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On invex functions with the same mapping η in single- and multivalued nonsmooth optimization*

by

Ville Rinne, Yury Nikulin* and Marko M. Mäkelä

Department of Mathematics and Statistics, University of Turku,
Vesilinnantie 5, FI-20014 Turku, Finland
*corresponding author: yurnik@utu.fi

Abstract: In this paper, we investigate nonsmooth unconstrained single- and multiobjective optimization problems, in which the objective functions are locally Lipschitz continuous (LLC) and possibly nondifferentiable. Using the Clarke subdifferential, we study invexity properties of such functions and focus on the characterization of finite families of functions that are invex with respect to the same mapping η. We establish the necessary and sufficient optimality conditions for weak Pareto and Pareto optimal solutions under the assumption of V-invexity. Furthermore, we derive new equivalence results, linking the invexity of individual component functions with the invexity of their scalarized counterparts in the nonsmooth setting. We show that a straightforward extension of known differentiable results is not always possible and illustrate this limitation through a carefully constructed counterexample. The results obtained generalize the existing characterizations of invexity with respect to the same mapping to the class of nonsmooth LLC functions, thereby extending the applicability of invexity theory in nonsmooth and multiobjective optimization.

Keywords: invexity, nonsmooth analysis, Clarke subdifferential, single- and multivalued optimization

1. Introduction: related work and background

Invexity and generalized convexity have been widely investigated in mathematical programming, particularly in Euclidean space settings and multiobjective optimization. Important contributions include invexity-based optimality and duality results for multiobjective programming (Osuna-Gómez, 1998; Rufián-Lizana and Ruiz-Canales, 1998), as well as generalized invexity frameworks for

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minimax and fractional programming problems (Antczak, Mishra amd Upadhyay, 2016; Jayswal, Kummari and Ahmad, 2015). These studies demonstrate the effectiveness of invex-type assumptions in deriving first-order optimality conditions beyond classical convexity. The concept of invexity broadens the scope of optimization by enabling the solution of nonconvex problems under more relaxed conditions, making it highly valuable in fields such as economics, engineering, and operations research (Craven, 2008). In essence, invexity provides a powerful tool for solving a wider array of optimization problems without sacrificing the assurance of finding optimal solutions.

In recent years, several authors have extended invexity concepts to nonsmooth and variational settings. In particular, nonsmooth semi-infinite and minimax programming problems involving generalized invexity have been studied in Upadhyay and Mishra (2015) and Upadhyay et al. (2022), while higher-order and variational-like inequalities under invexity assumptions have been analyzed in Upadhyay and Mishra (2020). Related optimization questions via variational inequalities have also been investigated in Treanţă and Guo (2023). Some recent developments in multiobjective optimization algorithms under uncertainty can also be found in Kumar et al. (2024).

These works primarily focus on specific problem classes (e.g., minimax, fractional, or variational inequalities) and often rely on problem-dependent generalized invexity notions.

In nonsmooth optimization, LLC functions constitute a natural and practically relevant class, as they arise frequently in applications involving max-type functions, absolute values, piecewise smooth models, and vector-valued objectives. For such functions, classical derivatives may fail to exist on sets of nonzero measure, making generalized derivative concepts essential. Among the available notions, the Clarke subdifferential provides a particularly suitable framework, due to its robustness, stability, and strong calculus properties (Clarke, 1983). Using the Clarke subdifferential framework, we had a look at a nonsmooth theory of invexity with respect to the same mapping η that applies to both single- and multiobjective optimization problems. Unlike existing nonsmooth invexity results (see Reiland, 1990; Upadhyay and Mishra, 2015), our analysis establishes equivalence conditions linking individual invexity, scalarization, and global optimality properties. Moreover, we demonstrate through a counterexample that naive extensions of the differentiable theory fail without additional assumptions.

2. Contributions and organization of the paper

Despite the extensive body of work on invexity in both smooth and nonsmooth optimization, some fundamental structural question remains insufficiently addressed: how to characterize families of objective functions that are invex with respect to a common mapping η in the presence of nonsmoothness and vector-valued objectives. In this paper, we follow the idea of Martínez-Legaz (2009) to address the issue. The motivation for that comes from the fact that invexity with respect to the same mapping η is a desirable theoretical property for a finite set of LLC functions, unifying them. It can efficiently be used in optimality conditions, algorithm design, and so on.

The paper is organized as follows. Section 3 contains all the necessary preliminaries. Section 4 discusses invex nonsmooth functions for single and multivalued functions. Two theorems providing a link between optimality and invexity are highlighted. Section 5 contains the main result of the paper. We start by presenting the known result of Martínez-Legaz (2009) for differentiable cases. Then, we give a counterexample, showing that a straightforward generalization of the result is not possible for the case of nonsmooth LLC functions without the assumption of the Clarke subdifferential regularity. Then, we formulate the main result of the paper (Theorem 7) showing the equivalence of four certain conditions characterizing V-invex functions with respect to the same mapping η.

3. Preliminaries: basic concepts and definitions

In this section, we discuss functions that are neither continuously differentiable nor convex. They have points in their domain where their gradient is not continuous. In these points, we define the Clarke generalized directional derivative and subgradient.

We start by defining classical concepts from nonsmooth analysis - local Lipschitz continuity (LLC) and Lipschitz continuity (LC).

Definition 1 (Bagirov, Karmitsa and Mäkelä, 2014) *A function $f : \mathbb{R}^n \to \mathbb{R}$ is* LLC at a point $\boldsymbol{x} \in \mathbb{R}^n$ *if there exist scalars $L_{\boldsymbol{x}} > 0$ and $\delta_{\boldsymbol{x}} > 0$ such that*

$$|f(\boldsymbol{y}) - f(\boldsymbol{z})| \leq L_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{z}\|, \ \forall \ \boldsymbol{y}, \boldsymbol{z} \in B(\boldsymbol{x}; \delta_{\boldsymbol{x}}),$$

where $B(\boldsymbol{x}; \delta_{\boldsymbol{x}})$ denotes the open ball with the center $\boldsymbol{x}$ and radius $\delta_{\boldsymbol{x}}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is LLC *if it is LLC at every point belonging to $\mathbb{R}^n$.*

Definition 2 (Bagirov, Karmitsa and Mäkelä, 2014) *A function $f : \mathbb{R}^n \to \mathbb{R}$ is* LC *if there exists a scalar L such that*

$$|f(\boldsymbol{y}) - f(\boldsymbol{z})| \leq L \|\boldsymbol{y} - \boldsymbol{z}\|, \ \forall \ \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n.$$

Definition 3 *A vector-valued function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^p$ is LC if each vector component is LC.*

We continue by defining the Clarke generalized directional (sometimes referred to as the limiting directional) derivative for LLC functions and discuss its properties.

Definition 4 (Clarke, 1983) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be LLC at* $\boldsymbol{x}^* \in \mathbb{R}^n$*. The* Clarke generalized directional derivative *of* f *at* $\boldsymbol{x}^*$ *in direction* $\boldsymbol{d} \in \mathbb{R}^n$ *is defined as follows:*

$$f^o(\boldsymbol{x}^*; \boldsymbol{d}) = \limsup_{\boldsymbol{y} \to \boldsymbol{x}^*,\ t \to 0_+} \frac{f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y})}{t},$$

or equivalently,

$$f^o(\boldsymbol{x}^*; \boldsymbol{d}) = \inf_{\delta > 0} \sup_{\substack{\|\boldsymbol{y} - \boldsymbol{x}^*\| \leq \delta, \\ 0 < t < \delta}} \frac{f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y})}{t}.$$

Note that for LLC functions, we always have $f^o(\boldsymbol{x}^*; \boldsymbol{d}) < \infty$. In the definition of the classical directional derivative, the base point for taking differences is a fixed vector $\boldsymbol{x}^*$. In the generalized directional derivative, they are taken from a variable vector $\boldsymbol{y}$, which approaches $\boldsymbol{x}^*$.

Next, we define the Clarke generalized subdifferential (and subgradient as its element) using Clarke's generalized directional derivative and discuss its properties.

