obtain

$$\mathrm{cap}\ (G, \omega(h)) \ = \ 2\frac{K}{K'}$$

It is well known (see Whittaker and Watson, 1996, **n** 22.737) that

$$K(k) = \frac{1}{2}\pi + O(k^2), \quad K'(k) = \ln\big(\frac{4}{k}\big) + o(1) \ \text{ for } \ k \in (0, 1/3],$$

where $o(1) \to 0$ as $k \to 0$. In particular, we have

$$\text{cap }(G, \omega(h)) \;=\; 2\frac{K}{K'} = \pi \,\ln\Big(\,\frac{4}{k}\,\Big) + o(1) \quad \text{for} \quad k \in (0, 1/3].$$

Next, relations (4.10) and (4.11) imply

$$k = \frac{1-h}{1+h}, \quad k \in (0, 1/3] \quad \text{for} \quad h \in [1/2, 1).$$

We thus get

$$\text{cap }(G, \omega(h)) \;=\; \pi \,\ln\Big(\,\frac{4(1+h)}{1-h}\,\Big) + o(1) \quad \text{for} \quad h \in [1/2, 1),$$

where $o(1) \to 0$ as $h \to 1$. This completes the proof of Lemma 3. ■

For a fixed $a \in (0, 1)$, denote by $\omega(a, h) \Subset B$ the compact segment

$$\omega(a, h) = \big\{z = x_1 + i0,\, x_1 \in [a, h]\,\big\}, \quad a < h < 1.$$

Now our task is to estimate the capacity of the condenser $(B, \omega(a, h))$. The result is given by the following lemma.

Lemma 4 *Let $0 < a < 1$. Furthermore, assume that a moving parameter h satisfies the inequalities*

$$a < \frac{1+2a}{2+a} \le h < 1. \tag{4.12}$$

Then

$$\text{cap }\big(B, \omega(a, h)\big) = \pi \ln\Big(4\frac{1-a}{1-h}\;\frac{1+h}{1+a}\Big) + o(1), \tag{4.13}$$

where $o(1) \to 0$ as $h \to 1$.

Proof Note that the conformal automorphism

$$z' = \frac{z-a}{1-az}$$

of the unit circle B takes the segment $\omega(a, h)$ onto the segment

$$\omega(h'), \quad h' = \frac{h-a}{1-ah}.$$

It follows from this, the conformal invariance of the condenser capacity, and Lemma 3 that

$$\text{cap }\big(\,B, \omega(a, h)\,\big) = \text{cap }\big(\,B, \omega(h')\,\big) = \pi \,\ln\Big(\,\frac{4(1+h')}{1-h'}\,\Big) + o(1), \tag{4.14}$$

where $o(1) \to 0$ as $h' \to 1$. On the other hand, we have

$$\pi \ln\Big(\frac{4(1+h')}{1-h'}\Big) = \pi \ln\Big(4\frac{1-a}{1-h}\ \frac{1+h}{1+a}\Big).$$

Combining this result with (4.14) we finally obtain

$$\text{cap}\ \big(B, \omega(a,h)\big) = \pi \ln\Big(4\frac{1-a}{1-h}\ \frac{1+h}{1+a}\Big) + o(1).$$

Here

$$o(1) \to 0 \ \text{ as } \ = \frac{h-a}{1-ah} \to 1,$$

or, equivalently, $o(1) \to 0$ as $h \to 1$. This completes the proof of Lemma 4. ■

Lemma 5 *Let a connected compact set $\omega \Subset B$ satisfy the conditions*

$$0 < \theta \le |\omega|, \quad \varepsilon \equiv \text{dist}\ (\omega, \partial B) > 0, \tag{4.15}$$

where $\theta \in (0,1)$ be a fixed positive constant. Then the inequality

$$\text{cap}\ (B, \omega) \ge \Phi_\theta(\varepsilon) \tag{4.16}$$

holds for all

$$0 < \varepsilon \le \lambda = \frac{\theta}{6-\theta}. \tag{4.17}$$

Here, the function $\Phi_\theta : (0, \lambda] \to \mathbb{R}^+$ is defined by the equality

$$\Phi_\theta(\varepsilon) = \pi\ \ln\Big(2\frac{\theta}{\varepsilon}\ \frac{2-\varepsilon}{2-\frac{\theta}{2}}\Big) + o(1), \tag{4.18}$$

where $o(1) \to 0$ as ε. In particular, $\Phi_\theta(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Proof There exist two points, $z_0 \in \omega$ and $z^* \in \partial B$, such that $|z_0 - z^*| = \epsilon$. After rotation we may assume that $z^* = 1$. Denote by $\Pi\omega$ the orthogonal projection of ω onto the real axis. We have

$$\theta \le |\omega| = \int_{\Pi\omega} \Big\{ \int_{B_{x_1}} dx_2 \Big\} dx_1 \le 2 \int_{\Pi\omega} dx_1 = 2\text{meas}\ \Pi\omega.$$

It is easy to see that $\Pi z_0 \ge 1 - \varepsilon$. We thus get

$$\frac{\theta}{2} \le \text{meas}\ \Pi\omega, \quad \Pi z_0 \in [1-\varepsilon, 1). \tag{4.19}$$

Since ω is connected, the projection $\Pi\omega$ is also connected, then there are $\alpha, \beta \in (-1, 1)$ such that

$$\Pi\omega = \{z = x_1 + i0 : x_1 \in [\alpha, \beta]\} \equiv \omega_{\alpha,\beta}.$$

It follows from inequalities (4.19) that

$$1 > \beta \geq \Pi z_0 \geq 1 - \varepsilon, \quad \alpha \leq \beta - \frac{\theta}{2}.$$

We thus get

$$\Pi\omega = \omega(\alpha, \beta) \supset \omega(a, h), \quad a = 1 - \frac{\theta}{2},\ h = 1 - \varepsilon. \tag{4.20}$$

It easy to see that the Steiner symmetrization ω_{sym} contains the segment $\omega(\alpha, \beta)$. From this, and the monotonicity properties of the capacity, we obtain

$$\text{cap}\ (B, \omega) \geq \text{cap}\ (B, \omega_{sym}) \geq \text{cap}\ \big(B, \omega(\alpha, \beta)\big) \geq \text{cap}\ \big(B, \omega(a, h)\big).$$

This result, along with inequality (4.13) in Lemma 4, implies the estimate

$$\text{cap}\ (B, \omega) \geq \pi \ln\Big(4\frac{1-a}{1-h}\ \frac{1+h}{1+a}\Big) + o(1), \quad o(1) \to 0 \text{ as } h \to 1, \tag{4.21}$$

which holds true for all h satisfying inequalities (4.12). By virtue of relations $a = 1 - \theta/2$ and $h = 1 - \varepsilon$, inequalities (4.12) in Lemma 4 are equivalent to inequalities (4.17) in Lemma 5. Moreover, we have

$$\pi \ln\Big(4\frac{1-a}{1-h}\ \frac{1+h}{1+a}\Big) = \ln\Big(2\frac{\theta}{\varepsilon}\ \frac{2-\varepsilon}{2-\frac{\theta}{2}}\Big), \ \ h \to 1 \text{ as } \varepsilon \to 0.$$

