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Optimality conditions and Lagrangian duality for vector optimization of invex set-valued functions

by

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Abstract: A constraint qualification and the equivalence between the vector-valued Lagrangian condition and the Kuhn-Tucker condition are •presented. By using them, a Lagrangian duality theorem for the weak minimality of vector optimization for invex set-valued functions is proved. A necessary optimality condition and a duality theorem for the proper minimality are also given.

Keywords: Invex set-valued function, vector optimization, Lagrangian condition, Kuhn-Tucker condition, duality.

1. Introduction

In this paper, we investigate vector optimization problems when, objective and the constraints are set-valued functions. Such problems have been discussed by several authors (see Corley, 1987, 1988, Luc, 1989, and Sach and Craven, 1991, Sach, Yen and Craven, 1994). In particular, the Lagrangian duality theorem was proved for convex set-valued functions by Corley (1987) (see also Luc, 1989) and for nearly convexlike set-valued functions by Song (1997) and Song (1996); the Wolfe and Mond-Weir type duality theorems for invex set-valued functions were proved by Sach and Craven (1991b) and Sach, Yen and Craven (1994); the Fritz John and the Kuhn-Tucker type optimality conditions for the weak minimality were also established by Corley (1988) and Sach and Craven (1991a).

In this note, we present a constraint qualification and prove the equivalence between the vector-valued Lagrangian condition and the Kuhn-Tucker condition. By using them, we prove a Lagrangian duality theorem for the weak minimality of vector optimization for invex set-valued functions. We also prove a necessary optimality condition and a duality theorem for the prbper minimality.

2. Preliminaries

Let X, Y, Z be normed spaces with topological dual spaces X^*, Y^*, Z^*. Let $S \subset Y$, $Q \subset Z$ be pointed closed convex cones. The dual cone S^+ and its quasi-interior S^{+i} are defined as

$$S^+ = \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0, \text{ for all } y \in S\},$$

$$S^{+i} = \{y^* \in Y^* \mid \langle y^*, y\rangle > 0, \text{ for all } y \in S \setminus \{0\}\},$$

where $\langle , \rangle$ is the canonical bilinear form with respect to the duality between Y^* and Y.

We say that a subset B of S is a base for S if B is convex, $0 \notin B$, and

$$S = \operatorname{cone}(B) = \{\lambda b \mid \lambda \geq 0, b \in B\}.$$

It is easy to show that if S has a base, then S^{+i} is nonempty (see Jahn, 1986).

Let $F : X \to Y$ be a set-valued function. Denote by gr F, dom F, the graph and domain of F, that is

$$\operatorname{gr} F = \{(x, y) \mid y \in F(x)\},$$

$$\operatorname{dom} F = \{x \mid F(x) \neq \emptyset\}.$$

We are concerned with the following vector optimization problem

$$\min F(x) \qquad (1)$$
$$\text{s.t. } x \in A, \quad G(x) \cap (-Q) \neq \emptyset,$$

where $F: X \to Y$, $G: X \to Z$ are set-valued functions, A is a subset of X.

Let E denote the set of all feasible points for problem (1), i.e.,

$$E = \{x \in A \mid G(x) \cap (-Q) \neq \emptyset\}.$$

A point (x_0, y_0) is said to be a global (resp. local) weak minimum solution for problem (1) if $x_0 \in E$, $y_0 \in F(x_0)$ and there is no $x \in E$ (resp. no $x \in E \cap U$) such that

$$(F(x) - y_0) \cap (-\operatorname{int} S) \neq \emptyset,$$

where U is a neighborhood of x_0 and S is assumed to have a nonempty interior. In this case, we call y_0 a global (resp. local) weak minimum value for (1). These definitions are consistent with those of Corley (1987;1988) (see also Luc, 1989, Sach, 1991a;b, Sach, Yen and Craven, 1994).

If $(x_0, y_0) \in \operatorname{gr} F = \{(x,y) \mid y \in F(x)\}$ satisfies

$$\overline{\operatorname{cone}}(F(E) + S - y_0) \cap (-S) = \{0\},$$

we say that (x_0, y_0) is a Benson proper minimum solution of (1) (see Benson, 1979). In the sequel, we briefly call (x_0, y_0) a proper minimum solution of (1).

Let $A \subset X$ be a subset. For a given point $x \in A$, the contingent cone $T_A(x)$ is defined by

$$T_A(x) = \{v \in X \mid \liminf_{h \to 0+} h^{-1} d_A(x + hv) = 0\},$$

here $d_A(x) = \inf_{y \in A} \|x - y\|$. The Clarke tangent cone $C_A(x)$ is defined by

$$C_A(x) = \{v \in X \mid \limsup_{x' \to x, h \to 0+} h^{-1} d_A(x' + hv) = 0\}.$$

We denote the Clarke normal cone by $N_A(x) = (C_A(x))^-$, which is the negative dual cone of the Clarke tangent cone $C_A(x)$, i.e.,

$$N_A(x) = \{x^* \in X^* \mid \langle x^*, x\rangle \leq 0, \quad \text{for all } x \in C_A(x)\}.$$

For $(x, y) \in \text{gr } F$, define the set-valued mapping $CF(x, y) : X \to Y$ as follows

$$\text{gr } CF(x, y) = C_{\text{gr} F}(x, y).$$

When F is single-valued $CF(x, y) = CF(x, F(x))$.