Definition 5 (Clarke, 1983) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be LLC function at a point* $\boldsymbol{x}^* \in \mathbb{R}^n$*. The* Clarke subdifferential *of* f *at* $\boldsymbol{x}^*$ *is the set* $\partial f(\boldsymbol{x}^*)$ *of vectors* $\boldsymbol{\xi} \in \mathbb{R}^n$ *such that*

$$\partial f(\boldsymbol{x}^*) = \{\boldsymbol{\xi} \mid f^o(\boldsymbol{x}^*; \boldsymbol{d}) \geq \boldsymbol{\xi}^T \boldsymbol{d},\ \forall\ \boldsymbol{d} \in \mathbb{R}^n\}.$$

Each $\boldsymbol{\xi} \in \partial f(\boldsymbol{x}^*)$ *is called a* subgradient *of* f *at* $\boldsymbol{x}^*$*.*

For a convex function, the above definition is equal to the subdifferential of a convex function (Clarke, 1983). The Clarke subdifferential has the so-called classical plus-minus symmetry, in other words, we have

$$\partial(-f(\boldsymbol{x}^*)) = -\partial f(\boldsymbol{x}^*). \tag{1}$$

In some books (Mordukhovich, 2018; Mordukhovich and Nguyen, 2022) plus-minus symmetry (1) for the Clarke subdifferential of LLC functions was mentioned as a drawback, leading to non-distinguishing between convex and concave functions, as well as local minima and maxima. However, the symmetry property might be crucial and desirable in many situations, because it assures that the Clarke subdifferential behaves "nicely" under the negation of the function, maintaining its consistency and robustness as a generalization of the derivative concept to nonsmooth contexts.

A well-known result formulated and proven by Rademacher (see Theorem 9.60 in Rockafellar and Wets, 2009) states that an LLC function f is differentiable almost everywhere. In particular, every neighborhood of $\boldsymbol{x}$ contains a point $\boldsymbol{y}$ such that $\nabla f(\boldsymbol{y})$ exists. So, we have an equivalent definition for the Clarke subdifferential of LLC function

$$\partial f(\boldsymbol{x}^*) = \text{conv}\{\boldsymbol{\xi} \in \mathbb{R}^n \mid \exists\, \boldsymbol{x}^k \to \boldsymbol{x}^*,\ \nabla f(\boldsymbol{x}^k) \text{ exists and } \nabla f(\boldsymbol{x}^k) \to \boldsymbol{\xi}\},\ (2)$$

where conv denotes the *convex hull* of a set. Let us notice that (2) can be conveniently used in practice to compute the Clarke subdifferential of an LLC function. Note that this result is valid only in finite dimensional spaces.

The *subdifferential in convex analysis*, also called the *convex subdifferential*, is defined for a *convex function* $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\boldsymbol{x^*}$ as:

$$\partial_{conv} f(\boldsymbol{x^*}) = \{\boldsymbol{\xi} \in \mathbb{R}^n : f(\boldsymbol{y}) \geq f(\boldsymbol{x^*}) + \boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x^*}), \forall\, \boldsymbol{y} \in \mathbb{R}^n\}. \qquad *$$

Naturally, any convex f is also LLC (Wayne State..., 1972).

For a general LLC function f, the Clarke subdifferential is typically larger than the convex subdifferential:

$$\partial f(\boldsymbol{x}^*) \supseteq \partial_{conv} f(\boldsymbol{x}^*),$$

with equality only when f is convex (Clarke, 1983).

The next theorem shows that subgradients are generalizations of the classical gradient.

THEOREM 1 (BAGIROV, KARMITSA AND MÄKELÄ, 2014) *If $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at $\boldsymbol{x} \in \mathbb{R}^n$, then*

$$\partial f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}.$$

PROOF See, e.g., Bagirov, Karmitsa and Mäkelä (2014), Theorem 3.7. ∎

The Jacobian matrix of a vector-valued LLC function can be similarly generalized for the nonsmooth case.

DEFINITION 6 (CLARKE, 1983) *The* generalized Jacobian matrix $J(\boldsymbol{F}((\boldsymbol{u}))$ *of a nonsmooth vector-valued LLC function $\boldsymbol{F} = (f_1, \ldots, f_p)$ at $\boldsymbol{u}$ is defined as*

$$J(\boldsymbol{F}(\boldsymbol{u})) = \Big(\partial f_i(\boldsymbol{u})\Big)_{i=1,\ldots,p},$$

where $\partial f_i(\boldsymbol{u})$ is the Clarke subdifferential of f_i at $\boldsymbol{u}$. In other words, the i^{th} row of the Jacobian is the Clarke subdifferential of f_i at $\boldsymbol{u}$.

4. Single and vector-valued invex LLC functions

In this section, we discuss invex nonsmooth functions, which are generalizations of convex functions. Invex functions are rather large groups of functions that retain the useful property of convex functions, namely that, under mild conditions, we have the following (see Theorem 1 in Reiland, 1990):

- for invex functions defined on $\mathbb{R}^n$ every stationary point $\boldsymbol{u} \in \mathbb{R}^n$ (that is, $\mathbf{0} \in \partial f(\boldsymbol{u})$) is a global minimum.

Invexity has been extensively studied as a structural generalization of convexity, aimed at preserving global optimality properties under weaker assumptions; see, for example, the foundational discussion in Martin (1985).

First, we define invexity for single-valued nonsmooth functions and then generalize the concept of invexity for vector-valued nonsmooth functions.

Let us present the definition for an invex nonsmooth function, which was first conceived by Hanson (1981) for differentiable functions. The following definition from Reiland (1990) generalizes the notion of invexity for nonsmooth functions.

Definition 7 (Reiland, 1990) *An LLC function $f : \mathbb{R}^n \to \mathbb{R}$ is* invex *if there exists a function $\boldsymbol{\eta} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for all $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n$ and for all $\boldsymbol{\xi} \in \partial f(\boldsymbol{u})$ we have*

$$f(\boldsymbol{x}) - f(\boldsymbol{u}) \geq \boldsymbol{\xi}^T \boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u}),$$

or, equivalently,

$$f(\boldsymbol{x}) - f(\boldsymbol{u}) \geq f^o(\boldsymbol{u}; \boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u})).$$

Here, $f^o(\boldsymbol{u}; \boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u}))$ is the Clarke generalized directional derivative, defined at point $\boldsymbol{u}$ along the direction $\boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u})$ (see Definition 4). By setting $\boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u}) = \boldsymbol{u} - \boldsymbol{x}$, we can see that convexity is a special case of invexity.

Hanson (1999) also gave a slightly relaxed definition for invexity. These types of functions are called *Type I invex functions*, and they form the most general class of functions, for which the KKT conditions are necessary and sufficient for a global minimum. Unlike invex functions, invexity for Type I invex functions is defined relative to a fixed point $\boldsymbol{u}$, and for this reason, they are not necessarily invex for every point in $\mathbb{R}^n$.

The utility of invex functions was established when Hanson showed that if the objective and constraint functions of a nonlinear programming model are invex with respect to the same $\boldsymbol{\eta}$, then the weak duality and sufficiency of the KKT conditions still hold. The word invex descends from a contraction of 'invariant convex', and Craven proposed it in Craven (1981). Craven and Glover (1985) as well as Ben-Israel and Mond (1986) showed that the class of invex functions under the usual "closed cone" assumption is equivalent to the class of functions whose stationary points are global minima in the unconstrained case.

Notice that invexity can also be introduced for non-LLC functions. For example, in Singh and Laha (2023), invexity is introduced for vector quasidifferentiable (Denyanov and Rubinov, 1980) functions using quasidifferential instead of the Clarke subdifferential. More details on quasidifferential can be found in Craven (1986), Demyanov (1986), Demyanov, Nikulin and Shablinskaya (1986), and Demyanov and Polyakov (1980).

Let us now recall the classical definition of weakly continuous functions.

