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Memoryless equilibrium strategies in multilevel decision processes of discrete-time descriptor systems

by

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Abstract: Applying a general theorem developed by Ba§ar (1981), providing a set of sufficient conditions for a triple of strategies to be in hierarchical equilibrium, to games in normal (strategic) form, we study three-player Stackelberg games for linear quadratic discrete-time descriptor systems with three levels of hierarchy in decision making. We derive explicitly sufficient conditions for the existence of the memoryless hierarchical equilibrium strategies for the player (called Pi) at the top of the hierarchy, and for the player (called A) at the second level of the hierarchy. Since the resulting hierarchical equilibrium strategies do not depend on the memory information, Pi's original optimal team cost remains the tight (attainable) lower bound for Pi's cost function no matter whether the players at the lower levels of hierarchy act or not at the last two (or one) stages. Moreover, the resulting strategies have the advantages of simpler structure and higher credibility. A numerical example is solved to illustrate the validity of the results.

Keywords: dynamics games, description system, memoryless equilibrium strategies, discrete-time system

1. Introduction

Memory strategy concept is fundamental to the study of Stackelberg dynamic game problems with multi-levels of hierarchy in state space systems (Ba§ar and Olsder, 1982). Within the context of linear-quadratic dynamic games defined in discrete time, two kinds of the closed-loop Stackelberg strategies have been obtained in (Ba§ar and Helbuz, 1979; Tolwinski, 1981). Since the essential structure of those strategies contains the memory information on state vector, different team optimal control problems have been introduced depending on whether the follower acts at the last stage of the game or not. For the latter case, the tight lower bound for the leader's Stackelberg cost in the dynamic game will be determined completely by the related leader's team optimal control problem. Otherwise, the original game must be changed to a new game with transformed cost functions for the leader and the follower in order that the method developed for the case when the follower does not act at the last stage can be applied to solve the problem. Moreover, the closed-loop Stackelberg strategy obtained in Ba§ar and Helbuz (1979) is the linear or affine one-step memory strategy which, as indicated by Tolwinski (1981), is rather ill-suited for the possible nonoptimal behavior of the players at some stages of the game. In other words, if the follower plays nonoptimally at one stage, then the role of the leader's strategy in the remaining stages is to penalize the follower for his nonoptimal behavior. The leader's strategy is in no sense optimal when considered in the remaining stages. On the other hand, the closed-loop Stackelberg strategy proposed by Tolwinski (1981) is the nonlinear memory strategy which has the property of penalizing the follower's one stage nonoptimal behavior only at the next stage. The results of Ba§ar and Helbuz (1979) is further extended to the three-player dynamic game with three levels of hierarchy in decision making Ba§ar (1981), in which the so-called hierarchical equilibrium solution concept is defined. Conceptually, the method of Tolwinski (1981) can also be extended to the three-level Stackelberg games. However, such an extension seems not be an easy work because of the complicated structure of the proposed strategies.

With the properties stated above in mind, we now consider the Stackelberg game for discrete-time descriptor systems. In a recent paper (Xu and Mizukami, 1995), the team-optimal closed-loop Stackelberg game for two-player descriptor systems has been studied. An important feature has been found that the closed-loop memoryless information on descriptor vector is sufficient for the leader to construct the team-optimal closed-loop Stackelberg strategy for a large class of discrete-time descriptor systems. Moreover, since the resulting strategy for the leader dose not involve the memory information on descriptor vector, it is not necessary to assume that the follower does not act at the last stage of the game. In this paper, we extend the results of Xu and Mizukami (1995) to the three-player Stackelberg games for linear quadratic discrete-time descriptor systems with three levels of hierarchy in decision making. Applying a general theorem developed by Ba§ar (1981), which provides a set of sufficient conditions for a triple of strategies to be in hierarchical equilibrium to games in normal (strategic) form, we derive explicitly the memoryless hierarchical equilibrium strategies for the player (called A) at the top of the hierarchy, and for the player (called P2) at the second level of the hierarchy. Since the resulting hierarchical equilibrium strategies do not depend on the memory information, A 's original optimal team cost remains the tight (attainable) lower bound for A 's cost function no matter whether the players at the lower levels of hierarchy act or not at the last two (or one) stages. Moreover, the resulting strategies have the advantages of the simpler structure and the higher credibility. More precisely, the hierarchical equilibrium strategies are realized in linear feedback form. And, any nonoptimal behavior (at one stage) of the players at the lower levels of hierarchy is penalized only at that stage. In consequence, the original hierarchical equilibrium strategies still constitute the hierarchical equilibrium strategies for the remaining game starting at the next stage.

The paper is organized as follows. In Section 2, the three-player Stackelberg games for linear quadratic discrete-time descriptor systems with three levels of hierarchy in decision making are formulated. A related team optimal solutions for P_1's cost function are given. Section 3 is devoted to the derivations of the sufficient conditions such that the memoryless hierarchical equilibrium strategies for P_1 and P_2 exist. A numerical example is included in Section 4 to illustrate the results of the paper. The advantages of the memoryless hierarchical equilibrium strategies over the memory strategies are also discussed in the same section. Section 5 contains some conclusions.

2. Problem formulation

Consider a linear discrete-time descriptor system

$$Ex_{k+1} = Ax_k + Bu_k + Cv_k + Dw_k, \quad x_{k=0} = x_0, \qquad (k = 0, 1, 2, \dots, N-1) \tag{1}$$

where $x_k \in R^n$ is the descriptor vector, $u_k \in R^m$, $v_k \in R^\ell$ and $w_k \in R^q$ are the control vectors of player 1 (P_1), player 2 (P_2) and player 3 (P_3), respectively. E is a square matrix of rank $r \leq n$. The pencil $(sE - A)$ is assumed to be regular (i.e., $|sE - A| \neq 0$). Each player is assumed to have a quadratic cost function, respectively,

$$J_1 = \frac{1}{2} x_N^T E^T Q_N^1 E x_N + \frac{1}{2} \sum_{k=0}^{N-1} \{x_k^T Q^1 x_k + u_k^T R_{11} u_k + v_k^T R_{12} v_k + w_k^T R_{13} w_k\}, \tag{2a}$$

$$J_2 = \frac{1}{2} x_N^T E^T Q_N^2 E x_N + \frac{1}{2} \sum_{k=0}^{N-1} \{x_k^T Q^2 x_k + u_k^T R_{21} u_k + v_k^T R_{22} v_k + w_k^T R_{23} w_k\}, \tag{2b}$$

$$J_3 = \frac{1}{2} x_N^T E^T Q_N^3 E x_N + \frac{1}{2} \sum_{k=0}^{N-1} \{x_k^T Q^3 x_k + u_k^T R_{31} u_k + v_k^T R_{32} v_k + w_k^T R_{33} w_k\}, \tag{2c}$$

where $R_{ij} > 0$, $i, j = 1, 2, 3$, $i \leq j$ and all other weighting matrices being nonnegative definite. We assume that each player has access to memoryless information on X_k and can utilize it in constructing his strategy. The typical strategies for player i, i=1,2,3, at stage k, k = 0, 1, $\cdots$, N - 1, are denoted by ,'f E r1, 1' E r and 1' E r t respectively; their open-loop realizations are U_k E U_k ' V_k E vk and W_k E w $_k$ ' respectively. The strategy spaces rt of player i, i=1,2,3, are restricted in the class of linear feedback strategies of X_k. U_k, V_k and Wk are the decision spaces of P_1, P_2 and P_3, respectively. Moreover, we denote the entire collection {,'b, r'L $\cdots$, 1':hr-1} as r'^{i} E r^i for all the game. Finally, we denote the value of Ji, i=1,2,3, given by (2) for a triple of strategies (1'^{1},,,^{2},1'^{3}) by Ji(,'^{1},1'^{2} ' 1'^{3}).

Within the framework of the dynamic game described above, it is further stipulated that (a) player 1 (A) announces his strategy ahead of time and enforce it on the other two players, and (b) player 2, in view of the announced strategy of player 1, announces his strategy to player 3, then (c) player 3 decides his optimal strategy after knowing the announced strategies of player 1 and player 2. The game ordered in this way is called a three-level hierarchical dynamic game.