A set-valued function F is called locally-Lipschitz at $x_0 \in X$ if there exist a positive constant l and some neighborhood $U \subset \text{dom } F$ of x_0 such that for all $x_1, x_2 \in U$

$$F(x_1) \subset F(x_2) + l\|x_1 - x_2\| B_Y.$$

Let $y_0 \in F(x_0)$. F is called pseudo-Lipschitz at $(x_0, y_0) \in \text{gr } F$ (see Aubin, Frankowska, 1990) if there exist a positive constant l and some neighborhood $U \subset \text{dom } F$ of x_0 and V of y_0 such that for all $x_1, x_2 \in U$

$$F(x_1) \cap V \subset F(x_2) + l\|x_1 - x_2\| B_Y,$$

where B_Y denotes the unit ball of space Y.

Let $F(x) = F(x) + S$. The graph of F is called the epigraph of F and is denoted by epi F. Let A be a convex subset of X. We denote by $F|_A$ the restriction of F to A, defined by

$$F|_A(x) = \begin{cases} F(x), & \text{if } x \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$$

A set-valued function $F : X \to 2^Y$ is said to be S-convex on A, if the epigraph of $F|_A$, epi $F|_A$, is convex. That is, for any $x_1, x_2 \in A, \lambda \in [0, 1]$

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + S.$$

A set-valued function $F : X \to 2^Y$ is said to be S-nearly convexlike on A, if $\overline{F(A) + S}$ is convex.

It is obvious that if F is S-convex on A, then F is S-nearly convexlike on A. However, the converse is not true, i.e., an S-nearly convexlike set-valued function is not necessarily S-convex (see example 2.1 in Song, 1997).

A set-valued function $F: X \to Y$ is said to be invex at $(x_0, y_0) \in \mathrm{gr}\, F$ if for all $(x^*, -y^*) \in N_{\mathrm{epi}\, F}(x_0, y_0)$ (the Clarke normal cone of epi F at (x_0, y_0)) and $(x, y) \in \mathrm{gr}\, F$, there exists $\eta \in X$ such that

$$\langle y^*, y - y_0 \rangle \geq \langle x^*, \eta \rangle.$$

F is said to be strictly invex at $(x_0, y_0) \in \mathrm{gr}\, F$ if for all $(x^*, -y^*) \in N_{\mathrm{epi}\, F}(x_0, y_0) \setminus \{0\}$ and $(x, y) \in \mathrm{gr}\, F$, there exists $\eta \in X$ such that

$$\langle y^*, y - y_0 \rangle \geq \langle x^*, \eta \rangle$$

and the equality holds only for $x = x_0$. More precisely, for all $y \in F(x)$, the point $\eta = \eta(x, y, x^*, y^*)$ must be such that

$$\langle y^*, y - y_0 \rangle \geq \langle x^*, \eta \rangle, \quad \text{if } x = x_0,$$

$$\langle y^*, y - y_0 \rangle > \langle x^*, \eta \rangle, \quad \text{if } x \neq x_0.$$

It has been proved in Proposition 3.2 and 3.5 of Sach, Yen and Craven, 1994, that F is invex at (x_0, y_0) if and only if

$$F(X) - y_0 \subset \overline{CF(x_0, y_0)(X)},$$

and that if $\mathrm{int}\, \overline{CF(x_0, y_0)(X)} \neq \emptyset$, F is strictly invex at (x_0, y_0) if and only if

$$F(x_0) - y_0 \subset \overline{CF(x_0, y_0)(X)}$$

and

$$F(X \setminus \{x_0\}) - y_0 \subset \mathrm{int}\, \overline{CF(x_0, y_0)(X)}.$$

For set-valued functions $F: X \to Y$, $G: X \to Z$, let

$$H(x) = (F(x), G(x)), \quad \tilde{H}(x) = (F(x), G(x)) + S \times Q.$$

$F \times G$ is called invex (resp. strictly invex) at (x_0, y_0, z_0) if $\tilde{H}$ is invex (resp. strictly invex) at (x_0, y_0, z_0).

If a set-valued function $F: X \to Y$ is convex on X, then F is invex at any point $(x_0, y_0) \in \mathrm{gr}\, F$. However, the converse is not true in general (see Example 2 of Sach and Craven, 1991a). For the definitions and the related results on invex set-valued functions, we refer to Sach and Craven (1991a;b), Sach, Yen and Craven (1994).

Throughout this paper, we assume that $A \subset \mathrm{dom}\, F \cap \mathrm{dom}\, G$ and $A + \emptyset = \emptyset$, $A \times \emptyset = \emptyset$.

Lemma 1 *Aubin, Prankowska {1991). Let A be a subset of X, let* F: X -, Y *be a set-vafoed function, and let x* E *A and y* E *F(x). If F is pseudo-Lipschitz at (x, y), then*

$$CFIA(x'y)(u) = \{ \begin{array}{ll} CF(x,y)(u), & \text{if } u \text{ E } <?A(x); \\ 0, & \text{otherwise.} \end{array}$$

When F is locally Lipschitz at x, Lemma 1 is a special case of Proposition 5.2.3 of Aubin, Frankowska (1990). The proof given in Aubin, Frankowska (1990), is still valid for Lemma 1.