Definition 8 (Rockafellar and Wets, 2009) *A function $f : \mathbb{R}^n \to \mathbb{R}$ is* weakly continuous *if it is continuous with respect to the weak topology on $\mathbb{R}^n$ (with possibly infinite n). This means that for any sequence $\{\boldsymbol{x}_k\} \in \mathbb{R}^n$ that converges weakly to $\boldsymbol{x}$, i.e.,*

$$\ell(\boldsymbol{x}_k) \to \ell(\boldsymbol{x}) \quad \text{for all } \ell \in (\mathbb{R}^n)^*,$$

it follows that

$$f(\boldsymbol{x}_k) \to f(\boldsymbol{x}),$$

where ℓ is a continuous linear functional, and $(\mathbb{R}^n)^*$ is the dual space of $\mathbb{R}^n$.

Intuitively, this means the sequence converges when viewed through the "lens" of every bounded linear functional in the dual space. We have the following well-known result in finite-dimensional spaces $\mathbb{R}^n$, because the weak topology and the standard topology coincide.

Lemma 1 (Rockafellar and Wets, 2009) *Any LLC function $f : \mathbb{R}^n \to \mathbb{R}$ is also weakly continuous.*

We will use this fact later when establishing some equivalence between Lemma 2 and Lemma 3.

Now we introduce convexity and sublinearity with respect to an arbitrary closed convex cone.

Definition 9 (Craven and Glover, 1985) *A function $f : \mathbb{R}^n \to \mathbb{R}$ is S-*convex *if, whenever $0 < \alpha < 1$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we have*

$$\alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \in S \subseteq \mathbb{R},$$

where S is a closed convex cone.

Definition 10 (Craven and Glover, 1985) *A function $f : \mathbb{R}^n \to \mathbb{R}$ is S-*sublinear *if it is S-convex and positively homogeneous of degree one, that is, $f(\alpha \boldsymbol{x}) = \alpha f(\boldsymbol{x}) \quad \forall\, \alpha \geq 0$.*

Note that when $S = \mathbb{R}_{\geq}$, S-convexity transforms into classical convexity as well as S-sublinearity transforms into usual sublinearity (subadditivity and positive homogeneity).

In order to formulate the subsequent supplementary results, assume that $\partial(\lambda f) = \lambda \partial f$ is a subdifferential of the function λf, and

$$\operatorname{ray} S = \{\lambda : \ \lambda y \geq 0 \ \ \forall\, y \in S\},$$

is one-dimensional cone (ray) dual to S. It is obvious that $\operatorname{ray} S = \mathbb{R}_{\geq}$ if $S = \mathbb{R}_{\geq}$, and hence, $\lambda \geq 0$. We also use standart notation $\operatorname{cl}(\cdot)$ to denote the closure of a set.

The following lemma states two mutually exclusive properties for S-sublinear weakly continuous functions.

LEMMA 2 (CRAVEN AND GLOVER, 1985) *(Craven and Glover's lemma of alternative) Let $f : \mathbb{R}^n \to \mathbb{R}$ be S-sublinear and weakly continuous, and let $z \in \mathbb{R}$. Then, exactly one of the following is satisfied:*

(i) $\exists\, \boldsymbol{x} \in \mathbb{R}^n$ such that $-f(\boldsymbol{x}) + z \in S$;

(ii) $(\mathbf{0}, -1)^T \in \operatorname{cl}\Big(\bigcup_{\lambda \in ray\, S} \big(\lambda \partial f(\mathbf{0}) \times \{\lambda z\}\big)\Big)$.

The following known result is used as a characterization of invexity, generalized for nonsmooth functions.

THEOREM 2 (REILAND, 1990) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be LLC. For each $\boldsymbol{u} \in \mathbb{R}^n$, assume that for every $\boldsymbol{x} \in \mathbb{R}^n$, the convex cone in $\mathbb{R}^n \times \mathbb{R}$ (epigraphical tangent cone or a cone representation of the Clarke subdifferential):*

$$K_{\boldsymbol{x}}(\boldsymbol{u}) = \bigcup_{\lambda \geq 0} \Big(\lambda \partial f(\boldsymbol{u}) \times \big\{\lambda(f(\boldsymbol{x}) - f(\boldsymbol{u}))\big\}\Big) \tag{3}$$

is closed. Then, f is invex if and only if every stationary point is a global minimum of f over $\mathbb{R}^n$.

Let us note that the assumption on the closedness of $K_{\boldsymbol{x}}(\boldsymbol{u})$ is not very restrictive. It only allows for exclusion from the consideration of "trivial" cases, since a nontrivial convex cone (one that does not cover the whole space) cannot be open. It will always contain its boundary (e.g., the apex or "edges" of the cone), making it at least closed or not open.

REMARK 1 *In the differentiable case, the convex cone defined in (3) is always closed. For a differentiable single-valued function f, we obtain famous result of Craven and Glover (1985) that $f : \mathbb{R}^n \to \mathbb{R}$ is invex if and only if every stationary point (i.e. $\boldsymbol{u}$ such that $\nabla f(\boldsymbol{u}) = \mathbf{0}$) is a global minimum of f over $\mathbb{R}^n$.*

Now we are ready to adapt the Craven and Glover's lemma of alternative (Lemma 2) to LLC functions and use it later in the proof of Theorem 7.

LEMMA 3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be LLC, $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n$, and let $K_{\boldsymbol{x}}(\boldsymbol{u})$ defined in (3) be closed. Then, exactly one of the following is satisfied:*

(i) $\exists\ \boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u}) \in \mathbb{R}^n$ *such that*

$$f(\boldsymbol{x}) - f(\boldsymbol{u}) \geq \boldsymbol{\xi}^T \boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u}),$$

or equivalently,

$$f(\boldsymbol{x}) - f(\boldsymbol{u}) \geq f^o(\boldsymbol{u};\boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u}));$$

(ii) $(\boldsymbol{0}, -1)^T \in K_{\boldsymbol{x}}(\boldsymbol{u})$.

PROOF Taking into account Lemma 1, this result is obtained by setting $S = \mathbb{R}_{\geq}$ in Lemma 2 and substituting $\boldsymbol{x}$ with $\boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u})$, $f(\boldsymbol{x})$ with $f^o(\boldsymbol{u};\boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u}))$, and z with $f(\boldsymbol{x}) - f(\boldsymbol{u})$, respectively. The assumptions in Lemma 2 on sublinearity (subadditivity and positive homogeneity) of function f transform into assumptions of sublinearity of function $f^o(\boldsymbol{u};\boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u}))$ as a function of argument $\boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u})$, which always holds for all $\boldsymbol{u} \in \mathbb{R}^n$ in the case of the Clarke generalized directional derivative, Clarke (1983). We also use a theorem from Clarke (1983) stating that for an LLC function f at $\boldsymbol{u}$, we have $\partial f(\boldsymbol{u}) = \partial_{conv} f^o(\boldsymbol{u};\boldsymbol{0})$, that is

$$\partial f(\boldsymbol{u}) \ = \ \partial_{\mathrm{conv}}\Big(\boldsymbol{h} \ \mapsto \ f^{\circ}(\boldsymbol{u};\boldsymbol{h})\Big)\Big|_{\boldsymbol{h}=\boldsymbol{0}}.$$

■

A careful reader may notice that condition (i) in Lemma 3 will be equivalent to invexity of LLC function f when fulfilled for all $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n$.

Now we consider the following vector-valued function minimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \ \boldsymbol{F}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_p(\boldsymbol{x}))^T \in \mathbb{R}^p. \tag{VMP}$$

Let us first define the most fundamental principle of optimality in vector optimization, known as Pareto optimality (Pareto, 1935).

DEFINITION 11 *A point $\boldsymbol{x}^* \in \mathbb{R}^n$ is said to be* Pareto optimal solution *of VMP if there exist no $\boldsymbol{x} \in \mathbb{R}^n$ such that*

$$f_i(\boldsymbol{x}) \leq f_i(\boldsymbol{x}^*) \text{ for all } i = 1, \ldots, p$$

and

$$f_i(\boldsymbol{x}) < f_i(\boldsymbol{x}^*) \text{ for some } i = 1, \ldots, p.$$

We can modify the Pareto optimality principle by requiring strict inequality for all components in the vector dominance condition. This produces a milder optimality principle known as weak Pareto optimality.