Before introducing the hierarchical equilibrium solution concept for three-player dynamic games with three levels of hierarchy, let us first define the admissible strategy concepts for P_1 and P_2, respectively. Define two strategy sets.

$$R_2(1'^1) := \{(.;2,e) \text{ E } r^2 \times r^3 :$$
$$\bar{J}_2(1'^1,.;2,e) \leq J_2(1'^1,1'^2,1'^3),\ \forall(1'^2,1'^3) \text{ E } r^2 \times r^3\} \tag{3}$$

and

$$R_3(,'^1,,'^2) := \{e \text{ E } r^3 : \bar{J}_3(,'^1,,'^2,e) \leq J_3(,'^1,,'^2,,'^3),\ \forall y^3 \text{ E } r^3\}. \tag{4}$$

Since the open-loop solution of the linear-quadratic optimal control problem for discrete-time descriptor systems is not necessarily unique, the open-loop realization of the above responding strategies may not be unique too.

DEFINITION 1 $R_2(1'^1)$ *(similarly,* $R_3(,'^1,,'^2)$) *is called the realization singleton if the strategy pair (strategy) in* $R_2(1'^1)$ ($R_3(1'^1,,'^2)$) *admits a unique open-loop realization pair (open-loop realization).*

DEFINITION 2 *A strategy* ,,^{1} E r; c r^1 *is called an admissible strategy of* Pi *if* $R_2(,'^1)$ *is the realization singleton.* r; *is called the admissible strategy space of* A-

DEFINITION 3 *For an admissible strategy* ,'^{1} E r;, *a strategy* ,'^{2} E r; C r^2 *is called an admissible strategy of* P2 *if* $R_3(1'^1,,'^2)$ *is the realization singleton.* r; *is called the admissible strategy space of* P_2.

We will show in the sequel whether a strategy is admissible, or not, can be determined by analyzing the regularity of appropriate matrices. Moreover, we stipulate that r is composed of those strategies 1'^{3} E r^3 such that the triplet (1'^{1},,,^{2},1'^{3}), where, ,,^{1} E r; and ,,^{2} E r;, leads to a unique solution of the descriptor system (1).

DEFINITION 4 *For the dynamic game posed above, an admissible strategy $\gamma^{1*} \in \Gamma$ constitutes a hierarchical equilibrium strategy for P_1 if*

$$J_1^* = \sup_{\gamma^2\in R_2(\gamma^{1*})} \sup_{\gamma^3\in R_3(\gamma^{1*},\gamma^2)} J_1(\gamma^{1*},\gamma^2,\gamma^3) = \min_{\gamma^1\in\Gamma_a^1} \sup_{\gamma^2\in R_2(\gamma^1)} \sup_{\gamma^3\in R_3(\gamma^1,\gamma^2)} J_1(\gamma^1,\gamma^2,\gamma^3), \tag{5}$$

where

$$R_2(\gamma^1) := \{ \in \Gamma : \sup_{\gamma^3\in R_3(\gamma^1,e)} J_2(\gamma^1, , \gamma^3) \le \sup_{\gamma^3\in R_3(\gamma^1,\gamma^2)} J_2(\gamma^1,\gamma^2,\gamma^3),\ \forall \gamma^2 \in \Gamma \} \tag{6}$$

and

$$R_3(\gamma^1,\gamma^2) := \{ \in \Gamma : \bar{J}_3(\gamma^1,\gamma^2,) \le J_3(\gamma^1,\gamma^2,\gamma^3),\ \forall \gamma^3 \in \Gamma \}. \tag{7}$$

Any strategy $\gamma^{2*} \in R_2(\gamma^{1*})$ is a corresponding equilibrium strategy for P_2, and any $\gamma^{3*} \in R_3(\gamma^{1*},\gamma^{2*})$ is an equilibrium strategy for P_3 corresponding to the strategy pair $(\gamma^{1*},\gamma^{2*})$.

The foregoing definition of the hierarchical equilibrium strategies also takes into account possible nonunique responses of the players at the lower levels of the hierarchy. An important feature of descriptor systems is that the sets $R_2(\gamma^1)$ and $R_3(\gamma^1,\gamma^2)$ are not singletons even if the admissible strategies are restricted in the class of the linear feedback strategies.

The hierarchical equilibrium strategies concept was initially introduced in a general framework of games in normal (strategic) form. Corresponding to the general definition of the hierarchical equilibrium strategies, Başar (1981) has developed a theorem which provided a set of sufficient conditions for a triple of strategies to be in hierarchical equilibrium. In order to introduce that theorem, we need some preliminary notations.

For each $\gamma^1 \in \Gamma$, define the subsets $S_1(\gamma^1) \subset \Gamma \times \Gamma$ and $S_2(\gamma^1) \subset \Gamma \times \Gamma$ by

$$S_i(\gamma^1) := \{(e, e) \in \Gamma \times \Gamma : J_i(\gamma^1, e, e) = \min_{\gamma^2\in\Gamma} \min_{\gamma^3\in\Gamma} \bar{J}_i(\gamma^1,\gamma^2,\gamma^3)\},\ i = 1, 2, \tag{8}$$

and introduce a subset $\bar{S}_2(\gamma^1) \subset S_2(\gamma^1)$ by

$$\bar{S}_2(\gamma^1) := \{(\gamma^2,\gamma^3) \in \bar{S}_2(\gamma^1) : \gamma^3 \in R_3(\gamma^1,\gamma^2)\}. \tag{9}$$

THEOREM 1 *Let there exist a $\gamma^{1*} \in \Gamma$ such that:*

(i) *$\bar{S}_2(\gamma^{1*})$ is nonempty and for every pair $(e, e) \in \bar{S}_2(\gamma^{1*})$*

$$\sup_{\gamma^3\in R_3(\gamma^{1*},e^2)} J_2(\gamma^{1*}, e, \gamma^3) = J_2(\gamma^{1*}, e, e). \tag{10}$$

(ii) 82(',^{1*}) c S$_1$ (,y^{1*}), *and,*
(iii) *for every* (e,e) E S1(',^{1*}),

arg $\min_{1^1 \mathrm{Er}}$ J1 *Cl, e,* e) = ·*l**. (11)

Then, -y^{1*} *is a hierarchical equilibrium strategy for* Pi, *and given any pair* (1'^{2*}, -y^{3*}) E 8$_2$(-y^{1*}), -y^{2*} *is a corresponding equilibrium strategy for* P$_2$, *and* -y^{3*} *is an equilibrium strategy for* P_3 *corresponding to the pair* (-y^{1*}, -y^{2*})'

Since the general framework of Theorem 1 pertains to games in normal (strategic) form, it can be applied to some specific hierarchical dynamic game problems. In the following, we will apply Theorem 1 to the dynamic game for multilevel discrete-time descriptor systems. Towards this end, we first provide team-optimal solution to the related team-optimal control problem

$$\min_{\gamma^1 \in \Gamma_a^1} \min_{\gamma^2 \in \Gamma_a^2} \min_{\gamma^3 \in \Gamma_a^3} J_1(\gamma^1, \gamma^2, \gamma^3) := J_1^t, \tag{12}$$

subject to the descriptor system (1).