Lemma 2 *Sach and Craven (1991a). Let F : X -, Y, G : X _., Z be set-valued functions. If either F is pseudo-Lipschitz at (x, y)* E gr *F or G is pseudo-Lipschitz at (x, z)* E gr G, *then for every u* E X

$$CF(x,y)(u) \times CG(x,z)(u) \subset C(F \times G)(x,y,z)(u).$$

If F is psev,do-Lipschitz at (x, y) E gr *F and G is pseudo-Lipschitz at (x, z) E* gr *G, then the converse inclusion holds.*

For locally Lipschitz set-valued functions, Lemma 2 coincides with Lemma 9 in Sach and Craven (1991a). The proof given in Sach and Craven (1991a) is still valid for Lemma 2. We observe that

$$\text{dom } C(F \times G)(x,y,z) = \text{dom } CF(:c,y) \cap \text{dom } CG(x,z).$$

For the problem (1), define as before $F(x) = F(x) + S$, $G(x) = G(x) + Q$.

Theorem 1 *(See also Corley, 1988 and Sach, Yen and Craven, 1994.) Let* int $S \neq 0$, *int* $Q \neq 0$. *If* (x_0, y_0) *is a local weak minimum solution of (1), then, for any* z_0 E $G(x_0) \cap (-Q)$, *there exist* y^* E s^+, z^* E $Q+$, *not both zero, such that*

$$(y^*,y) + (z^*,z) \quad 0, \tag{2}$$

for all (y,z) E $C(FIA \times GIA)(xo, Yo, zo)(X)$;

$$(z^*,zo)=0. \tag{3}$$

Proof. The proof can be obtained by slightly modifying the proof of Theorem 5.1 of Corley (1988) (see also Sach, Yen and Craven, 1994). Ill

Theorem 1 is a slightly more general form of Theorem 5.1 of Corley (1988) where set-valued functions F and G were used instead of F and G. The example 3 in Sach and Craven (1991a) shows that Theorem 1 can sometimes exclude a nonoptimal point, but Theorem 5.1 of Corley (1988) does not.

By Lemma 1 and Lemma 2, we can easily deduce from Theorem 1 the following result

Proposition 1 *Let* int $S \neq \emptyset$, *int* $Q \neq \emptyset$. *Assume that* F *is pseudo-Lipschitz at* $(x_0, y_0) \in \operatorname{gr} F$ *and* G *is pseudo-Lipschitz at* (x_0, z_0) *for some* $z_0 \in G(x_0) \cap (-Q)$. *If* (x_0, y_0) *is a local weak minimum solution of (1), then there exist* $y^* \in S^+$, $z^* \in Q^+$, *not both zero, satisfying*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0, \tag{4}$$

for all $(y,z) \in C(F \times G)(x_0,y_0,z_0)(CA(x_0))$

$$\langle z^*, z_0\rangle = 0. \tag{5}$$

We say that $(y^*, z^*) \in S^+ \times Q^+$ satisfies the Fritz John condition at (x_0, y_0, z_0) if $(y^*, z^*) \neq 0$ and the conditions (4) and (5) hold. If in addition $y^* \neq 0$, then we say that (y^*, z^*) satisfies the Kuhn-Tucker condition at (x_0, y_0, z_0) and that problem (1) is normal at (x_0, y_0, z_0).

3. Main results

We now present a constraint qualification, which ensures that the problem (1) is normal at (x_0, y_0, z_0).

Proposition 2 *Let* $x_0 \in A$, $y_0 \in F(x_0)$ *and* $z_0 \in G(x_0) \cap (-Q)$. *Let* dom $CF(x_0, y_0) \supset$ dom $CG(x_0, z_0) \cap CA(x_0)$. *Assume that*

(a) *either* F *is pseudo-Lipschitz at* (x_0, y_0) *or* G *is pseudo-Lipschitz at* (x_0, z_0);

(b) $0 \in \operatorname{int} \overline{z_0 + CG(x_0,z_0)(CA(x_0))}$.

If (y^*, z^*) $(\neq 0)$ *satisfies the conditions*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0, \tag{6}$$

for all $(y, z) \in C(F \times G)(x_0, y_0, z_0)(CA(x_0))$,

$$\langle z^*, z_0\rangle = 0, \tag{7}$$

then $y^* \neq 0$.

Proof. Assume on the contrary that $y^* = 0$. Then

$$\langle z^*, z\rangle \geq 0,$$

for all z in the projection of $C(F \times G)(x_0, y_0, z_0)(CA(x_0))$ on the space Z.

Lemma 2, together with dom $CF(x_0, y_0) \supset$ dom $CG(x_0, z_0) \cap CA(x_0)$, shows that

$$\langle z^*, z\rangle \geq 0,$$

for all $z \in CG(x_0,z_0)(CA(x))$.

Since $\langle z^*, z_0\rangle = 0$, one has that

$$\langle z^*, z\rangle \geq 0,$$

for all $z \in \overline{z_0 + CG(x_0, z_0)(CA(x))}$.