From this and (4.21) we finally obtain the desired estimate (4.16). ■

We are now in a position to complete the proof of Proposition 4. To this end, we employ Lemma 5 and the conformal invariance of the condenser capacity. Denote by $z' = w(z)$ the conformal mapping of G onto the unit circle B. Next, denote by $\omega' \Subset B$ the connected compact set $\omega' = w(\omega)$. The conformal invariance property of the condenser capacity implies

$$\text{cap}\ (G, \omega) = \text{cap}\ (B, \omega'). \tag{4.22}$$

Recall that G is a bounded simply connected domain with C^∞ boundary. Therefore, the derivatives of w and the inverse w^{-1} are uniformly bounded. It follows from this and relations $|\omega| > \theta$, dist $(\omega, \partial G) = \varepsilon$, that

$$|\omega'| = \int_\omega |w'(z)|^2\, dx_1 dx_2 \geq C_1|\omega|, \quad c^-\varepsilon \leq \varepsilon' \leq c^+\varepsilon, \tag{4.23}$$

where $\varepsilon' = \text{dist}\,(\omega', \partial B)$, the positive constants $C_1 \in (0,1)$ and $c^{\pm}$ depend only on G. Hence, ω' and ε' meet all requirements of Lemma (5) with θ replaced by $\theta' = C_1\theta$. It remains to note that the assertion of Proposition 4 is the straightforward consequence of (4.23), estimate (4.16) in Lemma 5 for the condenser (B, ω'), and identity (4.22).

5. The Dirichlet problem

Preliminaries

In this section, we consider in details the main boundary value problem for the harmonic function $u : \Omega \to \mathbb{R}$,

$$\begin{gathered} \Delta u = 0 \ \text{ in } \ \Omega, \\ u = g \ \text{ on } \ \partial G, \quad u = 0 \ \text{ on } \ \partial\omega. \end{gathered} \tag{5.1}$$

Hereinafter, we assume that G is a bounded simple connected domain with C^∞ boundary and $\omega \subset G$ is a compact set. We also assume that the set ω and the annulus $\Omega = G \setminus \omega$ are connected. This means that Ω is a doubly-connected domain in $\mathbb{R}^2$.

Recall Definition 3 of the weak solution to problem (5.1). Denote by $\mathscr{W}(\Omega)$, $\Omega = G \setminus \omega$, the space of functions $u \in W^{1,2}(G)$ such that every $u \in \mathscr{W}$ vanishes on some neighborhood of ω. We assume that $\mathscr{W}$ is endowed with the norm $W^{1,2}(G)$, which is equivalent to the norm

$$\|u\|_1^2 = \int_\Omega |\nabla u|^2 dx + \int_{\partial G} |u|^2 \, ds.$$

Denote by $\mathcal{H}^1$ the completion of $\mathscr{W}$ with respect to the norm $\|\cdot\|_1$. Every function $u \in \mathcal{H}^1$ has the extension $u^* : G \to \mathbb{R}$ such that

$$\|u^*\|_{W^{1,2}(G)} \le c\|u\|_1.$$

In particular, u has the trace $u\big|_{\partial G} \in W^{1/2}(\partial G)$.

Let $g : \partial G \to \mathbb{R}$ be an arbitrary function, satisfying the conditions

$$g \in C^\infty(\partial G), \quad g > c_g > 0 \ \text{ on } \ \partial G. \tag{5.2}$$

We say that u is a weak solution to problem (5.1) if

$$\begin{gathered} u \in \mathcal{H}^1(\Omega), \quad u = g \ \text{ on } \ \partial G, \\ \int_\Omega |\nabla u|^2 \, dx = \min_{\mathscr{I}} \int_\Omega |\nabla \tilde{u}|^2 \, dx, \end{gathered} \tag{5.3}$$

where $\mathscr{J} \subset \mathcal{H}^1$ is the set of all functions $\tilde{u} \in \mathcal{H}^1$ satisfying the boundary condition $\tilde{u} = g$ on ∂G.

Let us consider the following construction. It follows from Theorem 3 that there is a conformal diffeomorphism $w : \Omega \to D_\mu = \{\mu < |\zeta| < 1\}$, $0 \le \mu < 1$. For every $\lambda \in (\mu, 1)$, the mapping w^{-1} takes diffeomorphically the circumference $\{|\zeta| = \lambda\}$ onto an analytic Jordan curve $\Gamma_\lambda \Subset \Omega$. The disjoint curves Γ_λ and ∂G form the boundary of the doubly-connected domain $\Omega_\lambda = w^{-1}(D_\lambda)$. It is easy to check that

$$\bigcup_{\lambda>\mu} \Omega_\lambda = \Omega, \quad \bigcap_{\lambda>\mu} \Big(G \setminus \Omega_\lambda \Big) = \omega. \tag{5.4}$$

The following simple lemma will be used throughout this section:

LEMMA 6 *For every neighborhood $\mathcal{O}$ of ω, there is $\nu > \mu$ such that $G \setminus \Omega_\nu \subset \mathcal{O}$.*

PROOF Assume that the assertion of the lemma is false. Then, there is a neighborhood $\mathcal{O}$ of Ω, and sequences $\nu_n > \mu$, and $z_n \in G$ such that

$$\lambda_n \to \mu \text{ as } n \to \infty, \text{ and } z_n \in (G \setminus \Omega_{\nu_n}) \setminus \mathcal{O}.$$

After passing to a subsequence we may assume

$$\lambda_n \searrow \mu \text{ as } n \to \infty, \text{ and } (G \setminus \Omega_{\nu_n}) \setminus \mathcal{O} \ni z_n \to z^*, \tag{5.5}$$

as $n \to \infty$. We have

$$z^* \in \overline{\{z_k\}_{k \ge n}} \subset G \setminus \Omega_{\nu_n}$$

since $G \setminus \Omega_{\nu_n}$ is compact. It follows from this, (5.4), and the monotonicity of sequence Ω_{ν_n} that

$$z^* \in \bigcap_{\lambda>\mu} \Big(G \setminus \Omega_\lambda \Big) = \omega. \tag{5.6}$$

Hence dist $(z_n, \partial G) > \epsilon > 0$ for all sufficient large n. For such n, we have $z_n \in G_\epsilon \setminus \mathcal{O}$. Note that $G_\epsilon \setminus \mathcal{O}$ is a compact subset of $G \setminus \omega$. We thus get

$$z^* \in G_\epsilon \setminus \mathbf{O} \subset G \setminus \omega,$$

which contradicts (5.6). This completes the proof of Lemma 6. ∎

Denote by u_λ the solution to the boundary value problem

$$\Delta u_\lambda = 0 \text{ in } \Omega_\lambda, \quad u_\lambda = g \text{ on } \partial G, \quad u_\lambda = 0 \text{ on } \Gamma_\lambda. \tag{5.7}$$

Since the boundary $\partial\Omega_\lambda$ belongs to the class C^∞, we have $u_\lambda \in C^\infty(\overline{\Omega_\lambda})$.