Without loss of generality, we make use of some transformations on the system matrices throughout the paper. It is well-known that there always exist two nonsingular matrices M and H such that

$$MEH = \begin{bmatrix} Ir & 0 \\ 0 & 0 \end{bmatrix}, \quad r = rank\ E. \tag{13}$$

Using (13), we have

$$\begin{bmatrix} z! \\ zl \end{bmatrix} = H^{-1} Xk, \quad \begin{bmatrix} z_0^1 \\ z_0^2 \end{bmatrix} = H^{-1}x_0, \tag{14}$$

$$\mathrm{MAH} = \begin{bmatrix} \mathrm{An} & \mathrm{A12} \\ \mathrm{A\,21} & \mathrm{A\,22} \end{bmatrix}, \quad \mathrm{MB} = \begin{bmatrix} \mathrm{B\,1} \\ \mathrm{B\,2} \end{bmatrix},$$

$$\mathrm{MC} = [\ g\ ], \quad \mathrm{MD} = [\quad], \tag{15}$$

and

$$M^{-T}Q_N^i M^{-1} = \begin{bmatrix} Q_{11N}^i & Q_{12N}^i \\ Q_{12N}^{iT} & Q_{22N}^i \end{bmatrix},$$

$$H^T Q^i H = \begin{bmatrix} Q_{11}^i & Q_{12}^i \\ Q_{12}^{iT} & Q_{22}^i \end{bmatrix}, \quad i = 1, 2, 3. \tag{16}$$

Furthermore, let us define

Sn = B1R1}Bf+ C1R1}Cf + D1R;}Df, (17a)

S$_1$2 = B1R'i}B[+ C1R1}C[+ D1R'1}D!, (17b)

$$S_{22} = B_2R_{11}^{-1}B_2^T + C_2R_{12}^{-1}C_2^T + D_2R_{13}^{-1}D_2^T, \tag{17c}$$

$$T_1 = \begin{bmatrix} A_{11} & -S_{11} \\ Q^1_{11} & A_{11}^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_{12} & -S_{12} \\ Q^1_{12} & A_{21}^T \end{bmatrix},$$
$$T_3 = \begin{bmatrix} A_{21} & -S_{12}^T \\ -Q^{1T}_{12} & -A_{12}^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} A_{22} & -S_{22} \\ -Q^1_{22} & -A_{22}^T \end{bmatrix}, \tag{18}$$

and calculate

$$\begin{bmatrix} A_o & -S_o \\ Q_o & A_o^T \end{bmatrix} = T_1 - T_2T_4^{-1}T_3, \tag{19}$$

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = -T_4^{-1}T_3. \tag{20}$$

In above, T_4 is invertible if and only if the matrices $[A_{22}, B_2, C_2, D_2]$ and $[A_{22}^T, Q^1_{22}]$ are of full row rank respectively, where $Q^1_{22} = C_2^T C_2$. Suppose that $y_k = C_2x_k$ is the output of the system (1). Then, the existence of T_4^{-1} also means causal controllability and observability of the descriptor system (1), which guarantees the existence of the unique solution to the team-optimal control problem (12).

It is worth to note that the matrix (19) is a Hamiltonian matrix. A discrete matrix Riccati equation

$$P_k^1 = Q_o + A_o^T P_{k+1}^1[I + S_oP_{k+1}^1]^{-1}A_o, \qquad P_N^1 = Q_{11N}^1, \tag{21}$$

can be obtained from the Hamiltonian matrix (19).

LEMMA 1 *Suppose that T_4 is invertible and the discrete matrix Riccati equation (21) admits a unique nonnegative definite solution P_k^1. Then,*

(i) *the team-optimal control problem (12) admits uncountably many team optimal strategies in linear feedback form, with the family of the strategies given by*

$$\gamma_k^{1t} = -R_{11}^{-1}B^TK_k^1x_k, \tag{22a}$$
$$\gamma_k^{2t} = -R_{12}^{-1}C^TK_k^2x_k, \tag{22b}$$
$$\gamma_k^{3t} = -R_{13}^{-1}D^TK_k^3x_k, \tag{22c}$$

for $0 \le k \le N-1$, *where,*

$$K_k^1 = M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - F_k^1Z_k^2 & F_k^1 \end{bmatrix} H^{-1}, \tag{23a}$$

$$K_k^2 = M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - F_k^2Z_k^2 & F_k^2 \end{bmatrix} H^{-1}, \tag{23b}$$

$$K_k^3 = M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - F_k^3Z_k^2 & F_k^3 \end{bmatrix} H^{-1}, \tag{23c}$$

and

$$Z_k^2 = M_{11} + M_{12}L_k^1, \tag{24a}$$

$$L\%_{\sigma}\ M2_1 + M_{22}{}^{L}t, \qquad (24b)$$

$$Lt = {}_{P}f_{+}\ 1[I + Sa^{P}f_{+}\ l]^{-\ l}A_{o}. \qquad (24c)$$

$F\{$, FJ and Ff are arbitrary three $(n - r) \times (n - r)$ matrices making A_{22} - $B2K;\}B'JFl;$, - $C_2R1\}C!FJ$ - $D_2H1lD'JFf$ invertible.

(ii) *The open-loop realizations of 'Yft, 'Ykt and 'Yit are unique respectively, given by*

$$u_k^t = -R_{11}^{-1}[B_1^TL_k^1 + B_2^TL_k^2]z_k^{1t}, \qquad (25a)$$

$$v_k^t = -R_{12}^{-1}[c_1^rL_k^1 + c_2^rL_k^2]z_k^{1t}, \qquad (25b)$$

$$w_k^t = -R_{13}^{-1}[D_1^TL_k^1 + D_2^TL_k^2]z_k^{1t}, \qquad (25c)$$

where, $z!t$ is the unique solution of

$$z_{k+l}^1 = z\,l_{kzk}^{\ 1}, \qquad z_{k=O}^1 = z_O^1, \qquad (26)$$

$$zi = [I + Sa^{P}f_{+}\ 1]^{-1}A_{o}, \qquad (27)$$

and Zkt satisfies the algebraic equation

$$z_k^{2t} = z_k^2 z_k^{1t}. \qquad (28)$$

Proof. For the proof of Lemma 1, the reader is referred to the proof of Theorem 3 in Appendix A, where, instead of the three-player team optimal control problem, we have provided a standard technique to solve the general single-player optimal regulator problem. Lemma 1 is obtained in a similar way. ■

3. Derivations of memoryless equilibrium strategies for Pi and A

In contrast to the unique team solution in linear feedback form for state space systems, we have obtained the sets of the linear feedback team strategies for each player from (22a,b,c) respectively. We denote them by $r^{it} \subset r$, $i = 1, 2, 3$, respectively, where, the superscript 't' represents the terms related to the team optimal control problem (12).

The following result is obvious and little different from Proposition 2 in Ba§ar (1981).

PROPOSITION 1 *Jf y^1 is restricted to* Γ'^{1t}, *then* $S_1(Y^1) = \Gamma'^{2t}(Y^1) \times \Gamma'^{3t}(Y^1) \subset r^{2t} \times r^{3t}$.

The set $S_1(y^1)$ defined in (8) depends on the strategy y^1 since each strategy set rtt, $0 \leq k \leq N - 1$, i=1,2,3, is parameterized by the matrix Ft respectively. Whether the matrix A_{22} - $B2K;\}B'JFl;,$-$C2R_1/\ C'JFJ$ - D_2H11$D'JFf$ in Lemma 1 is invertible or not depends on the combined roles of $Fl;,$, FJ and Ff.

In order to apply Theorem 1 to find the hierarchical equilibrium strategy, we have to find a specific strategy $Y^{1*} \in r^{1t}$ such that $S_2(Y^{1*}) \subset \Gamma'^{2t} \times r^{3t}$. The sufficient conditions under which such a strategy exists will be given later in Lemma 2 through the following derivations.