This, together with condition (b), implies that $z^* = 0$, a contradiction. ∎

Remark 1 *When $G(x_0) \cap (-\text{int } Q) \neq \emptyset$, since (see Sach and Craven, 1991)*

$$CG(x,z)(\cdot) = CG(x,z)(\cdot) + Q, \text{ for } (x,z) \in \mathcal{G}rG,$$

there exists $z_0 \in G(x_0) \cap (-\text{int } Q)$ such that hypothesis (b) holds. When $G(x) = g(x)$ is a continuous differentiable single-valued function, we have $CG(x_0, z_0)(v) = g'(x_0)(v) + Q$ and the condition (b) takes the following form

$$0 \in \text{int } (g(x_0) + g'(x_0)(C_A(x_0)) + Q),$$

which is the Robinson regularity condition (see Robinson, 1976).

Corollary 1 *Let $A = X$, $x_0 \in A$, $y_0 \in F(x_0)$ and $z_0 \in G(x_0) \cap (-Q)$. Assume that $\text{dom } CF(x_0,y_0) \supset \text{dom } CG(x_0,z_0) \cap C_A(x_0)$. Assume that either F is pseudo-Lipschitz at (x_0,y_0) or G is pseudo-Lipschitz at (x_0,z_0). If $(y^*, z^*)\ (\neq 0)$ satisfies the conditions (6) and (7) of Proposition 2 (in this case $C_A(x_0) = X$)), then each of the following conditions is sufficient for $y^* \neq 0$.*
(c) *G is invex at (x_0, z_0) and $0 \in \text{int } G(X)$;*
(d) *G is strictly invex at (x_0, z_0), $\text{int } CG(x_0,z_0)(X) \neq \emptyset$ and the the feasible point set of (1) is not a singleton.*

Proof. Assume that (c) is true. Since G is invex at (x_0, z_0) and $0 \in \text{int } G(X)$, we have

$$0 \in \text{int } \overline{z_0 + CG(x_0,z_0)(X)}.$$

So the conclusion follows from Proposition 2.
If G is strictly invex at (x_0, z_0), then

$$G(X \setminus \{x_0\}) - z_0 \subset \text{int } \overline{CG(x_0,z_0)(X)}.$$

Since the feasible set of problem (1) is not a singleton, $0 \in G(X \setminus \{x_0\})$. It follows that

$$0 \in \text{int } \overline{z_0 + CG(x_0,z_0)(X)}.$$

So the conclusion follows from Proposltion 2. ■

Corollary 1 (c) generalizes Theorem 3.1 of Sach and Craven (1991) and Corollary 1 (d) is a special case of Proposition 3.6 of Sach, Yen and Craven (1994).

Now we prove the equivalence between the vector-valued Lagrangian condition and the Kuhn-Tucker condition. Let $L^+(Z,Y)$ denote the set of all linear continuous operator A from Z to Y such that $A(Q) \subset S$.

Proposition 3 *Let $\text{int } S \neq \emptyset$ and let $x_0 \in A$, $y_0 \in F(x_0)$ and $z_0 \in G(x_0) \cap (-Q)$. Assume that F is pseudo-Lipschitz at (x_0,y_0) and G is pseudo-Lipschitz at (x_0,z_0). If $F \times G|_A$ is invex at (x_0, y_0, z_0), then the following statements are equivalent*

(i) *there exist* $y^* \in S^+$ $(y^* \neq 0)$, $z^* \in Q^+$, *such that*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0,$$
for all $(y,z) \in C(F \times G)(x_0, y_0, z_0)(CA(x_0))$

and

$$\langle z^*, z_0\rangle = 0;$$

(ii) *there exists* $\Lambda \in L^+(Z, Y)$ *such that* (x_0, y_0) *is a global weak minimum solution of the following problem:*

$$\min_{x \in A} (F(x) + \Lambda G(x)) \tag{8}$$

and $\Lambda z_0 = 0$,

Proof. (i) $\Rightarrow$ (ii) Since $(F \times G)|A$ is invex at (x_0, y_0, z_0),

$$(F \times G)(A) - (y_0, z_0) \subset \overline{C((F \times G)|A)(x_0, y_0, z_0)(X)}$$

$$\subset \overline{C(F \times G)(x_0, y_0, z_0)(CA(x_0))}.$$

It follows from (i) that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq \langle y^*, y_0\rangle + \langle z^*, z_0\rangle = \langle y^*, y_0\rangle, \tag{9}$$
for all $y \in F(x)$, $z \in G(x)$, $x \in A$.

Fix $e \in \operatorname{int} S$ with $\langle y^*, e\rangle = 1$ since $y^* \neq 0$. Define $\Lambda : Z \to Y$ by

$$\Lambda z = \langle z^*, z\rangle e, \quad \text{for every } z \in Z,$$

then

$$y^* \circ \Lambda = z^*, \quad \Lambda z_0 = 0, \quad \Lambda(Q) \subset S,$$

Hence $\Lambda \in L^+(Z, Y)$. Replacing z^* by $y^* \circ \Lambda$ in (9), we have

$$\langle y^*, y + \Lambda z\rangle \geq \langle y^*, y_0 + \Lambda z_0\rangle = \langle y^*, y_0\rangle, \tag{10}$$
for all $y \in F(x)$, $z \in G(x)$, $x \in A$.

It follows that (x_0, y_0) is a weak minimum solution of the problem (8) since $y^* \neq 0$ and $y_0 \in F(x_0) + \Lambda G(x_0)$. Therefore (ii) is true.