Definition 12 *A point $\boldsymbol{x}^* \in \mathbb{R}^n$ is said to be* weakly Pareto optimal solution *of VMP if there exists no $\boldsymbol{x} \in \mathbb{R}^n$ such that*

$$f_i(\boldsymbol{x}) < f_i(\boldsymbol{x}^*) \text{ for all } i = 1, \dots, p.$$

Therefore, it is easy to see that every Pareto optimal solution is also weakly Pareto optimal, but the opposite may not necessarily be true.

The concept of V-invexity was generalized for nonsmooth LLC functions in Mishra and Giorgi (2008).

Definition 13 (Mishra and Giorgi, 2008) *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be LLC functions for all $i = 1, \dots, p$. A vector-valued function*

$$\boldsymbol{F}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \dots, f_p(\boldsymbol{x}))^T$$

is V-invex *(with respect to $\boldsymbol{\eta}$) if there exist a function $\boldsymbol{\eta} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and a vector $\boldsymbol{\beta}$ with components $\beta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_>$ such that for all $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n$, for all $i = 1, \dots, p$ and for all $\boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u})$ we have*

$$f_i(\boldsymbol{x}) - f_i(\boldsymbol{u}) \geq \beta_i(\boldsymbol{x}; \boldsymbol{u}) (\boldsymbol{\xi}^i)^T \boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u}),$$

or, equivalently,

$$f_i(\boldsymbol{x}) - f_i(\boldsymbol{u}) \geq \beta_i(\boldsymbol{x}; \boldsymbol{u}) f_i^o(\boldsymbol{u}; \boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u})).$$

Here we can write $\beta_i(\boldsymbol{x}; \boldsymbol{u})\boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u}) = \boldsymbol{\eta}_i(\boldsymbol{x}; \boldsymbol{u})$, which shows that every i-th component function has a different $\boldsymbol{\eta}$-function, i.e. $\boldsymbol{\eta}_i$. A vector function whose components are invex with the same $\boldsymbol{\eta}$ is V-invex. To see that, it suffices to set $\beta_i \equiv 1$ for all $i = 1, \dots, p$.

Note that Definition 13 was first introduced for differentiable functions in Jeyakumar and Mond (1992). It was shown by Jeyakumar and Mond (1992) that for a V-invex differentiable function $\boldsymbol{F} = (f_1, \dots, f_p)^T$ the point $\boldsymbol{u} \in \mathbb{R}^n$ is a weak Pareto optimal solution of the VMP if and only if there exists $\boldsymbol{\lambda} \in \mathbb{R}^p_{\geq} \backslash \{\boldsymbol{0}\}$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\boldsymbol{u}) = \boldsymbol{0}. \tag{4}$$

Let us extend (4) to nonsmooth functions. We will first need the following result (known as Gordan's lemma of alternative).

Lemma 4 (Gordan, 1873) *(Gordan's lemma of alternative I). For a given $m \times n$ matrix A exactly one of the following two systems (but never both) has a solution:*

(i) primal system $A\boldsymbol{x} < (>)\, \mathbf{0}$ *has a solution* $\boldsymbol{x} \in \mathbb{R}^n$*;*
(ii) dual system $A^T\boldsymbol{\lambda} = \mathbf{0},\ \boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{\lambda} \neq \mathbf{0}$ *has a solution* $\boldsymbol{\lambda} \in \mathbb{R}^m_{\geq}\backslash\{\mathbf{0}\}$*.*

We will use Lemma 4 to prove the following result, specifying the criteria for weak Pareto optimality for VMP with V-invex functions.

THEOREM 3 *Let* $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^p$ *be V-invex and let* $f_i : \mathbb{R}^n \to \mathbb{R}$ *be LLC functions for all* $i = 1, \ldots, p$*. Given* $\boldsymbol{u} \in \mathbb{R}^n$*, assume that the cone* $K_{\boldsymbol{x}}(\boldsymbol{u})$*, defined in* (3)*, is closed for every* $\boldsymbol{x} \in \mathbb{R}^n$*. Then,* $\boldsymbol{u}$ *is a weakly Pareto optimal solution of the VMP, associated with* $\boldsymbol{F}$*, if and only if there exists*

$$\boldsymbol{\xi} := (\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^p), \qquad \boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u}),\ i = 1, \ldots, p,$$

and there exists $\boldsymbol{\lambda} \in \mathbb{R}^p_{\geq}\backslash\{\mathbf{0}\}$ *such that*

$$\sum_{i=1}^{p} \lambda_i \boldsymbol{\xi}^i = \mathbf{0}. \tag{5}$$

PROOF ($\Rightarrow$) Assume that $\boldsymbol{u}$ is a weakly Pareto optimal solution of the VMP. Then there is no $\boldsymbol{x} \in \mathbb{R}^n$ such that $f_i(\boldsymbol{x}) < f_i(\boldsymbol{u})$ for all $i = 1, \ldots, p$. Equivalently, the system

$$f_i(\boldsymbol{x}) - f_i(\boldsymbol{u}) < 0, \qquad i = 1, \ldots, p,$$

is inconsistent.

Since $\boldsymbol{F}$ is V-invex, there exist mappings $\boldsymbol{\eta}(\cdot;\boldsymbol{u})$ and coefficient functions $\beta_i(\cdot;\boldsymbol{u}) > 0$ such that for every $\boldsymbol{x} \in \mathbb{R}^n$ and every i,

$$f_i(\boldsymbol{x}) - f_i(\boldsymbol{u}) \geq \beta_i(\boldsymbol{x};\boldsymbol{u})\, (\boldsymbol{\xi}^i)^T \boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u}), \qquad \forall\, \boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u}).$$

Hence, if $f_i(\boldsymbol{x}) - f_i(\boldsymbol{u}) < 0$ holds for all i, then necessarily

$$(\boldsymbol{\xi}^i)^T \boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u}) < 0, \qquad i = 1, \ldots, p,$$

for at least one selection $\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^p)$ with $\boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u})$. Therefore, for this selection $\boldsymbol{\xi}$, the system

$$(\boldsymbol{\xi}^i)^T \boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u}) < 0, \qquad i = 1, \ldots, p,$$

is inconsistent.

Fix $\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^p)$ with $\boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u})$ and define the matrix $A \in \mathbb{R}^{p\times n}$ by $A = \big((\boldsymbol{\xi}^1)^T, \ldots, (\boldsymbol{\xi}^p)^T\big)^T$. Applying Gordan's lemma of alternative I (Lemma 4) to matrix A, we conclude that there exists $\boldsymbol{\lambda} \in \mathbb{R}^p_{\geq}\backslash\{\mathbf{0}\}$ such that

$$\sum_{i=1}^{p} \lambda_i \boldsymbol{\xi}^i = \mathbf{0}.$$

Thus, (5) holds.