Substituting (22a) into (1) and (2b) yields a team-optimal control problem to be faced by P_2 and P_3, that is,

$$\min_{\gamma^2\in\Gamma^2}\min_{\gamma^3\in\Gamma^3}\Big\{\frac{1}{2}x_N^TE^TQ_N^2Ex_N+\frac{1}{2}\sum_{k=0}^{N-1}(x_k^T[Q^2+K_k^TBR_{11}^{-1}R_{21}R_{11}^{-1}B^TK_k]x_k+v_k^TR_{22}v_k+w_k^TR_{23}w_k)\Big\}, \tag{29}$$

subject to

$$Ex_{k+1}=[A-BR_{11}^{-1}B^TK_k]x_k+Cv_k+Dw_k,\quad x_{k=0}=x_0, \tag{30}$$

(29),(30) can be further transformed to

$$\min_{\gamma^2\in\Gamma_a^2}\min_{\gamma^3\in\Gamma_a^3}\Big\{\frac{1}{2}z_N^{1T}Q_{11N}^2z_N^1+\frac{1}{2}\sum_{k=0}^{N-1}\Big([z_k^{1T}\ z_k^{2T}]\begin{bmatrix}\hat{Q}_{11k}^2 & \hat{Q}_{12k}^2\\ \hat{Q}_{12k}^{2T} & \hat{Q}_{22k}^2\end{bmatrix}\begin{bmatrix}z_k^1\\ z_k^2\end{bmatrix}$$

$$+v_k^TR_{22}v_k+w_k^TR_{23}w_k\Big)\Big\}, \tag{31}$$

subject to

$$z_{k+1}^1=\hat{A}_{11k}z_k^1+\hat{A}_{12k}z_k^2+C_1v_k+D_1w_k,\qquad z_{k=0}^1=z_0^1, \tag{32a}$$

$$0=\hat{A}_{21k}z_k^1+\hat{A}_{22k}z_k^2+C_2v_k+D_2w_k, \tag{32b}$$

by using (13), where, the corresponding terms are defined in Appendix B.

Similar to the derivations in Appendix A, define

$$S_{11}=C_1R_{22}^{-1}C_1^T+D_1R_{23}^{-1}D_1^T, \tag{33a}$$

$$S_{12}=C_1R_{22}^{-1}C_2^T+D_1R_{23}^{-1}D_2^T, \tag{33b}$$

$$S_{22}=C_2R_{22}^{-1}C_2^T+D_2R_{23}^{-1}D_2^T, \tag{33c}$$

and

$$T_{1k}=\begin{bmatrix}\hat{A}_{11k} & -S_{11}\\ -\hat{Q}_{11k}^2 & -\hat{A}_{11k}^T\end{bmatrix},\quad T_{2k}=\begin{bmatrix}\hat{A}_{12k} & -S_{12}\\ -\hat{Q}_{12k}^2 & -\hat{A}_{21k}^T\end{bmatrix},$$
$$T_{3k}=\begin{bmatrix}\hat{A}_{21k} & -S_{12}^T\\ -\hat{Q}_{12k}^{2T} & -\hat{A}_{12k}^T\end{bmatrix},\quad T_{4k}=\begin{bmatrix}\hat{A}_{22k} & -S_{22}\\ -\hat{Q}_{22k}^2 & -\hat{A}_{22k}^T\end{bmatrix}. \tag{34}$$

And then, calculate

$$\begin{bmatrix}M_{11k} & M_{12k}\\ M_{21k} & M_{22k}\end{bmatrix}=-T_{4k}^{-1}T_{3k}, \tag{35}$$

$$\begin{bmatrix}A_{0k} & -S_{0k}\\ -Q_{0k} & -A_{0k}^T\end{bmatrix}=T_{1k}-T_{2k}T_{4k}^{-1}T_{3k}. \tag{36}$$

REMARK 1 *From Appendix B, we know that the corresponding terms in (34) contain the unknown parameter matrix F^1_k. Therefore, calculating (35) and (36) is not a simple numerical computation. We need the help of some computer algebra system (for example, REDUCE, Copyright of The RAND Corporation 1985, 1993) to "calculate" (35) and (36).*

From the Hamiltonian matrix (36), we can arrive at a discrete matrix Riccati equation

$$P_k^2 = \hat{Q}_{0k} + \hat{A}_{0k}^T P_{k+1}^2 [I + \hat{S}_{0k} P_{k+1}^2]^{-1} \hat{A}_{0k}, \qquad P_N^2 = Q_{11N}^2. \tag{37}$$

Moreover, we have

$$\hat{Z}_k^2 = \hat{M}_{11k} + \hat{M}_{12k}\hat{L}_k^1, \qquad \hat{L}_k^2 = \hat{M}_{21k} + \hat{M}_{22k}\hat{L}_k^1, \tag{38}$$

$$\hat{Z}_k^1 = [I + \hat{S}_{0k} P_{k+1}^2]^{-1} \hat{A}_{0k}, \qquad \hat{L}_k^1 = P_{k+1}^2 [I + \hat{S}_{0k} P_{k+1}^2]^{-1} \hat{A}_{0k}, \tag{39}$$

PROPOSITION 2 *A strategy $\gamma^{1t} \in \Gamma^1$ is admissible if the matrix T_{4k} is invertible at each stage k, $0 \le k \le N-1$.*

Proof. It can be deduced from Appendix A that the regularity of the matrix T_{4k} at each stage k, $0 \le k \le N-1$, means that the team optimal control problem (29),(30) admits a unique open-loop solution. Therefore, $\gamma^{1t} \in \Gamma^1$ is admissible according to Definition 2. ■

Solving the team optimal control problem (29), (30), we have

PROPOSITION 3 *Suppose that T_{4k} is invertible at each stage k, $0 \le k \le N-1$, and that the discrete matrix Riccciti equation (37) admits a unique nonnegative definite solution P_k^2. Then,*

(i) *the team optimal control problem formulated by (29), (30) admits uncountably many team optimal strategies in linear feedback form, with the family of the strategies given by*

$$\gamma_k^{2t} = -R_{22}^{-1} C^T K_k^2 x_k, \tag{40a}$$

$$\gamma_k^{3t} = -R_{33}^{-1} D^T K_k^3 x_k, \tag{40b}$$

for $0 \le k \le N-1$, where,

$$K_k^2 = M^T \begin{bmatrix} \hat{L}_k^2 - F_k^2 \hat{Z}_k^2 & F_k^2 \end{bmatrix} H^{-1} \tag{41a}$$

$$K_k^3 = M^T \begin{bmatrix} \hat{L}_k^2 - F_k^3 \hat{Z}_k^2 & F_k^3 \end{bmatrix} H^{-1}, \tag{41b}$$

F_k^2 and F_k^3 are arbitrary two $(n-r) \times (n-r)$ matrices making $A_{22} - B_2 R_{11}^{-1} B_2' [F_k^1] - C_2 R_{22}^{-1} C_2' [F_k^2] - D_2 R_{33}^{-1} D_2' [F_k^3]$ invertible.

(ii) *The open-loop realizations of γ^{2t} and γ^{3t} are unique, given by*

$$v_k^t = -R_{22}^{-1} [C_1' \hat{L}_k^1 + C_2' \hat{L}_k^2] x_k^t, \tag{42a}$$

$$w_k^t = -R_{33}^{-1} [D_1' \hat{L}_k^1 + D_2' \hat{L}_k^2] x_k^t, \tag{42b}$$

respectively, where, z_k^{1t} *is the unique solution of*

$$z_{k+1}^{1} = z_k^{1} z_k^{1}, \qquad z_{k=0}^{1} = z_0, \tag{43}$$

and z_k^{2t} *satisfies the algebraic equation*

$$z_k^{2t} = Z_k^{2} z_k^{1t}. \tag{44}$$

For an arbitrary admissible strategy $\gamma^1 \in \Gamma^{1t}$, we have obtained the responding strategy set $S_2(\gamma^1)$, which is described by (40). The next problem is to find the conditions for a specific strategy γ^{1*} to exist such that $S_2(\gamma^{1*}) \subset \Gamma^{2t} \times \Gamma^{3t}$.

CONDITION 1 *There exists at least one matrix sequence* $\{F_{N-1}^{1*}, F_{N-2}^{1*}, \cdots, F_0^{1*}\}$*, such that,*

$$z_k^{1} = \hat{z}_k^{1*}, \tag{45a}$$

$$R_{22}^{-1}(C_1^T L_k^1 + C_2^T L_k^2) = R_{22}^{-1}(C_1^T \hat{L}_k^{1*} + C_2^T \hat{L}_k^{2*}), \tag{45b}$$

$$R_{23}^{-1}(D_1^T L_k^1 + D_2^T L_k^2) = R_{23}^{-1}(D_1^T \hat{L}_k^{1*} + D_2^T \hat{L}_k^{2*}), \tag{45c}$$

where, the matrices with the superscript '' represent the corresponding matrices obtained when* F_k^1, $0 \le k \le N-1$, *in (23a) is substituted by* F_k^{1*}.