(ii) $\Rightarrow$ (i) We shall show that if (ii) is true, then

$$[CF(x_0, y_0) + \Lambda CG(x_0, z_0)](CA(x_0)) \cap (-\operatorname{int} S) = \emptyset \tag{11}$$

Indeed, in the contrary case, there exist $v \in -\operatorname{int} S$, $u \in CA(x_0)$, $v_1 \in CF(x_0, y_0)(u)$, $w \in CG(x_0, z_0)(u)$ such that $v = v_1 + \Lambda w$. Thus, for any $h_n \to 0^+$ there exist $u_n \to u$, $u_n^i \to u$, $i = 1, 2$, $v_n \to v_1$ and $w_n \to w$ such that

$$x_0 + h_n u_n \in A,$$

$$y_0 + h_n v_n \in \hat{F}(x_0 + h_n u_n^1),$$

$$z_0 + h_n w_n \in \hat{G}(x_0 + h_n u_n^2).$$

Since P is pseudo-Lipschitz at (x_0, y_0), there exists a positive constant l_1 such that

$$y_0 + h_n v_n \in \hat{F}(x_0 + h_n u_n) + l_1 h_n \|u_n - u_n^1\| B_Y$$

for n sufficiently large. So there exists $V_n \to v_1$ such that

$$Y_0 + h_n V_n \in F(x_0 + h_n u_n).$$

Since G is pseudo-Lipschitz at (x_0, z_0), there exists $W_n \to w$ such that

$$Z_0 + h_n w_n \in G(x_0 + h_n u_n),$$

Hence

$$y_n = y_0 + \Lambda z_0 + h_n(\bar{v}_n + \Lambda \bar{w}_n) \in (\hat{F} + \Lambda \hat{G})(x_0 + h_n u_n).$$

Since $A z_0 = 0$ and

$$\frac{Y_n - Y_0}{h_n} = V_n + A W_n \to V \in -\operatorname{int} S,$$

we get

$$\frac{Y_n - Y_0}{h_n} \in -\operatorname{int} S'$$

for n large enough, and then $Y_n - y_0 \in -\operatorname{int} S$. This is not possible since (x_0, y_0) is a weak minimum solution of (8) and it is also a weak minimum solution for

$$\min_{x \in A} (F + AG)(x).$$

Thus (11) is true. Since $[CF(x_0, y_0) + ACG(x_0, z_0)](\cdot)$ is a convex process and $C_A(x_0)$ is a closed convex cone, by standard separation arguments, there exists $y^* \in S^+ \setminus \{0\}$ such that

$$(y^*, y + Az) \geq 0, \tag{12}$$
for all $y \in CF(x_0, Y_0)(u)$, $z \in CG(x_0, z_0)(u)$, $u \in CA(x_0)$,

Let $z^* = y^* A$. (12) is equivalent to

$$(y^*, y) + (z^*, z) \geq 0, \tag{13}$$
for all $y \in CF(x_0, Y_0)(u)$, $z \in CG(x_0, z_0)(u)$, $u \in CA(x_0)$.

Since

$$CF(x_0, y_0)(u) \times CG(x_0, z_0)(u) = C(F \times G)(x_0, Y_0, z_0)(v,),$$

from (13), we obtain (i). ■

Remark 2 *In the case $A = X$, we only need to assume that either $\hat{F}$ is pseudo-Lipschitz at (x_0, y_0) or G is pseudo-Lipschitz at (x_0, z_0), and a result analogous to the implication* (i) $\Rightarrow$ (ii) *has been proved in Theorem 4.3 of Sach and Craven (1991b) under more restrictive assumptions. Jahn (1986) proved a similar result for single-valued Frechet differentiable mappings F and G. On the other hand, It is easy to show that the condition* (ii) *is a sufficient condition for (x_0, y_0) to be a global weak minimum solution of (1) and so is the condition (i) if $(F \times G)|_A$ is invex at (x_0, y_0, z_0) (similar sufficient conditions were given in Sach and Craven, 1991a;b, Sach, Yen and Craven, 1994).*

As a direct consequence of Propositions 1, 2 and 3, we have

Theorem 2 *Let* $\text{int } S \neq \emptyset$, $\text{int } Q \neq \emptyset$. *Let $x_0 \in A$, $y_0 \in F(x_0)$ and $z_0 \in G(x_0) \cap (-Q)$, and let* $\text{dom } CF(x_0, y_0) \supset \text{dom } CG(x_0, z_0) \cap CA(x_0)$. *Assume that*
(a) *$\hat{F}$ is pseudo-Lipschitz at (x_0, y_0) and G is pseudo-Lipschitz at (x_0, z_0);*
(b) $0 \in \text{int } z_0 + CG(x_0, z_0)(CA(x_0))$.
If (x_0, y_0) is a local weak minimum solution of (1), then there exists $(y^*, z^*) \in S^+ \times Q^+$ satisfies the Kuhn-Tucker condition at (x_0, y_0, z_0), i. e., the condition (i) of Proposition 3 holds.