LEMMA 7 *The function u_λ, being extended by zero to G, belongs to the space $\mathcal{W}$.*

PROOF It suffices to show that the set $G\backslash\Omega_\lambda$ contains some neighborhood $\mathcal{O}$ of ω. Note that the set Ω has no common points with ω. The same is true for $\Gamma_\lambda \Subset \Omega$. We obviously have $\partial G \cap \omega = \emptyset$. Hence, the compact set $K = \Omega_\lambda \cup \partial G \cup \Gamma_\lambda$ has no common points with the compact set ω. It follows that dist $(K, \omega) > d > 0$ for some $d > 0$. Since $\Omega_\lambda \subset K$, we have dist $(\Omega_\lambda, \omega) > d$. Therefore, $d/2$ neighborhood of ω belongs to $G \setminus \Omega_\lambda$ and the lemma follows. ■

COROLLARY 2 *Let $0 < \mu < \lambda < \lambda' < 1$. Then*

$$\int_\Omega |\nabla u|^2 dx \le \int_{\Omega_\lambda} |\nabla u_\lambda|^2 dx \le \int_{\Omega_{\lambda'}} |\nabla u_{\lambda'}|^2 dx. \tag{5.8}$$

PROOF The first inequality in (5.8) obviously follows from Lemma 7 and variational principle (5.3). In order to prove the second, extend the function $u_{\lambda'}$ by zero to Ω_λ. The extended function satisfies the same boundary conditions as u_λ. Hence, the desired estimate follows from equations (5.7) and the Dirichlet principle. ■

Along with the functions $u : \Omega \to \mathbb{R}$ and $u_\lambda : \Omega_\lambda \to \mathbb{R}$ we will consider the auxiliary functions $v : D_\mu \to \mathbb{R}$ and $v_\lambda : D_\lambda \to \mathbb{R}$, $\mu < \lambda < 1$, defined by the equalities

$$v = u \circ w^{-1}, \quad v_\lambda = u_\lambda \circ w^{-1}, \tag{5.9}$$

where $w : \Omega \to D_\mu$ is a conformal diffeomorphism. The normalization conditions are not essential at this stage. However, we will assume that the conformal modulus satisfies the inequalities

$$0 < \mu^- \le \mu \le \mu^+ < 1 \tag{5.10}$$

with fixed constants $\mu^\pm$. Note that Ω_λ is a curvilinear annulus with c^∞ boundary $\partial G \cup \Gamma_\lambda$. It follows from this and Theorem 3 that $w \in C^\infty(\overline{\Omega_\lambda})$, $w^{-1} \in C^\infty(\overline{D_\lambda})$, which yields $v_\lambda \in C^\infty(\overline{D_\lambda})$. Finally, note that the function v_λ satisfies the equations and boundary conditions

$$\begin{gathered} \Delta v_\lambda = 0 \ \text{ in } \ D_\lambda, \\ v_\lambda(\zeta) = f(\zeta) \ \text{ for } \ |\zeta| = 1, \quad v_\lambda(\zeta) = 0 \ \text{ for } \ |\zeta| = \lambda, \end{gathered} \tag{5.11}$$

where $f(e^{i\theta}) = g(w^{-1}(e^{i\theta}))$ belongs to the class $C^\infty(\mathbb{S}^1)$.

Basic inequality

THEOREM 5 *Assume that the conformal modulus satisfies inequalities* (5.10). *Then there is a constant C, depending only on the constants $\mu^\pm$ in* (5.10), *such that*

$$\int_\Omega |\nabla u|^2\,dx \le \int_{\Omega_\lambda} |\nabla u_\lambda|^2\,dx \le \big(1 + C(\lambda-\mu)\big)\int_\Omega |\nabla u|^2\,dx \tag{5.12}$$

for all $\lambda \in (\mu, (1+\mu^+)/2)$.

PROOF The lower estimate in (5.12) is a consequence of Corollary 2. The proof of the upper estimate falls into three steps.

Step 1. First we prove that for every $\varepsilon \in (0,1)$, there is $\nu(\varepsilon) \in (\mu, 1)$ such that

$$\int_{\Omega_{\nu(\varepsilon)}} |\nabla u_{\nu(\varepsilon)}|^2\,dx \le \int_\Omega |\nabla u|^2\,dx + \varepsilon,\ \ \nu(\varepsilon)\to\mu \ \text{ as } \varepsilon \to 0. \tag{5.13}$$

Recall Definition 2 of the class $\mathcal{W}$. Since $\mathcal{W}$ is dense in $\mathcal{H}^1$ and u is a solution to variational problem (1.6), for every $\varepsilon > 0$ there is a function $\tilde u \in \mathcal{W}$ satisfying the conditions

$$\int_\Omega |\nabla u|^2\,dx \le \int_\Omega |\nabla \tilde u|^2\,dx \le \int_\Omega |\nabla u|^2\,dx + \varepsilon, \quad \tilde u\Big|_{\partial G} = g. \tag{5.14}$$

By the definition of the class $\mathcal{W}$, there exists a neighborhood $\mathcal{O}(\varepsilon)$ such that $\tilde u = 0$ in $\mathcal{O}(\varepsilon)$. By virtue of Lemma 6, there is $\nu(\varepsilon) \in (\mu, 1)$ such that $G\backslash\Omega_{\nu(\varepsilon)} \subset \mathcal{O}(\varepsilon)$. In particular, $\tilde u$ vanishes in a neighborhood of $\Gamma_{\nu(\varepsilon)}$. We thus get

$$\Delta u_{\nu(\varepsilon)} = 0 \ \text{ in } \ \Omega_{\nu(\varepsilon)},$$
$$u_{\nu(\varepsilon)} = g \ \text{ on } \ \partial G, \quad u_{\nu(\varepsilon)} = 0 \ \text{ on } \ \Gamma_{\nu(\varepsilon)}.$$

and

$$\tilde u \in W^{1,2}(\Omega_{\nu(\varepsilon)}), \quad \tilde u = g \ \text{ on } \ \partial G, \quad \tilde u = 0 \ \text{ on } \ \Gamma_{\nu(\varepsilon)}.$$

It follows from this and the Dirichlet principle that

$$\int_{\Omega_{\nu(\varepsilon)}} |\nabla u_{\nu(\varepsilon)}|^2 dx \le \int_{\Omega_{\nu(\varepsilon)}} |\nabla \tilde u|^2 dx \le \int_\Omega |\nabla \tilde u|^2 dx.$$

By combining this result with (5.14) we get the desired estimate (5.13). Note that if (5.13) holds for some $\nu(\varepsilon)$, then it also holds for all $\nu \in (\mu, \nu(\varepsilon))$. Hence, we may assume that $\nu(\varepsilon) \to \mu$ as $\varepsilon \to 0$.