LEMMA 2 *Let Condition 1 be satisfied and the obtained strategy* γ^{1*} *be an admissible strategy. Then, we have*

$$S_2(\gamma^{1*}) = \Gamma^{2t}(\gamma^{1*}) \times \Gamma^{3t}(\gamma^{1*}) \subset \Gamma^{2t} \times \Gamma^{3t} \tag{46}$$

by choosing

$$\gamma_k^{1*} = -R_{11}^{-1} B^T K_k^{1*} x_k, \quad 0 \le k \le N-1, \tag{47}$$

with

$$K_k^{1*} = M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - F_k^{1*} Z_k^2 & F_k^{1*} \end{bmatrix} H^{-1}. \tag{48}$$

Moreover, $S_2(\gamma^{1*}) = S_1(\gamma^{1*})$.

Proof. From Proposition 3, the set $S_2(\gamma^{1*})$ can be described by the strategies

$$\gamma_k^{2t*} = -R_{22}^{-1} C^T K_k^{2*} x_k, \tag{49a}$$

$$\gamma_k^{3t*} = -R_{23}^{-1} D^T K_k^{3*} x_k, \tag{49b}$$

where $0 \le k \le N-1$. Since γ^{1*} is admissible, the set $S_2(\gamma^{1*})$ is the realization singleton with the open-loop realization pair described uniquely by

$$\hat{v}_k^{t*} = -R_{22}^{-1}[C_1^T \hat{L}_k^{1*} + C_2^T \hat{L}_k^{2*}] \hat{z}_k^{1t*}, \tag{50a}$$

$$\hat{w}_k^{t*} = -R_{23}^{-1}[D_1^T \hat{L}_k^{1*} + D_2^T \hat{L}_k^{2*}] \hat{z}_k^{1t*}, \tag{50b}$$

where z!h is the solution of

$$z_{k+1}^1 = \hat{Z}_k^{1*} z_k^1, \qquad zk=O : Zo \tag{51}$$

Similar to the statement in Condition 1, the terms with the superscript '*' denote the corresponding ones obtained in Proposition 3 when 1'1* is substituted into (1) and (2b). Comparing (51) with (26) and (50a,b) with (25b,c) yields the relations z!t* = z!t, i{* = vi and u,t* = wL hence z t* = z t, because of Condition 1, which implies that the strategy triplet (1'1*,i t*,i t*) is equivalent to the strategy triplet (1'1*,1' \ ,)' t). Therefore, we have S2(1'1*) : r 2t (,'1*) x r 3t (,'1*) C r 2t x r 3t according to Proposition 1. S2(1'1*) : S1(1'1*) because S1(1'1*) = r 2t b 1*) x r 3t b 1*). ■

Similar to Ba§ar (1981), the next step in the derivation now is to determine a ')'2* E r 2t (1'1*) such that R3(,,1*,,)'2*) C r 3t (1'1*), and {1'2*} x R3(1'1*,,)'2*) C S_2(1'1*). The basic derivation is similar to the above.

Substituting (47) and (22b) into (1) and (2c) and making a transformation using (13) yields a optimal control problem to be faced by P_3, that is,

min , 3 Er { 1/2 ZN 1T Q 3 1lNzN 1 +

½}; [zF z T 1 [2r: ii::] [:i] + wf R33Wk)} , (52)

subject to

$$Zk+1 : A11kZk^1 + A12kZk^2 + D1 Wk, \qquad z_{k=0}^1 = z_0^1, \tag{53a}$$

$$0 : A21kzk^1 + A22kzk^2 + D2wk, \tag{53b}$$

where, the corresponding terms are defined in Appendix C.

Similar to the derivations in Appendix A, define

$$\tilde{S}_{11} = D_1 R_{33}^{-1} D_1^T, \tag{54a}$$

$$\tilde{S}_{12} = D_1 R_{33}^{-1} D_2^T, \tag{54b}$$

$$\tilde{S}_{22} = D_2 R_{33}^{-1} D_2^T, \tag{54c}$$

and

$$\tilde{T}_{1k} : [tr\ Ai::], \quad \tilde{T}_{2k} = \begin{bmatrix} A12k & -\tilde{S}_{12} \\ Q_{12k}^3 & \tilde{A}_{21k}^T \end{bmatrix},$$

$$T_{3k} : \begin{bmatrix} A21k & -S'f.i \\ -Q_{12k}^{3T} & -A_{12k}^T \end{bmatrix}, \quad \tilde{T}_{4k} = \begin{bmatrix} A22k & -\tilde{S}_{22} \\ -Q22k & -\tilde{A}_{22k}^T \end{bmatrix}. \tag{55}$$

Then, calculate (see also Remark 1)

$$\begin{bmatrix} \tilde{M}_{11k} & M_{12k} \\ \tilde{M}_{21k} & M22k \end{bmatrix} = - r\sim_{4k}^{-1} T_{3k}, \tag{56}$$

$$\begin{bmatrix} \tilde{A}_{0k} & -\tilde{S}_{0k} \\ \tilde{Q}_{0k} & \tilde{A}_{0k}^T \end{bmatrix} = \tilde{T}_{1k} - \tilde{T}_{2k}\tilde{T}_{4k}^{-1}\tilde{T}_{3k}. \tag{57}$$

From the Hamiltonian matrix (57), we can arrive at a discrete matrix Riccati equation

$$P_k^3 = \tilde{Q}_{0k} + \tilde{A}_{0k}^T P_{k+1}^3 [I + \tilde{S}_{0k} P_{k+1}^3]^{-1} \tilde{A}_{0k}, \qquad P_N^3 = Q_{11N}^3. \tag{58}$$

Moreover, we have

$$\tilde{Z}_k^2 = \tilde{M}_{11k} + \tilde{M}_{12k}\tilde{L}_k^1, \qquad \tilde{L}_k^2 = \tilde{M}_{21k} + \tilde{M}_{22k}\tilde{L}_k^1, \tag{59}$$

$$\tilde{Z}_k^1 = [I + \tilde{S}_{0k} P_{k+1}^3]^{-1} \tilde{A}_{0k}, \qquad \tilde{L}_k^1 = P_{k+1}^3 [I + \tilde{S}_{0k} P_{k+1}^3]^{-1} \tilde{A}_{0k}, \tag{60}$$

PROPOSITION 4 *Corresponding to the strategy γ^{1*}, a strategy $\gamma^2 \in \Gamma^{2t}(\gamma^{1*})$ is admissible if the matrix T_{4k} is invertible at each stage k, $0 \leq k \leq N-1$.*

Proof. The result follows from the same reasoning as in the proof of Proposition 2. ■

PROPOSITION 5 *Suppose that T_{4k} is invertible at each stage k, $0 \leq k \leq N-1$, and that the discrete matrix Riccati equation (58) admits a unique nonnegative definite solution P_k^3. Then,*

(i) *the optimal control problem formulated by (52), (53) admits uncountably many optimal strategies in linear feedback form, with the family of the strategies given by*

$$\tilde{\gamma}_k^{3t} = -R_{33}^{-1} D^T K_k^3 x_k, \qquad 0 \leq k \leq N-1, \tag{61}$$

where,

$$K_k^3 = M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - F_k^2 Z_k^2 & F_k^2 \end{bmatrix} H^{-1}, \tag{62}$$

F_k^2 are any $(n-r) \times (n-r)$ matrices making $A_{22} - B_2 R_{22}^{-1} B_2^T F_k^2 - C_2 R_{11}^{-1} C_2^T F_k^2 - D_2 R_{33}^{-1} D_2^T F_k^2$ invertible.