Moreover, if $(F \times G)|_A$ is invex at (x_0, y_0, z_0), then the vector-valued Lagrangian condition holds, i. e., condition (ii) *of Proposition 3 holds.*

We now consider a Lagrangian dual problem to (1). Define $H: L^+(Z, Y) \to 2^Y$ by

$$H(\Lambda) = \{y \mid \exists x \in A$$

s.t. (x,y) is a global weak minimum solution of (8)

Consider the following maximization problem

$$\max H(\Lambda) \qquad (14)$$
$$\text{s. t. } \Lambda \in L^+(Z, Y).$$

A point Λ is said to be a feasible point of (14) if $\Lambda \in L^+(Z, Y)$ and $H(\Lambda) \neq \emptyset$. The set of all such points will be denoted by E'. (Λ_0, y_0) is called a global weak maximum solution of (14) if $\Lambda_0 \in E'$, $y_0 \in H(\Lambda_0)$, and there is no $\Lambda \in E'$ such that

$$(y_0 - H(\Lambda)) \cap (-\text{int } S) \neq \emptyset.$$

Theorem 3 *Let* $\text{int } S \neq \emptyset$, $\text{int } Q \neq \emptyset$. *Let $x_0 \in A$, $y_0 \in F(x_0)$ and $z_0 \in G(x_0) \cap (-Q)$, and let* $\text{dom } CF(x_0, y_0) \supset \text{dom } CG(x_0, z_0) \cap CA(x_0)$. *Assume that*
(a) *$(F \times G)|_A$ is invex at (x_0, y_0, z_0);*
(b) *$\hat{F}$ is pseudo-Lipschitz at (x_0, y_0) and G is pseudo-Lipschitz at (x_0, z_0);*
(c) $0 \in \text{int } z_0 + CG(x_0, z_0)(CA(x_0))$.

If (xo, y_0) is a local weak minimv,m solv,tion of (1), then there exists an Ao *E L+(Z,* Y) *such that* (Ao, y_0) *is a global weak maximum solv,tion of (14).*

Proof. By Theorem 2, there exists Ao *E* L^+ *(z ,*Y) such that *(x_0, y_0)* is a global weak minimum solution of (8) corresponding to Ao. This means that Ao is a feasible point of (14) and *Yo E* H(Ao).

For any feasible point A of (14) and anyy *E H(A),* there is *x E A* such that *(x, y)* is a global weak minimum solution of (8), then

[(F + *AG)(A) -* y] n (-int *S)* = *0.*

We shall prove that *(Yo - y) ¢* -int *S.* Indeed, if *Yo - y E* -int *S,* take z_0 *E* $G(x_0)$ n *(-Q),* then, Ao *E* L^+ *(z , Y)* implies that *Aozo E - S.* So,

Yo + Aozo - y E -int *S - S C* -int *S.*

This leads to a contradiction. Note that *y E H(A)* is arbitrary, we get

(Yo - H(A)) n (-int *S)* = *0.*

Therefore, (Ao, *Yo)* is a global weak maximum solution of (14). *II*

Remark 3 *In the case when A = X, we only need to assume that either F is pseudo-Lipschitz at (x_0,y_0) or G is pseudo-Lipschitz at (x_0,z_0). Similar Lagrangian dual,ity results for the weak minimality were proved v,nder the Slater condition for S-convex set-valued functions by Corley (1987), and for S-nearly convexl,ike set-valued functions by Song (1997).*

Now we present a necessary optimality condition for (.1:0, y_0) to be a Benson proper minimum solution of problem (1).

Theorem 4 *Let*x_0 *E A,*y_0 *E F(xo) andzo E* G(xo)n(-Q). *Letdom CF(xo,Yo) :J* dom *CG(xo,* z_0) n *CA*(x_0). *Assume that*

(a) *either S has a weakly compact base and F is S-nearly convexlike on A or S has a compact base,*

(b) *F is pseudo-Lipschitz at (xo, Yo) and G is pseudo-Lipschitz at (xo, zo),*

(c)/ 0 *E* int z_0 + *CG(xo, zo)(CA(xo)) and* int Q # *0.*

If (x_0, y_0) is a prnper minimv,m solution of (1), then

(i) *there exist y* E s+i, z* E Q+ such that*

(y*, y) + (z*, z) ≥ 0,

for all *(y,z) E C(F x G)(xo,Yo,zo)(CA(xo))*

and

(z*,zo) = 0.

Moreover, if (F x *G)IA is invex at (xo,Yo,zo), then*

(ii) *there exists an* A *E* L^+ *(z ,*Y) *such that (x_0, yo) is a proper minimum solution of (8) and Azo* = 0.

Proof. By the definition of the Benson proper minimum solution, we have

$$\overline{\text{cone}}(F(E) + S - y_0) \cap (-S) = \{0\}.$$

If F is S-nearly convexlike on E, then $\overline{F(E) + S}$ is convex. Since $\overline{F(E) + S} - y_0 \subset \overline{\text{cone}}(F(E) + S - y_0)$, by Proposition 4.2.1 in Aubin, Frankowska (1990), we can deduce that

$$\overline{\text{cone}}(F(E) + S - y_0) = \overline{\text{cone}}(\overline{F(E) + S} - y_0)$$

is a weakly closed convex cone. This, with assumption (a) implies that the hypotheses of Theorem 2.3 in Dauer and Saleh (1993), are satisfied and hence there exists a pointed closed convex cone $C \subset Y$ such that $-S \setminus \{0\} \subset -\text{int}\, C$ and

$$\overline{\text{cone}}(F(E) + S - y_0) \cap (-C) = \{0\}.$$

We claim that

$$\overline{\text{cone}}[(F \times G)(A) - (y_0, 0)] \cap [-(\text{int}\, C \times \text{int}\, Q)] = \emptyset. \tag{15}$$