Step 2. Now fix an arbitrary $\lambda \in (\mu, (1+\mu^+)/2)$. Next, choose $\varepsilon > 0$ so small that

$$\mu < \nu(\varepsilon) < \lambda < 1.$$

Out task is to establish relations between Dirichlet's integrals of the functions u_λ and $u_{\nu(\varepsilon)}$. Recall the definition of harmonic functions

$$v_\lambda = u_\lambda \circ w^{-1} : D_\lambda \to \mathbb{R} \quad \text{and} \quad v_{\nu(\varepsilon)} = u_{\nu(\varepsilon)} \circ w^{-1} : D_{\nu(\varepsilon)} \to \mathbb{R}.$$

Note that $v_\lambda \in C^\infty(\overline{D_\lambda})$ and $v_{\nu(\varepsilon)} \in C^\infty(\overline{D_{\nu(\varepsilon)}})$. By virtue of the conformal invariance of the Dirichlet integral, we have

$$\int_{\Omega_\lambda} |\nabla u_\lambda|^2 \, dx = \int_{D_\lambda} |\nabla v_\lambda|^2 \, d\zeta, \quad \int_{\Omega_{\nu(\varepsilon)}} |\nabla u_{\nu(\varepsilon)}|^2 \, dx = \int_{D_{\nu(\varepsilon)}} |\nabla v_{\nu(\varepsilon)}|^2 \, d\zeta. \tag{5.15}$$

Introduce the mapping $\sigma : D_{\nu(\varepsilon)} \to D_\lambda$, defined by the equality

$$\sigma(\zeta) = \frac{1}{\nu(\varepsilon) - 1} \big((\lambda - 1)|\zeta| + \nu(\varepsilon) - \lambda \big) \frac{\zeta}{|\zeta|}.$$

It is clear that σ takes diffeomorphically $\overline{D_{\nu(\varepsilon)}}$ onto $\overline{D_\lambda}$. Moreover, σ and σ^{-1} belong to the class C^∞. Denote by $M(\zeta)$ the Jacobi matrix $\nabla_\zeta \sigma(\zeta)$. Calculations show that

$$|M| + |M^{-1}| \le C, \quad |M - I| \le C(\lambda - \nu(\varepsilon)) \;\text{ in }\; D_{\nu(\varepsilon)}. \tag{5.16}$$

Note that the function $v_{\nu(\varepsilon)}$ satisfies the equations and boundary conditions

$$\begin{gathered} \Delta v_{\nu(\varepsilon)} = 0 \;\text{ in }\; D_{\nu(\varepsilon)}, \\ v_{\nu(\varepsilon)}(\zeta) = f(\zeta) \;\text{ for }\; |\zeta| = 1, \quad v_{\nu(\varepsilon)}(\zeta) = 0 \;\text{ for }\; |\zeta| = \nu(\varepsilon). \end{gathered} \tag{5.17}$$

Now, set $\varphi_\varepsilon(\sigma) = v_{\nu(\varepsilon)}(\zeta(\sigma))$. It follows from (5.17) and the expression for $\sigma(\zeta)$ that

$$\varphi_\varepsilon \in W^{1,2}(D_\lambda), \quad \varphi_\varepsilon(e^{i\theta}) = f(e^{i\theta}), \quad \varphi_\varepsilon(\sigma) = 0 \text{ for } |\sigma| = \lambda \tag{5.18}$$

Direct calculations lead to the identity

$$\int_{D_\lambda} |\nabla \varphi_\varepsilon|^2 \, d\sigma = \int_{D_{\nu(\varepsilon)}} \det M \, M^{-1} M^{-\top} \nabla v_{\nu(\varepsilon)} \cdot \nabla v_{\nu(\varepsilon)} \, d\zeta. \tag{5.19}$$

Next, inequalities (5.16) imply the estimate

$$\big| \det M \, M^{-1} M^{-\top} - I \big| \le C(\lambda - \nu(\varepsilon)) \le C(\lambda - \mu),$$

where C depends only on $\mu^{\pm}$. By combining this result with (5.19) we obtain

$$\int_{D_\lambda} |\nabla \varphi_\varepsilon|^2 \, d\sigma \le \big(1 + C(\lambda - \mu) \big) \int_{D_{\nu(\varepsilon)}} |\nabla v_{\nu(\varepsilon)}|^2 \, d\zeta. \tag{5.20}$$

It follows from relations (5.18) and equations (5.11) for the harmonic function v_λ that this function satisfies the same boundary conditions as φ_ε. From this and the Dirichlet principle we conclude that

$$\int_{D_\lambda} |\nabla v_\lambda|^2 \, d\zeta \le \int_{D_\lambda} |\nabla \varphi_\varepsilon|^2 \, d\sigma.$$

By substituting this inequality in (5.20) we obtain

$$\int_{D_\lambda} |\nabla v_\lambda|^2 \, d\zeta \le \big(1 + C(\lambda - \mu) \big) \int_{D_{\nu(\varepsilon)}} |\nabla v_{\nu(\varepsilon)}|^2 \, d\zeta.$$

Since the Dirichlet integral is invariant with respect to a conformal transform, we can rewrite this inequality in the equivalent form

$$\int_{\Omega_\lambda} |\nabla u_\lambda|^2 \, dx \le \big(1 + C(\lambda - \mu) \big) \int_{\Omega_{\nu(\varepsilon)}} |\nabla u_{\nu(\varepsilon)}|^2 \, dx. \tag{5.21}$$

On the other hand, inequality (5.13) yields

$$\int_{\Omega_{\nu(\varepsilon)}} |\nabla u_{\nu(\varepsilon)}|^2 \, dx \le \int_\Omega |\nabla u|^2 \, dx + \varepsilon.$$

By substituting this inequality into (5.21) we obtain

$$\int_{\Omega_\lambda} |\nabla u_\lambda|^2 \, dx \le \big(1 + C(\lambda - \mu) \big) \Big(\int_\Omega |\nabla u|^2 \, dx + \varepsilon \Big). \tag{5.22}$$

Letting $\varepsilon \to 0$ we arrive at the desired estimate (5.12). This completes the proof of Theorem 5. ∎