(ii) *The open-loop realization of $\tilde{\gamma}^{3t}$ is given by*

$$w_k^t = -R_{33}^{-1} [D_1^T L_k^1 + D_2^T L_k^2] z_k^{1t}, \tag{63}$$

where, z_k^{1t} is the unique solution of

$$z_{k+1}^1 = \tilde{Z}_k^1 z_k^1, \qquad z_{k=0}^1 = z_0^1, \tag{64}$$

and z^{2t} satisfies the algebraic equation

$$z_k^{2t} = \tilde{Z}_k^2 z_k^{1t}. \tag{65}$$

From Proposition 5, we can obtain the responding strategy set $R_3(\gamma^{1*}, \gamma^2)$, which is described by (61), for a fixed strategy γ^{1*} and an arbitrary strategy $\gamma^2 \in \Gamma^{2t}(\gamma^{1*})$. The problem now is to find a specific strategy $\gamma^{2*} \in \Gamma^{2t}(\gamma^{1*})$ such that $R_3(\gamma^{1*}, \gamma^{2*}) \subset \Gamma^{3t}(\gamma^{1*}) \subset \Gamma^{3t}$.

CONDITION 2 *There exists at least one matrix sequence* $\{F_{N-1}^{*}, F_{N-2}^{*}, \cdots, F_0^{*}\}$, *such that,*

$$Z_k^1 = \tilde{Z}_k^{1*}, \tag{66a}$$

$$R_{13}^{-1}(C_1^T L_k^1 + C_2^T L_k^2) = R_{33}^{-1}(D_1^T \tilde{L}_k^{1*} + D_2^T \tilde{L}_k^{2*}), \tag{66b}$$

where, the matrices with the superscript '' represent the corresponding matrices obtained when* F_k^2, $0 \le k \le N-1$, *in (23b) is substituted by* F_k^{2*}.

LEMMA 3 *Let Condition 2 be satisfied and the obtained strategy* γ^{2*} *be an admissible strategy. Then, we have*

$$R_3(\gamma^{1*}, \gamma^{2*}) \subset \Gamma^{3t}(\gamma^{1*}) \subset \Gamma^{3t} \tag{67}$$

by choosing (47) and

$$\gamma_k^{2*} = -R_{22}^{-1} B^T K_k^{2*} x_k, \tag{68}$$

with

$$K_k^{2*} = M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - F_k^{2*} Z_k^2 & F_k^{2*} \end{bmatrix} H^{-1}. \tag{69}$$

Moreover, $\{\gamma^{2*}\} \times R_3(\gamma^{1*}, \gamma^{2*}) \subset S_2(\gamma^{1*})$.

Proof. When Condition 2 is satisfied, the fact that the strategy triplet $(\gamma^{1*}, \gamma^{2*}, \gamma^3) \in \{\gamma^{1*}\} \times \{\gamma^{2*}\} \times R_3(\gamma^{1*}, \gamma^{2*})$ minimizes J_1 and J_3 simultaneously is easy to prove in a similar way to the proof of Lemma 2. Hence, we have $R_3(\gamma^{1*}, \gamma^{2*}) \subset \Gamma^{3t}(\gamma^{1*})$. Furthermore, since $S_1(\gamma^{1*}) = S_2(\gamma^{1*})$ (Lemma 2), we arrive at $\{\gamma^{2*}\} \times R_3(\gamma^{1*}, \gamma^{2*}) \subset S_1(\gamma^{1*}) = S_2(\gamma^{1*})$. ■

Summarizing the results of Lemma 2 and Lemma 3, we finally have

THEOREM 2 *For the three-player Stackelberg games of linear quadratic discrete-time descriptor systems formulated in Section 2, suppose that Conditions 1 and 2 are satisfied. Then,*

(i) γ^{1*} *as defined by (47) provides a memoryless hierarchical equilibrium strategy for* P_1, γ^{2*}, *as defined by (68), is a corresponding memoryless equilibrium strategy for* P_2, *and any* $\gamma^3 \in R_3(\gamma^{1*}, \gamma^{2*})$ *is a memoryless equilibrium strategy for* P_3.

(ii) *The obtained game values for* P_1, P_2 *and* P_3 *are given, respectively, by*

$$J_1^* = \frac{1}{2} z_0^{1T} P_0^1 z_0^1, \tag{70a}$$

$$J_2^* = \frac{1}{2} z_0^{1T} P_0^{2*} z_0^1, \tag{70b}$$

$$J_3^* = \frac{1}{2} z_0^{1T} P_0^{3*} z_0^1 \tag{70c}$$

Proof. Condition (i) of Theorem 1 is fulfilled since {1'2*} x R3(,y1*,,-y2*) C S2(,y1*). Condition (ii) is also fulfilled since S'2(1*) C S2('-„1*) = S1(,y1*). Finally, Condition (iii) is fulfilled by noting Proposition 1. Therefore, (i) of this theorem follows directly from Theorem 1. (ii) follows directly from Lemma 1 and Propositions 3 and 5. ■

4. An illustrative example

Consider a linear discrete-time descriptor system

$$
\left[\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] 11 \left[\begin{matrix} Z_{k+1} \\ Z_{+1} \\ Z_{k,1} \end{matrix}\right] 1 =
$$

$$
\left[\begin{matrix} \bullet & 0 \\ & 1 \\ \bullet & 1 \end{matrix}\right] 1 [;1 + [\,n\,u, + [n + [: 1 w,, \qquad (71)
$$

(0 k N - 1).

The cost functions are given as

$$
J1 = \frac{5}{2}(z1)^2 + \frac{1}{2}\sum_{k=0}^{N-1}\{x_f \left[\begin{matrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{matrix}\right] x,+2u/+v/+wl), \qquad (72a)
$$

$$
J2 = \frac{3}{2}(z1)^2 + \frac{1}{2}\sum_{k=0}^{N-1}\{x_f \left[\begin{matrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{matrix}\right] x, +u(+2vl+w1), \qquad (72b)
$$

$$
J3 = \frac{1}{2}(z1)^2 + \frac{1}{2}\sum_{k=0}^{N-1}\{x_f \left[\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}\right] x, +ul+v/+2wi\}, \qquad (72c)
$$

where x f = [zfT, ZkT, zfT]. In the following, we make use of a computer algebra system called REDUCE as stated in Remark 1 to solve the example. The relevant terms in Lemma 1 are determined from

$$
\begin{aligned}
Z_k^1 &= 12/(11Pf+i+6), \\
L_k^1 &= 12Pf+i/(11Pf+i+6), \\
Z_k^2 &= \left[\begin{matrix} Zk1 \\ Zk2 \end{matrix}\right] - \left[\begin{matrix} (-5Pf+i+6)/(11P\{+1+6) \\ (-12Pf+i+6)/(11Pf+i+6) \end{matrix}\right], \\
L_k^2 &= \left[\begin{matrix} L\%1 \\ Lk_2 \end{matrix}\right] - \left[\begin{matrix} 10Pk+i/(11P\{t_1+6) \\ -16P_{k+}i/(11P_{k+1}+6) \end{matrix}\right], \\
P_k^1 &= 24Pf+_1/(11Pf+_1+6), \qquad Pf, = 5,
\end{aligned}
$$

for O k N - 1. Their numerical results are given in Table 1.

k	P_k^{*}	L_{k1}^{*}	L_{k2}^{*}	L_k^{*}	Z_{k1}^{*}	Z_{k2}^{*}
N	5	-	-	-	-	-
N-1	1.96721	0.81967	-1.31148	0.98361	-0.98361	-0.50820
N-2	1.70819	0.71174	-1.13876	0.85409	-0.85409	-0.57295
N-3	1.65375	0.68906	-1.10250	0.82687	-0.82687	-0.58656
N-4	1.64068	0.68361	-1.09378	0.82034	-0.82034	-0.58983
N-5	1.63744	0.68227	-1.09163	0.81872	-0.81872	-0.59064
N-6	1.63663	0.68193	-1.09169	0.81832	-0.81832	-0.59084
N-7	1.63643	0.68165	-1.09095	0.81822	-0.81822	-0.59089
N-8	1.63638	0.68183	-1.09092	0.81819	-0.81819	-0.59090
N-9	1.63637	0.68182	-1.09091	0.81818	-0.81818	-0.59091
N-10	1.63636	0.68182	-1.09091	0.81818	-0.81818	-0.59091
N-11	1.63636	0.68182	-1.09091	0.81818	-0.81818	-0.59091
N-12	1.63636	0.68182	-1.09091	0.81818	-0.81818	-0.59091