($\Leftarrow$) Assume that there exist $\boldsymbol{\xi}$ with $\boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u})$ and $\boldsymbol{\lambda} \in \mathbb{R}^p_{\geq} \setminus \{\mathbf{0}\}$ such that $\sum_{i=1}^p \lambda_i \boldsymbol{\xi}^i = \mathbf{0}$. Suppose, to the contrary, that $\boldsymbol{u}$ is not weakly Pareto optimal. Then, there exists $\boldsymbol{x}^* \in \mathbb{R}^n$ such that $f_i(\boldsymbol{x}^*) < f_i(\boldsymbol{u})$ for all $i = 1, \ldots, p$. By V-invexity (with the same $\boldsymbol{\eta}$ and $\beta_i > 0$), we obtain

$$0 > \frac{1}{\beta_i(\boldsymbol{x}^*; \boldsymbol{u})} \big(f_i(\boldsymbol{x}^*) - f_i(\boldsymbol{u}) \big) \geq (\boldsymbol{\xi}^i)^T \boldsymbol{\eta}(\boldsymbol{x}^*; \boldsymbol{u}), \qquad i = 1, \ldots, p.$$

Thus, $(\boldsymbol{\xi}^i)^T \boldsymbol{\eta}(\boldsymbol{x}^*; \boldsymbol{u}) < 0$ for all i, which means that the primal system in Gordan's lemma of alternative I is feasible for the matrix A with rows $(\boldsymbol{\xi}^i)^T$. However, the existence of $\boldsymbol{\lambda} \geq \mathbf{0}$, $\boldsymbol{\lambda} \neq \mathbf{0}$ with $\sum_{i=1}^p \lambda_i \boldsymbol{\xi}^i = \mathbf{0}$ is exactly the feasibility of the dual system in Gordan's lemma. This contradicts Lemma 4. Hence, $\boldsymbol{u}$ must be weakly Pareto optimal. ∎

Note that, when $p = 1$, Definition 13 coincides with the classical definition of invexity given in Definition 7. In this case, the weak Pareto optimality condition in Theorem 3 reduces to the scalar optimality condition, presented in Theorem 2.

We illustrate Theorem 3 with two examples below.

Example 1 *Consider the following VMP*

$$\min_{x \in \mathbb{R}} \boldsymbol{F}(x) = \big(f_1(x), f_2(x) \big)^T, \qquad f_1(x) = |x|, \quad f_2(x) = |x - 1|.$$

Both f_1 and f_2 are LLC and convex on $\mathbb{R}$. Hence, $\boldsymbol{F}$ is V-invex with respect to

$$\eta(x; u) = x - u, \qquad \beta_1(x; u) = \beta_2(x; u) \equiv 1.$$

Let $u = 0$. Then

$$f_1(0) = 0, \qquad f_2(0) = 1.$$

First, we check weak Pareto optimality at $u = 0$. If $f_1(x) < f_1(0) = 0$, then $|x| < 0$, which is impossible. Hence, there exists no $x \in \mathbb{R}$ such that $f_i(x) < f_i(0)$ for both $i = 1, 2$. Therefore, $u = 0$ is a weakly Pareto optimal solution of the VMP.

Now we verify the stationarity condition. The Clarke subdifferentials at $u = 0$ are

$$\partial f_1(0) = [-1, 1], \qquad \partial f_2(0) = \{-1\}.$$

Choose the admissible subgradient selection

$$\xi^1 = 1 \in \partial f_1(0), \qquad \xi^2 = -1 \in \partial f_2(0).$$

Then, with

$$\boldsymbol{\lambda} = (1,1) \in \mathbb{R}^2_{\geq} \setminus \{\mathbf{0}\},$$

we obtain

$$\lambda_1 \xi^1 + \lambda_2 \xi^2 = 1 \cdot 1 + 1 \cdot (-1) = 0.$$

Hence, there exist $\boldsymbol{\xi}$ *and* $\boldsymbol{\lambda}$ *satisfying*

$$\sum_{i=1}^{2} \lambda_i \xi^i = 0,$$

which verifies the condition of Theorem 3 at $u = 0$.

Example 2 *Consider the following VMP*

$$\min_{x \in \mathbb{R}} \boldsymbol{F}(x) = \big(f_1(x), f_2(x)\big)^T, \qquad f_1(x) = |x|, \quad f_2(x) = |x|.$$

Both f_1 *and* f_2 *are LLC and convex on* $\mathbb{R}$. *Hence* $\boldsymbol{F}$ *is V-invex with respect to*

$$\eta(x; u) = x - u, \qquad \beta_1(x; u) = \beta_2(x; u) \equiv 1,$$

since for all $x, u \in \mathbb{R}$ *and all* $\xi^i \in \partial f_i(u)$ *the convex subgradient inequality yields*

$$f_i(x) - f_i(u) \geq \xi^i (x - u) = \beta_i(x; u)\, \xi^i\, \eta(x; u), \qquad i = 1, 2.$$

Let $u = 1$. *Then*

$$f_1(1) = 1, \qquad f_2(1) = 1.$$

Taking $x = 0$ *gives*

$$f_1(0) = 0 < 1 = f_1(1), \qquad f_2(0) = 0 < 1 = f_2(1),$$

and therefore $u = 1$ *is* not *a weakly Pareto optimal solution of the VMP.*

Now we verify that the condition of Theorem 3 fails at $u = 1$. *Since* $u > 0$, *we have*

$$\partial f_1(1) = \{1\}, \qquad \partial f_2(1) = \{1\}.$$

Hence, the only admissible choice of subgradients is

$$\xi^1 = 1 \in \partial f_1(1), \qquad \xi^2 = 1 \in \partial f_2(1).$$

For any $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2_{\geq} \setminus \{\mathbf{0}\}$,

$$\lambda_1 \xi^1 + \lambda_2 \xi^2 = \lambda_1 + \lambda_2 > 0,$$

so, it is impossible to have

$$\lambda_1 \xi^1 + \lambda_2 \xi^2 = 0.$$

Consequently, there does not exist $\boldsymbol{\xi}$ *with* $\xi^i \in \partial f_i(1)$ *and* $\boldsymbol{\lambda} \in \mathbb{R}^2_{\geq} \setminus \{\mathbf{0}\}$ *such that* $\sum_{i=1}^{2} \lambda_i \xi^i = 0$. *Hence the condition of Theorem 3 does not hold at the non-optimal point* $u = 1$.

To formulate one more result, it is handy to use another version of Gordan's lemma of alternative, given below.

LEMMA 5 (GORDAN, 1873) *(Gordan's lemma of alternative II). For a given* $m \times n$ *matrix* A *exactly one of the following two systems (but never both) has a solution:*

(i) primal system $A\boldsymbol{x} \leq (\geq)\, \mathbf{0}$ *has a solution* $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$;
(ii) dual system $A^T \boldsymbol{\lambda} = \mathbf{0}$ *has a solution* $\boldsymbol{\lambda} \in \mathbb{R}^m_{>}$.

The following result can be proven by analogy with the proof of Theorem 3 and the help of Gordan's lemma of alternative II (Lemma 5). It specifies the Pareto optimality criteria for VMP with V-invex functions.

THEOREM 4 *Let* $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^p$ *be V-invex and let* $f_i : \mathbb{R}^n \to \mathbb{R}$ *be LLC for all* $i = 1, \ldots, p$. *Fix* $\boldsymbol{u} \in \mathbb{R}^n$ *and assume that, for every* $\boldsymbol{x} \in \mathbb{R}^n$, *the cone* $K_{\boldsymbol{x}}(\boldsymbol{u})$ *defined in* (3) *is closed. Then* $\boldsymbol{u}$ *is a Pareto optimal solution of the VMP associated with* $\boldsymbol{F}$ *if there exist*

$$\boldsymbol{\xi} := (\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^p), \qquad \boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u}),\ i = 1, \ldots, p,$$

and $\boldsymbol{\lambda} \in \mathbb{R}^p_{>}$ *such that*

$$\sum_{i=1}^{p} \lambda_i \boldsymbol{\xi}^i = \mathbf{0}.$$

EXAMPLE 3 *Consider the same VMP as in Example 1*

$$\min_{x \in \mathbb{R}} \boldsymbol{F}(x) = \big(f_1(x), f_2(x)\big)^T, \qquad f_1(x) = |x|, \quad f_2(x) = |x - 1|.$$

Both f_1 *and* f_2 *are LLC and convex on* $\mathbb{R}$. *Thus,* $\boldsymbol{F}$ *is V-invex with respect to*

$$\eta(x; u) = x - u, \qquad \beta_1 = \beta_2 \equiv 1.$$

Let $u = \frac{1}{2}$. Then

$$f_1(u) = f_2(u) = \tfrac{1}{2}.$$

Since $u \neq 0, 1$, the Clarke subdifferentials are singletons:

$$\partial f_1\big(\tfrac{1}{2}\big) = \{1\}, \qquad \partial f_2\big(\tfrac{1}{2}\big) = \{-1\}.$$

Thus, the unique admissible selection is

$$\xi^1 = 1, \qquad \xi^2 = -1.$$

By choosing

$$\boldsymbol{\lambda} = (1,1) \in \mathbb{R}^2_{>0},$$

we obtain

$$\lambda_1\xi^1 + \lambda_2\xi^2 = 1 - 1 = 0.$$

Hence the stationarity condition with $\boldsymbol{\lambda} \in \mathbb{R}^2_{>}$ holds at point $u = \frac{1}{2}$.