Stability

At the end of this sections, we study the domain dependence of solutions to the Dirichlet problem (5.1). Recall Definition 4 of the set $\mathscr{C}_h$. Consider doubly connected domains $\Omega = G \setminus \omega$, $\Omega_n = G \setminus \omega_n$, $n \ge 1$, where connected compact sets $\omega, \omega_n \in \mathscr{C}_h$. Denote by u and u_n the corresponding weak solutions of the boundary value problems

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial G, \quad u = 0 \text{ on } \partial\omega. \tag{5.23}$$

$$\Delta u_n = 0 \text{ in } \Omega_n, \quad u_n = g \text{ on } \partial G, \quad u_n = 0 \text{ on } \partial\omega_n. \tag{5.24}$$

Assume that they satisfy the following conditions

C.1 There is $c > 0$ such that

$$\int_\Omega |\nabla u|^2\, dx + \int_{\Omega_n} |\nabla u_n|^2\, dx \le c, \quad n \ge 1. \tag{5.25}$$

C.2

$$0 < \mu^- \le \mu_n \le \mu^+ < 1, \quad \mu_n \to \mu \text{ as } n \to \infty, \tag{5.26}$$

where $1/\mu_n$ and $1/\mu$ are the conformal modulus of Ω_n and Ω.

C.3 The domains Ω_n (the compact sets ω_n) converge to the domain Ω (the compact set ω) in the sense of Caratheodory-Hausdorff.

Theorem 6 *Let conditions* **C.1**-**C.3** *be satisfied. Then*

$$\int_{\Omega_n} |\nabla u_n|^2\, dx \to \int_\Omega |\nabla u|^2\, dx \quad \textit{as} \quad n \to \infty. \tag{5.27}$$

Proof Denote by $w_n : \Omega_n \to D_{\mu_n}$ and $w : \Omega \to D_\mu$ the conformal mappings, satisfying the normalization conditions

$$w_n(z_0) = \zeta_0, \quad w(z_0) = \zeta_0, \quad w_n'(z_0) > 0, \quad w'(z_0) > 0,$$

with fixed $z_0 \in \mathcal{A}(h)$, $\zeta_0 \in D_{\mu^+}$. For every $\lambda \in (\mu, 1)$, introduce the domain $\Omega_\lambda = w^{-1}(D_\lambda)$. Since $\mu_n \to \mu$, the inequality $\mu_n < \lambda$ holds for all large n. Further, we will assume that n is sufficiently large. Set $\Omega_{n,\lambda} = w_n^{-1}(D_\lambda)$. The doubly-connected domain Ω_λ is bounded by the C^∞ Jordan curves ∂G and $\Gamma_\lambda = w^{-1}(\{|\zeta| = \lambda\})$. In its turn, the doubly-connected domain $\Omega_{n,\lambda}$ is bounded by the C^∞ Jordan curves ∂G and $\Gamma_{n,\lambda} = w_n^{-1}(\{|\zeta| = \lambda\})$.

We also introduce the harmonic functions $u_\lambda \in C^\infty(\overline{\Omega_\lambda})$ and $u_{n,\lambda} \in C^\infty(\overline{\Omega_{n,\lambda}})$, satisfying the equations

$$\Delta u_\lambda = 0 \text{ in } \Omega_\lambda, \quad u_\lambda = g \text{ on } \partial G, \quad u_\lambda = 0 \text{ on } \Gamma_\lambda. \tag{5.28}$$

$$\Delta u_{n,\lambda} = 0 \text{ in } \Omega_{n,\lambda}, \quad u_{n,\lambda} = g \text{ on } \partial G, \quad u_{n,\lambda} = 0 \text{ on } \Gamma_{n,\lambda}. \tag{5.29}$$

Note that Ω_n, Ω and μ_n, μ meet all the requirements of Theorem 5 which yields

$$\begin{gathered}\int_\Omega |\nabla u|^2\, dx \le \int_{\Omega_\lambda} |\nabla u_\lambda|^2\, dx \le \big(1 + C(\lambda - \mu)\big) \int_\Omega |\nabla u|^2\, dx, \\ \int_{\Omega_n} |\nabla u_n|^2\, dx \le \int_{\Omega_{n,\lambda}} |\nabla u_{n,\lambda}|^2\, dx \le \big(1 + C(\lambda - \mu)\big) \int_{\Omega_n} |\nabla u_n|^2\, dx\end{gathered} \tag{5.30}$$

for $\lambda < (1 + \mu^+)/2$ and $\lambda > \mu$, $\lambda > \mu_n$. Here, the constant C depends only on $\mu^\pm$. Now, fix an arbitrary $\varepsilon > 0$ and choose λ such that

$$0 < C(\lambda - \mu) < \varepsilon, \quad \lambda < (1 + \mu^+)/2.$$

Since $\mu_n \to \mu$ as $n \to \infty$, we have

$$0 < C(\lambda - \mu_n) < \varepsilon \quad \text{for all large} \quad n.$$

From this and (5.30) we conclude that the inequalities

$$\int_{\Omega} |\nabla u|^2\, dx \le \int_{\Omega_\lambda} |\nabla u_\lambda|^2\, dx \le \big(1+\varepsilon\big) \int_{\Omega} |\nabla u|^2\, dx,$$
$$\int_{\Omega_n} |\nabla u_n|^2\, dx \le \int_{\Omega_{n,\lambda}} |\nabla u_{n,\lambda}|^2\, dx \le \big(1+\varepsilon\big) \int_{\Omega_n} |\nabla u_n|^2\, dx$$

hold for all sufficiently large n. In particular, we have

$$\frac{1}{1+\varepsilon} \int_{\Omega_\lambda} |\nabla u_\lambda|^2\, dx \le \int_{\Omega} |\nabla u|^2\, dx. \tag{5.31}$$

We thus get

$$\begin{aligned} \int_{\Omega_n} |\nabla u_n|^2\, dx - \int_{\Omega} |\nabla u|^2\, dx &\le \int_{\Omega_{n,\lambda}} |\nabla u_{n,\lambda}|^2\, dx - \frac{1}{1+\varepsilon} \int_{\Omega_\lambda} |\nabla u_\lambda|^2\, dx \\ &= \mathbf{I}_n + \frac{\varepsilon}{1+\varepsilon} \int_{\Omega_\lambda} |\nabla u_\lambda|^2\, dx \le \mathbf{I}_n + \varepsilon \int_{\Omega} |\nabla u|^2\, dx, \end{aligned} \tag{5.32}$$

where

$$\mathbf{I}_n = \int_{\Omega_{n,\lambda}} |\nabla u_{n,\lambda}|^2\, dx - \int_{\Omega_\lambda} |\nabla u_\lambda|^2\, dx. \tag{5.33}$$