Since $\text{int}\, C \times \text{int}\, Q$ is an open cone, for this we only need to show that

$$[(F \times G)(A) - (y_0, 0)] \cap [-(\text{int}\, C \times \text{int}\, Q)] = \emptyset.$$

If it is not the case, then there exist $x \in A$, $y \in F(x)$, $z \in G(x)$, $s \in S$ and $q \in Q$ such that

$$y + s - y_0 \in -\text{int}\, C, \quad z + q \in -\text{int}\, Q.$$

Hence

$$z \in -\text{int}\, Q - Q \subset -Q$$

and hence $x \in E$, $y \in F(E)$. Thus

$$\overline{\text{cone}}(F(E) + S - y_0) \cap (-C) = \{y + s - y_0\}.$$

This is a contradiction.

Since F is pseudo-Lipschitz at (x_0, y_0) and G is pseudo-Lipschitz at (x_0, z_0), from the proof of Proposition 5.3.1 of Aubin, Frankowska (1990), we can deduce that

$$C(F \times G)(x_0, y_0, z_0)(C_A(x_0)) \subset T_{(F \times G)(A)}(y_0, z_0). \tag{16}$$

We next show that

$$(0, z_0) + T_{(F \times G)(A)}(y_0, z_0) \subset \overline{\text{cone}}[(F \times G)(A) - (y_0, 0)]. \tag{17}$$

Let $(u, v) \in T_{(F \times G)(A)}(y_0, z_0)$. Then there exist $h_n \to 0^+$, $(u_n, v_n) \to (u, v)$ and $x_n \in A$ such that for any $n \geq 1$

$$(y_0, z_0) + h_n(u_n, v_n) \in (F \times G)(x_n).$$

Since z_0 E $G(x_0)$ n (-Q), we have that

$$h_n (V_n + zo) = Zo + h_n V_n - (1 - h_n) Zo \text{ E } G(x_n) + QC\ G(x_n),$$

Hence

$$(Yo, 0) + h_n (u_n, V_n + zo) \text{ E } (F \text{ x } G)(A)$$

and so

$$(v_n, v + z_0) \text{ E } \overline{\text{cone}}[(F \text{ x } G)(A) - (yo,0)].$$

Therefore, it follows from (15)-(17) that

$$[(0, zo) + C(F \text{ x } G)(xo, Yo, zo)(C_A (xo))] \text{ n } (-(\text{int } C \text{ x int } Q)) = 0. \quad (18)$$

Since $C(F \text{ x } G)(x_0, y_0, z_0)$ is a closed convex process and $C_A (x_0)$ is a closed convex cone, $C(F \text{ x } G)(xo, Yo, zo)(C_A (xo))$ is convex. By standard separation arguments, there exist y^* E c^+, z^* E Q^+, not both zero, such that

$$(y^*, y) + (z^*, z + zo) \ 2 \ 0, \quad (19)$$

for all (y,z) E $C(F \text{ x } G)(xo,Yo,zo)(C_A (xo))$.

Since z_0 E $G(x_0)$ n (-Q) and (0, 0) E $C(F \text{ x } G)(xo, Yo, zo)(C_A (xo))$, the inequality (19), with z^* E Q+, implies that

$$(z^*, zo) = 0.$$

Hence

$$(y^*, y) + (z^*, z) \ 2 \ 0, \quad (20)$$

for all (y, z) E $C(F \text{ x } G)(xo, Yo, zo)(C_A (xo))$.

We only need to show that y^* E s^{+i}. From the proof of Proposition 2, we see that $y^* \neq 0$. Hence

$$(y^*, y) > 0, \text{ for all } y \text{ E int } C.$$

Since S \ {0} C int C, we obtain that y^* E s^{+i}.

Moreover, since $(F \text{ x } G)|_A$ is invex at (xo,Yo,zo), we can deduce that

$$(F \text{ x } G)(A) - (Yo, zo) \text{ C } \overline{C(F \text{ x } G)|_A (xo, Yo, zo)(X)}$$

$$\text{C } \overline{C(F \text{ x } G)(xo, Yo, zo)(C_A (xo))}. \quad (21)$$

Hence, (i) implies that

$$(y^*,y) + (z^*,z) \ 2 \ (y^*,yo) + (z^*,zo) = (y^*,yo), \quad (22)$$

for all (y, z) E (F x G)(A).

Fix e E 8 \ {0} such that $(y^*, e) = 1$ (such an e exists, since y^* E s^{+i}). Define A : Z -, Y by

$$Az = (z^*, z)e, \quad \text{for all } z \text{ E } Z.$$

Then

$$y^* A = z^*, \quad A(Q) \text{ c } 8, \quad Azo = 0.$$

Replacing z^* by y^*A in (22), we obtain

$$(y^*, y + Az) \quad (y^*, yo), \tag{23}$$
for all x E A, y E F(x) and z E G(x).