Table 1. The numerical values of the relevant terms in Lemma 1

A matrix sequence $\{F_k^{1*}, 0 \leq k \leq N-1\}$ which satisfies Condition 1 is

$$P_k^{1*} = \begin{bmatrix} 1 & 2 \\ F_{21k}^{1*} & 2 \end{bmatrix}, \tag{73}$$

where,

$$F_{21k}^{1*} = 2(P_{k+1}^{1*} + 3P_{k+1}^{2*} + 3)/(3(P_{k+1}^{1*} - P_{k+1}^{2*})),$$

$$P_k^{1*} = (3(133(P_{k+1}^{1*})^2 + 84P_{k+1}^{2*} + 48P_{k+2}^{2*} + 36))/(11P_{k+1}^{2*} + 6)^2, \quad P_N^{1*} = 3.$$

The numerical values of P_k^{1*} and F_{21k}^{1*} are given in Table 2. Hence we have

$$\Gamma_k^{*} = -\frac{1}{2}[2 \ \ 0 \ \ 1] \begin{bmatrix} L_k^1 & 0 & 0 & z_k^1 \\ L_{k1}^1 - z_{k1}^1 - 2z_{k2}^1 & 1 & 2 & z_k^2 \\ L_{k2}^2 - F_{21k}^{1*} z_{k1}^2 - 2z_{k2}^2 & F_{21k}^{1*} & 2 & z_k^3 \end{bmatrix}, \tag{74}$$

a linear feedback strategy. Furthermore, a matrix sequence $\{F_k^{2*}, 0 \leq k \leq N-1\}$ which satisfies Condition 2 is

$$P_k^{2*} = \begin{bmatrix} 2 & 1 \\ F_{21k}^{2*} & 1 \end{bmatrix}, \tag{75}$$

where,

$$\begin{aligned} F_{21k}^{2*} &= -(37(P_{k+1}^{1*})^2 + 27P_{k+1}^{2*}P_{k+1}^{1*} - 48P_{k+1}^{2*}P_{k+1}^{2*} + 30P_{k+1}^{1*} + 18P_{k+1}^{2*} \\ &\quad -36P_{k+1}^{2*})/(24(P_{k+1}^{1*} - P_{k+1}^{2*})P_{k+1}^{1*}), \\ P_k^{2*} &= (2(269(P_{k+1}^{2*})^2 + 96P_{k+1}^{2*} + 72P_{k+1}^{2*} + 36))/(11P_{k+1}^{2*} + 6)^2, \\ P_N^{2*} &= 1. \end{aligned}$$

k	p_k^{2*}	p_k^{3*}	Fi{k	Fi{k	ITl:1	1tl:1
N	3	1	-	-	-	-
N-1	3.16447	3.93066	5.66667	-5.39167	18.04167	77.00569
N-2	3.40805	4.05499	-8.05208	0.99222	30.31347	42.98485
N-3	3.56924	4.15546	-5.85628	0.64939	18.86102	27.32615
N-4	3.63957	4.20234	-5.34639	0.58612	16.71896	24.10948
N-5	3.66533	4.22001	-5.18933	0.56680	16.09842	23.13807
N-6	3.67388	4.22597	-5.13947	0.56053	15.90531	22.82934
N-7	3.67656	4.22785	-5.12399	0.55855	15.84573	22.73292
N-8	3.67736	4.22842	-5.11933	0.55794	15.82781	22.70369
N-9	3.67759	4.22858	-5.11796	0.55776	15.82255	22.69507
N-10	3.67766	4.22863	-5.11756	0.55771	15.82104	22.69258
N-11	3.67768	4.22864	-5.11745	0.55770	15.82062	22.69187
N-12	3.67768	4.22865	-5.11742	0.55769	15.82050	22.69168

Table 2. The numerical values of the terms related to the equilibrium strategies

The numerical values of Pf+1 and Ff{k are also given in Table 2. Therefore, we have

$$\gamma_k^{2*} = -[1 \;\; 0 \;\; 1] \begin{bmatrix} L_k^1 & 0 & 0 \\ L_{k1}^2 - 2Z_{k1}^2 - Z_{k2}^2 & 2 & 1 \\ L_{k1}^2 - F_{21k}^{2*} Z_{k1}^2 - Z_{k2}^2 & F_{21k}^{2*} & 1 \end{bmatrix} \begin{bmatrix} z_k^1 \\ z_k^2 \\ z_k^3 \end{bmatrix}. \tag{76}$$

It is worthy to noting that the numerical values of the relevant terms above can be computed recursively (Xu and Mizukami, 1995).

Moreover, li'J:I $\neq 0$ and IT]kl $\neq 0$ in Table 2 verify that the obtained hierarchical equilibrium strategies are admissible (Propositions 2 and 4).

REMARK 2 *In contrast to the memory equilibrium strategies in state space systems Başar (1981), the resulting equilibrium strategies for descriptor systems in this paper are linear strategies involving only memoryless information on x_k. Therefore, the equilibrium strategies obtained in this paper have the advantages of the less information and the simpler structure. Moreover, Condition 1 and Condition 2, and thereby the equilibrium strategies (47), (68) and $\gamma^3 \in R_3(\gamma^{1*}, \gamma^{2*})$ do not depend on the initial state x_0. For this reason, the obtained equilibrium strategies are less sensitive to unintentional nonoptimal actions of the players at the lower levels of the hierarchy. For example, suppose that P_3 takes a nonoptimal strategy outside the set $R_3(\gamma^{1*}, \gamma^{2*})$ at stage s. Then, the obtained state X_{s+1} will deviates from the optimal state x^*_{s+1}, which may affect the cost values of all the players for the total game. However, considering a subgame starting at stage $s + 1$, the original equilibrium strategies* *obtained for the total game still constitute the equilibrium strategies for the remaining subgame on the interval* $[s+1, s+2, \cdots, N-1]$. *This property, as stated in Introduction, makes the equilibrium strategies have the higher credibility (Tolwinski, 1981).*

5. Conclusions

In this paper, we have considered the three-player Stackelberg games for linear quadratic discrete-time descriptor systems with three levels of hierarchy in decision making. We have derived explicitly sufficient conditions for the existence of the memoryless hierarchical equilibrium strategies for the player (called A) at the top of the hierarchy, and for the player (called P_2) at the second level of the hierarchy. Since the resulting hierarchical equilibrium strategies do not depend on the memory information, A 's original optimal team cost remains the tight (attainable) lower bound for A 's cost function no matter whether the players at the lower levels of hierarchy act or not at the last two (or one) stages. The resulting strategies have the advantages of the simpler structure and the higher credibility. Moreover, the method developed in this paper is easy to be extended to general $M (M > 3)$ player dynamic games with M-level of hierarchy in decision making. The equilibrium strategies can be determined sequentially by calculating parameter matrix sequence {Fj.;$_{-1}$, Fj.;$_{-2}$, $\cdots$, Fd*}, $i = 1, 2, \cdots, M$, from the top of the hierarchy. At each level of the hierarchy, the similar procedure is adopted to determine the equilibrium strategy.

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Appendix A

We provide below a useful technique for finding the solution of the linear quadratic optimal regulator problem for discrete-time descriptor systems, that is,

$$\min_{u_k}\{\frac{1}{2}x_N^T E^T Q_N E x_N + \frac{1}{2}\sum_{k=0}^{N-1}(x_k^T Q x_k + u_k^T R u_k)\}, \tag{77}$$

subject to

$$Ex_{k+1} = Ax_k + Bu_k, \quad x_{k=0} = x_0. \tag{78}$$

This technique is different from the well-known ones (Bender and Laub, 1987; Mantas and Krikelis, 1989), and has been applied to solve the main problem of this paper.