The permutation $(u_n, u) \to (u, u_n)$ gives the the symmetric inequality

$$\int_{\Omega} |\nabla u|^2\, dx - \int_{\Omega_n} |\nabla u_n|^2\, dx \le -\mathbf{I}_n + \varepsilon \int_{\Omega_n} |\nabla u_n|^2\, dx.$$

Upon combining this result with (5.32) we get the two-sided estimate

$$\mathbf{I}_n - \varepsilon \int_{\Omega} |\nabla u_n|^2\, dx \le \int_{\Omega_n} |\nabla u_n|^2\, dx - \int_{\Omega} |\nabla u|^2\, dx \le \mathbf{I}_n + \varepsilon \int_{\Omega} |\nabla u|^2\, dx,$$

which holds true for all sufficiently large n. From this and condition **C.1** we conclude that

$$\mathbf{I}_n - c\varepsilon \le \int_{\Omega_n} |\nabla u_n|^2\, dx - \int_{\Omega} |\nabla u|^2\, dx \le \mathbf{I}_n + c\varepsilon, \tag{5.34}$$

where c is independent of n and ε. Let us prove that $\mathbf{I}_n$ tends to 0 as $n \to \infty$. Introduce the harmonic functions $v_\lambda, v_{n,\lambda} : D_\lambda \to \mathbb{R}$, defined by the equalities

$$v_\lambda = u_\lambda \circ w^{-1}, \quad v_{n,\lambda} = u_{n,\lambda} \circ w_n^{-1}. \tag{5.35}$$

It follows from (5.33) and the conformal invariance of the Dirichlet integral that

$$\mathbf{I}_n = \int_{D_\lambda} \Big(|\nabla v_{n,\lambda}|^2 - |\nabla v_\lambda|^2 \Big)\, d\zeta.$$

From this and the identity

$$|\nabla v_{n,\lambda}|^2 - |\nabla v_\lambda|^2 = (\nabla v_{n,\lambda} - \nabla v_\lambda)\cdot(\nabla v_{n,\lambda} + \nabla v_\lambda)$$

we conclude that

$$\begin{aligned} |\mathbf{I}_n| \le c\|v_{n,\lambda} - v_\lambda\|_{C^1(D_\lambda)} \Big(\int_{D_\lambda} (|\nabla v_{n,\lambda}|^2 + |\nabla v_\lambda|^2)\, d\zeta \Big)^{1/2} = \\ c\|v_{n,\lambda} - v_\lambda\|_{C^1(D_\lambda)} \Big(\int_{\Omega_{n,\lambda}} |\nabla u_{n,\lambda}|^2 dx + \int_{\Omega_\lambda} |\nabla u_\lambda|^2\, dx \Big)^{1/2} \\ \le c(1+\varepsilon)^{1/2}\|v_{n,\lambda} - v_\lambda\|_{C^1(D_\lambda)} \Big(\int_{\Omega_n} |\nabla u_n|^2 dx + \int_{\Omega} |\nabla u|^2)\, dx \Big)^{1/2} \\ \le c\|v_{n,\lambda} - v_\lambda\|_{C^1(D_\lambda)}. \end{aligned} \tag{5.36}$$

Here we use inequality (5.31). Next, note that the function $v_{n,\lambda} - v_\lambda$ satisfies the equations

$$\begin{aligned} \Delta(v_{n,\lambda} - v_\lambda) = 0 \text{ in } D_\lambda, \\ v_{n,\lambda} - v_\lambda = f_n - f \text{ for } |\zeta| = 1, \;\; v_{n,\lambda} - v_\lambda = 0 \text{ for } |\zeta| = \lambda, \end{aligned} \tag{5.37}$$

where

$$f_n = g(w_n^{-1}(e^{i\theta})), \quad f = g(w^{-1}(e^{i\theta})).$$

We will consider ∂G and $\mathbb{S}^1$ as parametrized curves equipped with natural parametrization $z = Z(s)$, $0 \le s \le L$, and $\zeta = e^{i\theta}$, $0 \le \theta \le 2\pi$. We will write simply $g(s)$ instead of $g(Z(s))$. It is clear that the L-periodic function $g(s)$ belongs to the class $C^\infty[0, L]$. With this notation we may rewrite the expressions for f_n and f in the parametric form

$$f_n(e^{i\theta}) = g \circ \varphi_n^{-1}(\theta), \quad f(e^{i\theta}) = g \circ \varphi^{-1}(\theta). \tag{5.38}$$

Here, the mapping φ, determined by Definition 8, establishes the connection between parameters s and θ, induced by the conformal mapping w. The definition of φ_n is similar. Conditions **C.1-C.3** imply that the conformal mappings w_n and w meet all the requirements of Proposition 3. It follows from relation (3.11) in this proposition that

$$\varphi_n^{-1} \to \varphi^{-1} \text{ in } C^r[0, 2\pi] \;\text{ and hence }\; g \circ \varphi_n^{-1} - g \circ \varphi^{-1} \to 0 \;\text{ in } C^r[0, 2\pi]$$

as $n \to \infty$ for all integers $r \geq 0$. It follows that $f_n - f \to 0$ as $n \to \infty$ in every space $C^r(\mathbb{S}^1)$, $r \geq 0$. Note that λ and ε are independent of n. The standard a priori estimate for solutions to problem (5.37) implies the relation

$$\|v_{n,\lambda} - v_\lambda\|_{C^r(D_\lambda)} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad r \geq 0,$$

which, along with (5.36), yields the equality $\lim\limits_{n\to\infty} \mathbf{I}_n = 0$. Letting $n \to \infty$ in (5.34) we finally obtain

$$-c\varepsilon \leq \liminf_{n\to\infty} \Big(\int_{\Omega_n} |\nabla u_n|^2 \, dx - \int_\Omega |\nabla u|^2 \, dx \Big) \leq$$
$$\limsup_{n\to\infty} \Big(\int_{\Omega_n} |\nabla u_n|^2 \, dx - \int_\Omega |\nabla u|^2 \, dx \Big) \leq c\varepsilon,$$

where $\varepsilon > 0$ is an arbitrary small number and c is independent of ε. This completes the proof of Theorem 6 ■

6. Proof of Theorem 1

Recall the formulation of the main Theorem 1. We fix a bounded simply connected domain $G \subset \mathbb{R}^2$ with C^∞ boundary and two functions $g \in C^\infty(\partial G)$, $Q \in C^\infty(G)$, satisfying the conditions

$$g > c_g > 0 \quad \text{on} \quad \partial G, \quad Q > C^- > 0 \quad \text{in} \quad \Omega. \tag{6.1}$$