Assume that for some $x \in \mathbb{R}$ we have

$$f_1(x) \leq f_1(u) = \tfrac{1}{2} \quad \text{and} \quad f_2(x) \leq f_2(u) = \tfrac{1}{2}.$$

The first inequality gives $|x| \leq \frac{1}{2}$, hence $x \in [-\frac{1}{2}, \frac{1}{2}]$. The second inequality gives $|x-1| \leq \frac{1}{2}$, hence $x \in [\frac{1}{2}, \frac{3}{2}]$. Therefore, $x \in [-\frac{1}{2}, \frac{1}{2}] \cap [\frac{1}{2}, \frac{3}{2}] = \{\frac{1}{2}\}$, i.e., $x = u$. Consequently, there exists no $x \neq u$ such that $f_i(x) \leq f_i(u)$ for $i = 1, 2$ with at least one strict inequality, and hence $u = \frac{1}{2}$ is a Pareto optimal solution of the VMP.

5. Main results

In this section, we generalize a result from Martínez-Legaz (2009), concerning the characterization families of differentiable functions that are invex with respect to a common mapping $\boldsymbol{\eta}$. For the completeness of the paper, we start by reproducing the result of Martínez-Legaz (2009) along with the proof. First, we need the following known result.

Lemma 6 (Gale, 1960) *(Gale's lemma of alternative). For a given $m \times n$ matrix A and a given column vector $\boldsymbol{b} \in \mathbb{R}^m$, exactly one of the following two linear systems (but never both) has a solution:*

(i) The primal system $A\boldsymbol{x} \leq \boldsymbol{b}$ has a solution $\boldsymbol{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$.
(ii) The dual system $A^T\boldsymbol{\lambda} = \mathbf{0}$, $\boldsymbol{b}^T\boldsymbol{\lambda} = -1$ has a solution $\boldsymbol{\lambda} \in \mathbb{R}^m_{\geq}$.

The next theorem characterizes invexity with respect to a common mapping $\boldsymbol{\eta}$ for differentiable functions.

Theorem 5 (Martínez-Legaz, 2009) *Let $f_1, \dots, f_p$ be differentiable functions defined in $\mathbb{R}^n$. The following statements are equivalent:*

(i) The functions $f_1, \dots, f_p$ are invex with respect to the same function $\boldsymbol{\eta}$.
(ii) The function $\sum_{i=1}^p \lambda_i f_i$ is invex with respect to the same $\boldsymbol{\eta}$ as the individual functions f_i, when $\lambda_i \geq 0,\ i = 1, \dots, p$.
(iii) The function $\sum_{i=1}^p \lambda_i f_i$ is invex, when $\lambda_i \geq 0,\ i = 1, \dots, p$.
(iv) For all $\lambda_i \geq 0,\ i = 1, \dots, p$, every stationary point of $\sum_{i=1}^p \lambda_i f_i$ is a global minimum.

Proof The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious, so we only have to prove implication (iv) $\Rightarrow$ (i). Assume, by contradiction, that there is no function $\boldsymbol{\eta}$ such that

$$f_i(\boldsymbol{x}) - f_i(\boldsymbol{u}) \geq \nabla f_i(\boldsymbol{u})^T \boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u}),\ \ \boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n,\ \ i = 1, \dots, p.$$

In other words, there exist $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n$, such that the linear inequality system

$$\nabla f_i(\boldsymbol{u})^T \boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u}) \leq f_i(\boldsymbol{x}) - f_i(\boldsymbol{u}),\ \ i = 1, \dots, p$$

with the vector $\boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u})$, has no solution. Hence, by Gale's lemma of alternative (Lemma 6), there exists $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p_{\geq}$ such that

$$\sum_{i=1}^p \lambda_i \nabla f_i(\boldsymbol{u}) = \mathbf{0} \text{ and } \textstyle\sum_{i=1}^p \lambda_i (f_i(\boldsymbol{x}) - f_i(\boldsymbol{u})) = -1.$$

Therefore, $\sum_{i=1}^p \lambda_i f_i$ has a stationary point $\boldsymbol{u}$, which is not a global minimum, since $\sum_{i=1}^p \lambda_i f_i(\boldsymbol{x}) = \sum_{i=1}^p \lambda_i f_i(\boldsymbol{u}) - 1 < \sum_{i=1}^p \lambda_i f_i(\boldsymbol{u})$. This contradicts (iv). ∎

Assumption 1 *In the nonsmooth case, we must assume that the cone $K_{\boldsymbol{x}}$, defined in (3), is closed, and we keep the assumption in the following considerations.*

Remark 2 *In the nonsmooth case where for functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, p$, which are LLC at $\boldsymbol{u} \in \mathbb{R}^n$ and $\lambda_i \geq 0$ for all $i = 1, \dots, p$, we have*

$$\partial\Big(\sum_{i=1}^p \lambda_i f_i(\boldsymbol{u})\Big) = \sum_{i=1}^p \lambda_i \partial f_i(\boldsymbol{u}), \tag{6}$$

we can apply the proof in Theorem 5 for vector-valued LLC functions in a quite straightforward way by replacing gradients with subgradients.

The equality condition (6) is known as *subdifferential weighted sum rule.* It allows us to calculate the Clarke subdifferential of a weighted sum of functions as the weighted Minkowski sum of their individual Clarke subdifferentials. As shown in Clarke (1983), this condition is a consequence of Clarke subdifferential regularity, defined as follows:

Definition 14 (Bagirov, Karmitsa and Mäkelä, 2014) *The LLC function $f : \mathbb{R}^n \to \mathbb{R}$ is called* Clarke subdifferentially regular *at $\boldsymbol{u} \in \mathbb{R}^n$ if the classical directional derivative $f'(\boldsymbol{u};\boldsymbol{d})$ exists and coincides with the generalized directional derivative $f^o(\boldsymbol{u};\boldsymbol{d})$, that is*

$$f'(\boldsymbol{u};\boldsymbol{d}) = f^o(\boldsymbol{u};\boldsymbol{d})$$

for all $\boldsymbol{d} \in \mathbb{R}^n$. Geometrically, it corresponds to the equality between the contingent cone and the Clarke tangent cone, Clarke (1983)

The classical derivation rule for LLC functions is given as follows.

Theorem 6 (Bagirov, Karmitsa and Mäkelä, 2014) *Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$, be LLC at $\boldsymbol{u} \in \mathbb{R}^n$. Then*

$$\partial\Big(\sum_{i=1}^{p} \lambda_i f_i(\boldsymbol{u})\Big) \subseteq \sum_{i=1}^{p} \lambda_i \partial f_i(\boldsymbol{u}), \tag{7}$$

where $\lambda_i \in \mathbb{R}$ for all $i = 1, \ldots, p$. In addition, if f_i is subdifferentially regular in $\boldsymbol{u}$ and $\lambda_i \geq 0$ for all $i = 1, \ldots, p$, then equality occurs in (7), that is, (6) holds.

Thus, the Clarke subdifferential regularity allows us to move from the inclusion derivation rule (7) to the equation form of it, that is, the subdifferential weighted sum rule (6).