Since Yo E F(.To) + AG(xo) and y^* E s^{+i}, by Theorem 5.2.1 in Jahn (1986), we can conclude that (.To, Yo) is a proper minimum solution of the problem (8). Ill

In the case when *A = X,* (16) is true without assumption (b). Thus we only need to assume that either F is pseudo-Lipschitz at (xo, Yo) or G is pseudo-Lipschitz at (xo,zo). Clearly (ii) is a sufficient condition for (xo,Yo) E gr F to be a proper minimum solution of (1), and consequently (i) is also a sufficient condition under the assumption that (F x G)IA is invex at (xo,Yo,zo)-

Example 1 *Let X* = *Z* = , *Y* = ^{2} *and A* = [0,1]. *Let 8* = *i and* Q = +. *It is obvious that i has a compact base and* int + =/- 0. *Define set-valued mappings F and G as follows*

F(x) = { ,= (6,6) E R^2 I tr +t :::: x^2}, if x Q
if $x < 0$,

and

$$G(x) = \frac{1}{2} - x, \quad \text{for } x \text{ E } -$$

Since the feasible point set E = A n C^{-1}(- +) = [½, 1], the problem (1) takes the following form

$$\min_{x \in [½,1]} F(x). \tag{24}$$

Let xo = 1, y_0 = (-1, -1) and zo = -½. It is obvious that F is pseudo-Lipschitz at (xo,Yo) and G is pseudo-Lipschitz at (xo,zo)-

Since CG(xo, zo)(x) = - x + + and CA(xo) = {x E Ix:::; 0}, we have

$$\overline{z_0 + \hat{CG}(xo, zo)(CA(xo))} = [-\frac{1}{2}, +\infty).$$

Hence

$$0 \text{ E int } \overline{zo + CG(xo, zo)(CA(xo))}.$$

Note that

$$F(E) = \{y = (6,6) \in R^2 \; \text{Ilf} + (\; S \; 1\}.$$

It is easy to show that (x_0, y_0) is a proper minimum solution of (24).
We can easily verify that

$$C_{epi} \; p(xo,Yo) = C_{gr} \; p(xo,Yo) = \{(x,y) \mid Y = (6,6), \; 6 + 6 \geq -v2x\}.$$

By the definition of CF(xo, Yo)(·), we have

$$C\hat{F}(xo,Yo)(x) = \{y = (6,(2) \in {}^2 \; 16 + 6 \geq -v2^{1/n}x\}.$$

Hence

$$\text{dom } CF(xo, Yo) = \text{dom } CG(xo, zo) =$$

and

$$C(F \times G)(xo,Yo,zo)(CA(xo)) = \{(y,z) \mid Y = (6,6), 6 + (2 \geq 0, \; z \geq 0\}.$$

Let $y^* = (1, 1) \in \text{int } \mathbf{i}$, $z^* = 0 \in$ +- One see that

$$(y^*, y) + (z^*, z) \geq 0,$$
for all $(y,z) \in C(F \times G)(xo,Yo,zo)(CA(xo))$

and,

$$(z^*, z0) = 0.$$

It is easy to verify that

$$(F \times G)(A) - (yo, zo) \subset C(F \times G)(xo, Yo, zo)(CA(xo)).$$

Thus, $(F \times G)|A$ is invex at (xo, Yo, zo)-
Let $e = (\frac{1}{2}, \frac{1}{2}) \in \mathbf{1}$. It is clear that $(y^*, e) = 1$. Define the operator A : -+ ^{2} by

$$Az = (z^*, z)e = (0,0), \quad \text{for } z \in -$$

Then $A \in L^+(z, Y)$. Thus, the problem (8) is of the form

$$\min_{x \in [O,1]} F(x). \tag{25}$$

It is evident that (x_0, y_0) is also a proper minimum solution of (25).
Now we shall consider the Lagrangian duality for the proper minimality. Define the set-valued function $fI : L^+(z, Y) \to 2^Y$ by
$fI(A) = \{y \; I{:}l(\in A$ s.t. $((,y)$ is a proper minimum solution of (8)$\}$

Consider Lhe problem

maxfI(A) (26)

s. t. A E L+(z, Y).

A point A is said to be a feasible point of (26) if A E £+ (z ,Y) and fI(A) =/=0. (Ao, yo) is called a global maximum solution of (26) if Ao is a feasible point of (26), y_0 E fI(Ao), and there is no feasible point A of (26) such that

(Yo - fI(A)) n (- S \ {0})-=/= 0.

By using similar arguments as in the proof of Theorem 3, we can prove the following duality result

Theorem 5 *Let xo* E A, *Yo* E F(xo) *and zo* E G(xo) n (-Q). *Let* dom *CF(x0,* Yo):::) dom *CG(xo,zo)* n CA(xo). *Assume that*

(a) *either S has a weakly compact base and F is S-nearly convexlike on A or S has a compact base,*

(b) *F is pseudo-Lipschitz at* (xo,Yo) *and* G *is pseudo-Lipschitz at* (xo,zo),

(c) 0 E int *zo* + CG(xo,zo)(CA(xo)) *and* int Q=/= 0,

(d) (F x G)IA *is invex at (xo,Yo,zo)-*

If (x_0, y_0) is a proper minimum solution of (1), then there exists a Ao E L+ (z ,Y) such that (Ao, yo) is a global maximum solution of (26).

A similar duality result was proved by Song (1996) under a so-called image regular condition.

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