Denote by $\mathscr{C}^+$ the totality of all connected compact sets $\omega \Subset G$ of the positive measure $|\omega| > 0$ such that the set $\Omega = G \setminus \omega$ is open and connected. This means that Ω is doubly-connected. Introduce the functional

$$\mathcal{J}(\omega) = \int_\Omega |\nabla u|^2 dx + \int_\Omega Q^2 \, dx, \quad \Omega = G \setminus \omega, \tag{6.2}$$

where u is a solution to the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial G, \quad u = 0 \quad \text{on} \quad \partial\omega. \tag{6.3}$$

See Definition 3 for the explicit definition of the weak solution to problem (6.4). We also suppose that g and Q satisfy the following nondegeneracy condition: There is $\omega_0 \in \mathscr{C}^+$ such that

$$\mathcal{J}(\omega_0) < \int_G |\nabla U|^2 dx + \int_G Q^2 dx. \tag{6.4}$$

Here U is the solution to the Dirichlet problem

$$\Delta U = 0 \quad \text{in} \quad G, \quad U = g \quad \text{on} \quad \partial G.$$

Our main goal is to prove that the variational problem

$$\min_{\omega\in\mathscr{C}^+} \mathcal{J}(\omega) \tag{6.5}$$

has a solution $\omega \in \mathscr{C}^+$. Let us consider the minimizing sequence ω_n, $n \geq 1$, such that

$$\mathcal{J}(\omega_n) \to \inf_{\omega\in\mathscr{C}^+} \mathcal{J}(\omega). \tag{6.6}$$

We prove that this sequence contains a subsequence, which converges to a solution of problem (6.5) in the sense of Caratheodory-Hausdorff. Our considerations are based on Theorem 6. Therefore, it is necessary to show that the minimizing sequence meets all requirements of this theorem. The proof falls into a sequence of lemmas.

Lemma 8 *Let $\omega \in \mathscr{C}^+$ and $1/\mu$ be a conformal modulus of the domain $\Omega = G\backslash\omega$. Then, $|\omega| \leq c_G\ \mu^2$, where c_G depends only on G.*

Proof First, we consider the case of $G = B$, where B is a unit circle. It follows from identity (4.5) that

$$2\pi \Big(\ln \big(\frac{1}{\mu}\big)\Big)^{-1} = \text{ cap } (B, \omega). \tag{6.7}$$

On the other hand, the condenser capacity does not increase after the Steiner symmetrization of ω. This leads to the well known inequality

$$\text{cap } (B, \omega) \geq \text{cap } (B, B_\rho), \quad \text{where} \quad B_\rho = \{|z| \leq \rho\}, \ \rho = \pi^{-1/2}\, |\omega|^{1/2}. \tag{6.8}$$

It is easy to see that

$$\text{cap } (B, B_\rho) = \int_{B\backslash B_\rho} |\nabla\varphi|^2\, dx, \quad \text{where} \quad \varphi = \ln|z|/\ln\rho,$$

which yields

$$\text{cap } (B, B_\rho) = 2\pi\Big(-\ln\sqrt{\frac{|\omega|}{\pi}}\ \Big)^{-1}.$$

By combining this identity with (6.7) and (6.8) we finally obtain

$$2\pi \Big(\ln \big(\frac{1}{\mu}\big)\Big)^{-1} \geq 2\pi\Big(-\ln\sqrt{\frac{|\omega|}{\pi}}\Big)^{-1},$$

or, equivalently,

$$\frac{1}{\mu}\sqrt{\frac{|\omega|}{\pi}} \leq 1 \quad \text{and} \quad |\omega| \leq \pi\mu^2.$$

Let us turn to the general case of the simply connected domain G with c^∞ boundary. Note that there is C^∞ conformal diffeomorphism $w : \overline{G} \to \overline{B}$. It takes the connected compact set $\omega \subset G$ onto the connected compact set $w(\omega) \subset B$. Recall that the conformal modulus is invariant with respect to conformal transformations. It remains to note that $|\omega| \leq c|w(\omega)|$, and the lemma follows. ■

Lemma 9 *Let $\omega_n \in \mathscr{C}^+$ be a minimizing sequence, satisfying condition* (6.6). *Then, there are constants μ^- and θ, independent of n, such that*

$$0 < \mu^- \leq \mu_n, \quad 0 < \theta \leq |\omega_n| \ \ \textit{for all} \ \ n \geq 1. \tag{6.9}$$

Here, $1/\mu_n$ is the conformal modulus of the doubly-connected domain $\Omega_n = G \setminus \omega_n$.

Proof Suppose that the first inequality in (6.9) is false. After passing to a subsequence we may assume that $\mu_n \to 0$ as $n \to \infty$. It follows from this and Lemma 8 that

$$|\omega_n| \leq c_G \ \mu_n^2 \to 0 \ \text{ as } \ n \to \infty.$$

We thus get

$$\mathcal{J}(\omega_n) \geq \int_G |\nabla U|^2 \, dx + \int_{G\setminus\omega_n} Q^2 \, dx \to \int_G |\nabla U|^2 \, dx + \int_G Q^2 \, dx.$$

From this and nondegeneracy condition (6.4) we obtain $\liminf \mathcal{J}(\omega_n) > \inf \mathcal{J}$. Hence, ω_n is not a minimizing sequence. The contradiction proves the first estimate in (6.9). Repeating correspondingly these arguments gives the second. ■

Lemma 10 *Assume that a solution to problem* (6.3) *in the domain $\Omega = G \setminus \omega$, $\omega \in \mathscr{C}^+$, satisfies the inequality*

$$\int_\Omega |\nabla u|^2 \, dx \leq E.$$

Then the conformal modulus of Ω admits the estimate

$$\mu \leq e^{-\frac{2\pi c_g^2}{E}}, \tag{6.10}$$

where c_g is the constant in condition (6.1)

Proof Fix a conformal mapping $w : \Omega \to D_\mu$ and choose an arbitrary $\lambda \in (\mu, 1)$. Let us consider the doubly-connected domain $\Omega_\lambda = w^{-1}(D_\lambda)$. It is bounded by C^∞ Jordan curves $\partial G = w^{-1}(\mathbb{S}^1)$ and $\Gamma_\lambda = w^{-1}(\{|\zeta| = \lambda\})$. It follows from

Theorem 3 that $w \in C^\infty(\overline{\Omega_\lambda})$ and $w^{-1} \in C^\infty(\overline{D_\lambda})$. Introduce the harmonic function $u_\lambda \in C^\infty(\overline{\Omega_\lambda})$, satisfying the equations

$$\Delta u_\lambda = 0 \ \text{ in } \ \Omega_\lambda, \ \ u = g \ \text{ on } \ \partial G, \ \ u = 0 \ \text{ on } \ \Gamma_\lambda.$$

Since the distance between ω and ∂G is positive, we have $\omega \in \mathscr{C}_h$ for some $h > 0$. Obviously, μ satisfies inequalities (5.6) with $\mu^\pm = \mu$. Hence, u meets all the requirements of Theorem 5, which yields

$$\int_{\Omega_\lambda} |\nabla u_\lambda|^2 \, dx \le a_\lambda \int_\Omega |\nabla u|^2 \, dx \le a_\lambda E \ \text{ for } \ \lambda \le (1+\mu)/2.$$