Remark 3 *Another case where the subdifferential weighted sum rule (6) is valid is when at least $p-1$ of the functions f_i are continuously differentiable (see, for example, Clarke, 1983). However, when there are two or more LLC functions, the implication (iv) $\Rightarrow$ (i) in Theorem 5 does not necessarily hold without the assumption of the Clarke subdifferential regularity. Let us show this with the following example.*

Example 4 *[Counterexample showing the impossibility of straightforward generalization of Theorem 5 to nonsmooth case] Let us define LLC functions*

$$f_1(x) = \begin{cases} x\sin(\ln(x)) + x\cos(\ln(x)) + 5x, & x > 0 \\ 6x, & x \leq 0, \end{cases}$$

and $f_2(x) = 3|x|$. *For* f_1, *we have* $f_1'(x) = 2\cos(\ln(x)) + 5,\ x > 0$. *We have* $f_1'(x) \to [3,7]$, *as* $x \to 0_+$, *and* $f_1'(x) = 6$, *as* $x \to 0_-$. *Therefore, due to (2) we get* $\partial f_1(0) = [3,7]$. *For* f_2, *we have* $\partial f_2(0) = [-3,3]$.

First, let us show that in our example (iv) of Theorem 5 is true for every $\lambda_i \geq 0$, $i = 1,2$. *We consider three cases:*

- $\lambda_2 = 2\lambda_1$:
  In this case, we have $\sum_{i=1}^{2} \lambda_i f_i(x) = 0,\ x \leq 0$ *and* $\sum_{i=1}^{2} \lambda_i f_i(x) > 0,\ x \geq 0$. *So, all the stationary points* $x \leq 0$ *are the global minima.*

- $\lambda_2 > 2\lambda_1$:
  In this case, we have $\sum_{i=1}^{2} \lambda_i f_i(x) \geq 0$ *when* $x \leq 0$. *So, the only stationary point* 0 *is the unique global minimum for* $\sum_{i=1}^{2} \lambda_i f_i(x)$.

- $\lambda_2 < 2\lambda_1$:
  In this case $\sum_{i=1}^{2} \lambda_i f_i(x)$ *has no stationary points, hence (iv) is vacuously true.*

Thus, in all three cases, every stationary point of $\sum_{i=1}^{2} \lambda_i f_i(x)$ *is a global minimum.*

Let us now show that given (iv) is true, claim (i) does not hold. When we set $\lambda_1 = \lambda_2 = 1$, *we have* $0 \in \sum_{i=1}^{2} \lambda_i \partial f_i(0) = [0,10]$. *However, for* $\sum_{i=1}^{2} \lambda_i f_i(x)$, *we have* $\partial(\sum_{i=1}^{p} \lambda_i f_i)(0) = [3,10]$, *making* $u = 0$ *non-stationary. Let us take* $x = -\frac{1}{3}$. *Then we have* $\sum_{i=1}^{2} \lambda_i (f_i(x) - f_i(0)) = 1(-2+0) + 1(1+0) = -1$. *We can see that*

$$(0,-1)^T \in \Big(\sum_{i=1}^{2} \lambda_i \partial f_i(0), \sum_{i=1}^{2} \lambda_i (f_i(x) - f_i(0)) \Big).$$

Then, by Lemma 3), there exists no $\eta(x;u)$ *for which* $f_i(x) - f_i(u) \geq f_i^o(u;\eta(x;u))$ *for both* f_1 *and* f_2, *when* $x = -\frac{1}{3}$ *and* $u = 0$.

Indeed, for $x = -\frac{1}{3}$ *and* $u = 0$, *we have* $f_1(x) - f_1(u) = -2$. *For the condition* $f_1(x) - f_1(u) \geq f_1^o(u;\eta(x;u))$ *to hold, we need to have* $\eta(x;u) \leq -\frac{2}{3}$. *For* f_2, *we have* $f_2(x) - f_2(u) = 1$. *Since we need to have* $\eta(x;u) \leq -\frac{2}{3}$, *we have* $f_2^o(u;\eta(x;u)) \geq -3(-\frac{2}{3}) = 2$. *Then we can see that the condition* $f_2(x) - f_2(u) \geq f_2^o(u;\eta(x;u))$ *does not hold for* $x = -\frac{1}{3}$ *and* $u = 0$. *Therefore, functions* f_1 *and* f_2 *are not invex for the same mapping* η. *Thus, statement (i) of Theorem 5 does not hold.*

Similar phenomena, where generalized convexity-type properties hold for scalarized problems, but fail to extend to families of functions, have been observed in broader contexts of variational analysis and optimization; see, for example, Borwein (2016).

REMARK 4 *Theorem 5 can also be formulated for the nonsmooth case without the regularity assumptions, given in Remark 2, by rewriting (iv) as follows: Given that Assumption 1 holds, for every* $\lambda_i \geq 0,\ i = 1, \ldots, p$,

$$\sum_{i=1}^{p} \lambda_i \boldsymbol{\xi}^i = \boldsymbol{0},\ \boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u}) \tag{8}$$

implies that $\boldsymbol{u}$ *is a global minimum of* $\sum_{i=1}^{p} \lambda_i f_i$. *Indeed, since* $\boldsymbol{u}$ *is a global minimum, we have*

$$\boldsymbol{0} \in \partial\Big(\sum_{i=1}^{p} \lambda_i f_i\Big)(\boldsymbol{u}), \tag{9}$$

meaning that

$$\boldsymbol{0} \in \sum_{i=1}^{p} \lambda_i \partial f_i(\boldsymbol{u}) \implies \boldsymbol{0} \in \partial\Big(\sum_{i=1}^{p} \lambda_i f_i\Big)(\boldsymbol{u}). \tag{10}$$

Theorem 5 can also be generalized to V-invex functions (see Definition 13). The proof is a generalization of the proofs used in Theorem 3 in Reiland (1990) (see Theorem 2) and in Theorem 5 in this paper.

THEOREM 7 *Let* $f_1, \ldots, f_p$ *be LLC functions defined on* $\mathbb{R}^n$. *Let* $J(\boldsymbol{F}(\boldsymbol{u}))$ *be the generalized Jacobian of* $\boldsymbol{F} = (f_1, \ldots, f_p)^T$ *at* $\boldsymbol{u}$ *and*

$$\boldsymbol{b} := \boldsymbol{b}(\boldsymbol{x};\boldsymbol{u}) = \big(\beta_1(\boldsymbol{x};\boldsymbol{u})(f_1(\boldsymbol{x}) - f_1(\boldsymbol{u})), \ldots,\ \beta_p(\boldsymbol{x};\boldsymbol{u})(f_p(\boldsymbol{x}) - f_p(\boldsymbol{u}))\big)^T,$$

where $\boldsymbol{\beta} = (\beta_1, \ldots \beta_p)^T$ *is a vector of the coefficient functions* $\beta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_>$ *from Definition 13. For each* $\boldsymbol{u} \in \mathbb{R}^n$, *assume that for every* $\boldsymbol{x} \in \mathbb{R}^n$, *the convex cone*

$$K_{\boldsymbol{x}}(\boldsymbol{u}) = \bigcup_{\boldsymbol{\lambda} \geq 0} \Big(J(\boldsymbol{F}(\boldsymbol{u}))^T \boldsymbol{\lambda} \times \{\boldsymbol{\lambda}^T \boldsymbol{b}(\boldsymbol{x};\boldsymbol{u})\}\Big)$$

is closed. The following statements are equivalent:

(i) For a specified function coefficient vector $\boldsymbol{\beta} \in \mathbb{R}^p_>$, *the function* $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^p$ *is V-invex, that is, the functions* $f_1, \ldots, f_p$ *are invex with respect to the same* $\boldsymbol{\eta}$.

(ii) For a specified function coefficient vector $\boldsymbol{\beta} \in \mathbb{R}^p_>$, *the scalarized function* $\sum_{i=1}^{p} \lambda_i \beta_i f_i$ *with* $\lambda_i \geq 0,\ i = 1, \ldots, p$, *is invex with respect to the same* $\boldsymbol{\eta}$, *that is*

$$\sum_{i=1}^{p} \lambda_i \beta_i(\boldsymbol{x};\boldsymbol{u})((f_i(\boldsymbol{x}) - f_i(\boldsymbol{u})) \geq \sum_{i=1}^{p} \lambda_i (\boldsymbol{\xi}^i)^T \boldsymbol{\eta}(\boldsymbol{x};\boldsymbol{u})$$

$$\forall \lambda_i \geq 0,\ \boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n,\ \boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u}).$$

(iii) *For a specified function coefficient vector $\boldsymbol{\beta} \in \mathbb{R}^p_>$, the scalarized function $\sum_{i=1}^p \lambda_i \beta_i f_i$ is invex, when $\lambda_i \geq 0,\ i = 1, \dots, p$.*

(iv) *For every $\lambda_i \geq 0,\ i = 1, \dots, p$, and for a specified function coefficient vector $\boldsymbol{\beta} \in \mathbb{R}^p_>$, the fact that*

$$\sum_{i=1}^p \lambda_i \boldsymbol{\xi}^i = \boldsymbol{0}$$

for some $\boldsymbol{\xi}^i \in \partial f_i(\boldsymbol{u})$ implies that $\boldsymbol{u}$ is a global minimum of the scalarized function

$$\sum_{i=1}^p \lambda_i \beta_i f_i. \tag{11}$$

Proof The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious, so it suffices to prove (iv) $\Rightarrow$ (i). From (iv) we can see that $(\boldsymbol{0}, -1)^T \notin K_{\boldsymbol{x}}(\boldsymbol{u})$, because otherwise we would have

$$\sum_{i=1}^p \lambda_i \boldsymbol{\xi}^i = \boldsymbol{0}$$

and

$$\sum_{i=1}^p \lambda_i \beta_i(\boldsymbol{x}; \boldsymbol{u}) f_i(\boldsymbol{x}) < \sum_{i=1}^p \lambda_i \beta_i(\boldsymbol{x}; \boldsymbol{u}) f_i(\boldsymbol{x}) + 1 = \sum_{i=1}^p \lambda_i \beta_i(\boldsymbol{x}; \boldsymbol{u}) f_i(\boldsymbol{u}),$$

which is a contradiction to the implication that $\boldsymbol{u}$ is a global minimum of the scalarized function (11). Therefore, by Lemma 3, applied (considering $K_{\boldsymbol{x}}(\boldsymbol{u})$ is closed and may generally speaking be different for each $\boldsymbol{u}$) to the vector-valued function $\boldsymbol{\beta F}$ (each f_i is LLC, $\beta_i \in \mathbb{R}_>$), there exists $\boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u})$ such that

$$\boldsymbol{b}_i = \beta_i(\boldsymbol{x}; \boldsymbol{u})(f_i(\boldsymbol{x}) - f_i(\boldsymbol{u})) \geq \beta_i(\boldsymbol{x}; \boldsymbol{u}) f_i^o(\boldsymbol{u}; \boldsymbol{\eta}(\boldsymbol{x}; \boldsymbol{u})).$$

Thus, $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^p$ is V-invex. ■

Example 5 (Illustration of Theorem 7) *Let $n = 1$, $p = 2$, and define $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ by*

$$f_1(x) = |x|, \qquad f_2(x) = |x - 1|.$$

similar to those in Examples 1 and 3. Both functions are LLC on $\mathbb{R}$ and nonsmooth at $x = 0$ and $x = 1$, respectively. Define

$$\eta(x; u) = x - u, \qquad \beta_1(x; u) = \beta_2(x; u) \equiv 1.$$

Step 1: Verification of V-invexity (statement (i)). *Since f_1 and f_2 are convex, for all $x, u \in \mathbb{R}$ and for all $\xi^i \in \partial f_i(u)$, the convex subgradient inequality expressed before in Section 3 through (*), yields*

$$f_i(x) - f_i(u) \geq \xi^i(x-u) = \beta_i(x;u)\, \xi^i \eta(x;u), \qquad i = 1, 2.$$

Hence, $\boldsymbol{F} = (f_1, f_2)^T$ is V-invex with respect to η, and (i) holds.

Step 2: Scalarized invexity with common η (statement (ii)). *Let $\lambda_1, \lambda_2 \geq 0$. For all $x, u \in \mathbb{R}$ and $\xi^i \in \partial f_i(u)$, we have*

$$\sum_{i=1}^{2} \lambda_i \beta_i(x;u)\big(f_i(x) - f_i(u)\big) \geq \sum_{i=1}^{2} \lambda_i \xi^i \eta(x;u),$$

which shows that the scalarized function $\sum_{i=1}^{2} \lambda_i \beta_i f_i$ is invex with respect to the same η. Thus, (ii) holds.

Step 3: Invexity of scalarizations (statement (iii)). *Since each f_i is convex and $\lambda_i \beta_i \geq 0$, the scalarized function*

$$\sum_{i=1}^{2} \lambda_i \beta_i f_i = \lambda_1 |x| + \lambda_2 |x-1|$$

is convex and therefore invex. Hence, (iii) follows directly.

Step 4: Global optimality from stationarity (statement (iv)). *Let $u = \frac{1}{2}$. Then*

$$\partial f_1(u) = \{1\}, \qquad \partial f_2(u) = \{-1\}.$$

Choose $\xi^1 = 1 \in \partial f_1(u)$ and $\xi^2 = -1 \in \partial f_2(u)$. For $\lambda_1 = \lambda_2 > 0$, we obtain

$$\lambda_1 \xi^1 + \lambda_2 \xi^2 = 0.$$

Hence

$$0 \in \partial\big(\lambda_1 f_1 + \lambda_2 f_2\big)(u).$$

Since the scalarized function is convex, u is a global minimizer of

$$\sum_{i=1}^{2} \lambda_i \beta_i f_i,$$

which verifies statement (iv).

Step 5: Closedness of the cone $K_x(u)$***.*** *Since the component functions* f_i *are LLC and convex, the generalized Jacobian* $J(\boldsymbol{F}(u))$ *is a nonempty, compact, and convex set. Consequently, for fixed* x *and* u*, the set*

$$\left\{(\boldsymbol{A}^T\boldsymbol{\lambda}, \boldsymbol{\lambda}^T\boldsymbol{b}(x;u)) : \ \boldsymbol{A} \in J(\boldsymbol{F}(u)), \ \boldsymbol{\lambda} \geq 0\right\}$$

is the conical hull of a compact set in a finite-dimensional space. Hence, it is a closed convex cone. Therefore, the cone $K_x(u)$ *is closed for every* $x \in \mathbb{R}$*.*

Thus, all statements (i)–(iv) from Theorem 7 are satisfied for this example.

6. Conclusion and future research

We generalized the result of Martínez-Legaz (2009) for nonsmooth LLC invex single- and vector-valued functions. Straightforward generalization was not possible and this was shown by a counterexample. However, a small modification of one of the conditions did the trick. The characterization results, developed in this paper, are potentially applicable to a wide range of problems. In particular, nonsmooth invexity with respect to a common mapping η may be exploited in the design of algorithms for uncertain and multiobjective optimization, in the analysis of variational inequality formulations, and in PDE-constrained optimization models, where nonsmoothness and vector-valued objectives naturally arise. By abstracting the underlying invexity structure, the present framework provides theoretical support for extending existing algorithmic and variational approaches to broader nonsmooth and multiobjective settings.

While the results developed in this paper provide a comprehensive characterization of invexity with respect to the same mapping η for LLC functions, their applicability depends on certain structural assumptions. In particular, the closedness of the associated cones and the use of the Clarke subdifferential play a central role in the equivalence results. In situations, in which these assumptions fail, for instance, when subdifferential regularity does not hold or when the weighted sum rule for subdifferentials cannot be applied, the proposed criteria may no longer be valid. Moreover, the analysis is restricted to finite-dimensional settings and unconstrained problems, which excludes certain classes of variational and infinite-dimensional models. Possible mitigations include adopting alternative nonsmooth derivative concepts, such as quasidifferentials (Demyanov and Rubinov, 1980) or limiting subdifferentials (Mordukhovich, 2018), or strengthening the regularity assumptions in specific applications. Investigation of these extensions constitutes an important direction for future research and may broaden the range of problems, for which invexity with respect to a common mapping can be effectively characterized.

7. Conflict of interest and associated data statements

The authors declare no conflicts of interest related to this research. The manuscript has no associated data.

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