Here, $a_\lambda = 1 + C(\mu - \lambda)$, where C depends only on μ. It is clear that $a_\lambda \to 1$ as $\lambda \to \mu$. Next, introduce the harmonic function $v_\lambda : D_\lambda \to \mathbb{R}$, defined by the equality $v_\lambda = u_\lambda \circ w^{-1}$. Since the Dirichlet integral is invariant with respect to conformal transforms, we have

$$\int_{D_\lambda} |\nabla v_\lambda|^2 \, d\zeta = \int_{\Omega_\lambda} |\nabla u_\lambda|^2 \, dx \le a_\lambda \, E \ \text{ for } \ \lambda \le (1+\mu)/2. \tag{6.11}$$

Moreover, v_λ satisfies the equations and boundary conditions

$$\Delta v_\lambda = 0 \ \text{ in } \ D_\lambda,$$
$$v_\lambda(\zeta) = f(\zeta) \ \text{ for } \ |\zeta| = 1, \quad v_\lambda(\zeta) = 0 \ \text{ for } \ |\zeta| = \lambda,$$

where $f(e^{i\theta}) = g(w^{-1}(e^{i\theta}))$ belongs to the class $C^\infty(\mathbb{S}^1)$ and satisfies the inequality $f \ge c_g$. Here, c_g is the constant in (6.1). It follows from this and (6.11) that the function

$$\varphi(\zeta) = 1 - \min \Big\{ 1, \frac{1}{c_g} v_\lambda(\zeta) \Big\}, \quad \zeta \in D_\lambda,$$

satisfies the conditions

$$\varphi = 0 \text{ for } |\zeta| = 1, \quad \varphi = 1 \text{ for } |\zeta| = \lambda, \quad \int_{D_\lambda} |\nabla \varphi|^2 \le \frac{a_\lambda}{(c_g)^2} \, E.$$

It follows that

$$\text{cap } (B, B_\lambda) \le \frac{a_\lambda}{(c^-)^2} \, E, \quad B_\lambda = \{|\zeta| \le \lambda\}.$$

Since the conformal modulus of the annulus $D_\lambda = B \setminus B_\lambda$ equals $1/\lambda$, we have

$$2\pi \Big(\ln \frac{1}{\mu} \Big)^{-1} \le 2\pi \Big(\ln \frac{1}{\lambda} \Big)^{-1} = \text{cap } (B, B_\lambda) \le \frac{a_\lambda}{(c_g)^2} \, E.$$

From this we conclude that

$$\mu \le e^{-\frac{2\pi c_g^2}{a_\lambda E}} \to e^{-\frac{2\pi c_g^2}{E}} \ \text{ as } \ \lambda \to \mu, \tag{6.12}$$

and the lemma follows. ∎

Lemma 11 *Let $\omega_n \in \mathscr{C}^+$ be a minimizing sequence, satisfying condition* (6.6). *Then there is a constant $\mu^+ < 1$, independent of n, such that $\mu_n \le \mu^+$ for all $n \ge 1$. Here, $1/\mu_n$ is the conformal modulus of the doubly-connected domain $\Omega_n = G \setminus \omega_n$.*

Proof It suffices to note that the Dirichlet integrals of the functions u_n are uniformly bounded and to apply Lemma 10. ∎

Lemma 12 *Let $\omega_n \in \mathscr{C}^+$ be a minimizing sequence, satisfying condition* (6.6). *Then there is a constant $h > 0$, independent of n, such that* dist $(\omega_n, \partial G > h$ *for all $n \ge 1$. In other words, $\omega_n \in \mathscr{C}_h$.*

Proof Suppose, contrary to our claim, that there is a subsequence of the sequence ω_n, still denoted by ω_n, such that

$$\epsilon_n \equiv \text{ dist } (\omega_n, \partial G) \to 0 \ \text{ as } \ n \to \infty. \tag{6.13}$$

By virtue of Lemma 11, we have

$$\text{cap } (G, \omega_n) = 2\pi\Big(\ln \frac{1}{\mu_n} \Big)^{-1} \le 2\pi\Big(\ln \frac{1}{\mu^+} \Big)^{-1}. \tag{6.14}$$

On the other hand, it follows from Lemma 9 that $|\omega_n| \ge \theta > 0$, where θ is independent of n. Hence, ω_n meets all the requirements of Proposition 4. By virtue of this proposition, there are $\lambda \in (0,1)$ and a continuous function $\Phi : (0, \lambda) \to \mathbb{R}^+$, depending on G and θ, such that

$$\text{cap } (G, \omega_n) \ge \Phi(\epsilon_n) \to \infty \ \text{ as } \ \varepsilon_n \to 0.$$

It remains to note that this relation contradicts the estimate (6.14). ∎

We are now in a position to complete the proof of Theorem 1. Let us consider the minimizing sequence ω_n, satisfying condition (6.6). It follows from Lemmas 9, 11, and 12 that there are $\mu^\pm \in (0,1)$, $\theta > 0$, and $h > 0$ with the properties

$$0 < \mu^- \le \mu_n \le \mu^+ < 1, \quad |\omega_n| > \theta, \quad \omega_n \in \mathscr{C}_h. \tag{6.15}$$

After passing to a subsequence, we may assume that

$$\omega_n \to \omega \in \mathscr{C}_h \ \text{ in the sense of Caratheodory-Hausdorff and } \ \mu_n \to \mu$$

as $n \to \infty$. Relations (6.15) imply that the sequence ω_n satisfies conditions of the stability Theorem 6, which yields

$$\int_{\Omega_n} |\nabla u_n|^2 \, dx \to \int_{\Omega} |\nabla u|^2 \, dx. \tag{6.16}$$

It follows from Definition 7 of the Caratheodory-Hausdorff convergence that Ω_n converge to the kernel Ω in the sense of Caratheodory. Therefore, every compact subset of Ω belongs to Ω_n for all large n. We thus get

$$\int_K Q^2 dx \le \liminf \int_{\Omega_n} Q^2\, dx, \quad |K| \le \liminf |\Omega_n| \le |G| - \theta$$

for every compact set $K \Subset \Omega$, which yields

$$\int_\Omega Q^2 dx \le \liminf \int_{\Omega_n} Q^2\, dx, \quad |\omega| \ge \theta.$$

Combining this result with (6.16) we finally obtain

$$\mathcal{J}(\omega) \le \liminf \mathcal{J}(\omega_n) \ \text{ and } \ \omega \in \mathscr{C}^+.$$

Hence, ω is a solution to variational problem (6.5) and the theorem follows. □

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