The necessary conditions for u_k, $0 \leq k \leq N-1$, to be the optimal solution are (Bender and Laub, 1987; Mantas and Krikelis, 1989)

$$Ex_{k+1} = Ax_k + Bu_k, \tag{79a}$$

$$E^T\lambda_k = A^T\lambda_{k+1} + Qx_k, \tag{79b}$$

$$0 = Ru_k + B^T\lambda_{k+1}, \tag{79c}$$

with boundary conditions

$$x_{k=0} = x_0, \quad E^T\lambda_N = E^T Q_N E x_N. \tag{80}$$

The above necessary conditions are also sufficient for the linear-quadratic problem, and can be transformed as

$$z^1_{k+1} = A_{11}z^1_k + A_{12}z^2_k + B_1u_k, \tag{81a}$$

$$0 = A_{21}z^1_k + A_{22}z^2_k + B_2u_k, \tag{81b}$$

$$\lambda^1_k = A_{11}^T\lambda^1_{k+1} + A_{21}^T\lambda^2_{k+1} + Q_{11}z^1_k + Q_{12}z^2_k \tag{81c}$$

$$0 = A_{12}^T\lambda^1_{k+1} + A_{22}^T\lambda^2_{k+1} + Q_{21}z^1_k + Q_{22}z^2_k \tag{81d}$$

$$0 = Ru_k + B_1^T\lambda^1_{k+1} + B_2^T\lambda^2_{k+1}, \tag{81e}$$

by using (13). From (81e), we obtain

$$u_k^* = -R^{-1}B_1^T\lambda^1_{k+1} - R^{-1}B_2^T\lambda^2_{k+1}. \tag{82}$$

Substituting (82) into (81a,b) yields

$$\begin{bmatrix} z^1_{k+1} \\ \lambda^1_k \end{bmatrix} = \begin{bmatrix} A_{11} & -S_{11} \\ Q_{11} & A_{11}^T \end{bmatrix}\begin{bmatrix} z^1_k \\ \lambda^1_{k+1} \end{bmatrix} + \begin{bmatrix} A_{12} & -S_{12} \\ Q_{12} & A_{21}^T \end{bmatrix}\begin{bmatrix} z^2_k \\ \lambda^2_{k+1} \end{bmatrix}, \tag{83a}$$

$$0 = \begin{bmatrix} A_{21} & -S_{12}^T \\ -Q_{12}^T & -A_{12}^T \end{bmatrix} \begin{bmatrix} z_k^1 \\ \lambda_{k+1}^1 \end{bmatrix} + \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix} \begin{bmatrix} z_k^2 \\ \lambda_{k+1}^2 \end{bmatrix}, \tag{83b}$$

where

$$S_{11} = B_1 R^{-1} B_1^T, \quad S_{12} = B_1 R^{-1} B_2^T, \quad S_{22} = B_2 R^{-1} B_2^T. \tag{84}$$

Define the matrices in (83) as

$$T_1 = \begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_{12} & -S_{12} \\ -Q_{12} & -A_{21}^T \end{bmatrix}, \tag{85}$$

$$T_3 = \begin{bmatrix} A_{21} & -S_{12}^T \\ -Q_{12}^T & -A_{12}^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix}.$$

In above, $|T_4| \neq 0$ if and only if the system (78) is causally controllable and observable (Bender and Laab, 1987; Mantas and Krikelis, 1989). Hence, we have

$$\begin{bmatrix} z_k^2 \\ \lambda_{k+1}^2 \end{bmatrix} = -T_4^{-1} T_3 \begin{bmatrix} z_k^1 \\ \lambda_{k+1}^1 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} z_k^1 \\ \lambda_{k+1}^1 \end{bmatrix}, \tag{86}$$

and

$$\begin{bmatrix} z_{k+1}^1 \\ \lambda_k^1 \end{bmatrix} = (T_1 - T_2 T_4^{-1} T_3) \begin{bmatrix} z_k^1 \\ \lambda_{k+1}^1 \end{bmatrix} = \begin{bmatrix} A_0 & -S_0 \\ Q_0 & A_0^T \end{bmatrix} \begin{bmatrix} z_k^1 \\ \lambda_{k+1}^1 \end{bmatrix}, \tag{87}$$

from (83). (87) provides a two-point boundary value problem with the boundary conditions $z_{k=0}^1 = z_0^1$ and $\lambda_N^1 = Q_{11N} z_N^1$. Let $\lambda_k^1 = P_k z_k^1$, a matrix Riccati equation can be obtained from (87)

$$P_k = Q_0 + A_0^T P_{k+1} [I + S_0 P_{k+1}]^{-1} A_0, \quad P_N = Q_{11N}. \tag{88}$$

Using the solution of (88), we arrive at the following equations

$$z_{k+1}^1 = Z_k^1 z_k^1, \tag{89}$$

$$\lambda_{k+1}^1 = L_k^1 z_k^1, \tag{90}$$

where

$$Z_k^1 = [I + S_0 P_{k+1}]^{-1} A_0, \tag{91a}$$

$$L_k^1 = P_{k+1} [I + S_0 P_{k+1}]^{-1} A_0. \tag{91b}$$

Furthermore, substituting λ_{k+1}^1 into (86) yields

$$z_k^2 = Z_k^2 z_k^1, \tag{92}$$

$$\lambda_{k+1}^2 = L_k^2 z_k^1, \tag{93}$$

where,

$$Z_k^2 = M_{11} + M_{12} L_k^1, \tag{94a}$$

$$L_k^2 = M_{21} + M_{22} L_k^1. \tag{94b}$$

Based on the derivations given above, we arrive at the following conclusion.

THEOREM 3 *Suppose that the system (78} is causally controllable and observable and the discrete matrix Riccati equation (88} admits a unique nonnegative definite solution* $\bar{P}_k$. *Then,*

(i) the optimal regulator problem defined above admits uncountably many linear feedback solutions given by

Uk= - R - ^{l}B^{T} K$_k$X$_k$, 0 k N - 1, (95)

where,

K_k = MT [$_L$ ^{Li} -Azi $^{O}_{F_k}$] $_H$ $^{-1}$, (96)

and F_k is any (n - r) x (n - r) matrix making A$_{22}$ - B$_2$R - ^{l}B'[$\bar{F}_k$ invertible;

(ii) the open-loop control of uk is unique, given by

uk = - R - ^{1}[Bf B!] [ll]zl*, 0 k N - 1, (97)

where zf* is the unique solution of (89) with the initial condition zf$_{=0}$ = zi5.

Proof. The proof follows the derivations prior to the statements of the theorem and the reasoning similar to the one employed in the implementation of the optimal feedback control (Wang *et al.,* 1988). ■

Appendix B

$$\begin{bmatrix} \hat{Q}_{11k}^2 & \hat{Q}_{12k}^2 \\ \hat{Q}_{12k}^{2T} & \hat{Q}_{22k}^2 \end{bmatrix} := \begin{bmatrix} Q_{11}^2 & Q_{12}^2 \\ Q_{12}^{2T} & Q_{22}^2 \end{bmatrix} + \begin{bmatrix} \pounds lT_k & L_k^{2T} - Z_k^{2T} F_k^{1T} \\ 0 & F_k^{1T} \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - F_k^1 Z_k^2 & F_k^1 \end{bmatrix},$$

$$S_1 = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T, \; S_2 = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_2^T, \; S_3 = B_2 R_{11}^{-1} R_{21} R_{11}^{-1} B_2^T,$$

$$\hat{A}_{11k} = A_{11} - B_1 R_{11}^{-1} [B_1^T L_k^1 + B_2^T (L_k^2 - F_k^1 Z_k^2)],$$

$$\hat{A}_{12k} = A_{12} - B_1 R_{11}^{-1} B_2^T F_k^1,$$

$$\hat{A}_{21k} = A_{21} - B_2 R_{11}^{-1} [B_1^T L_k^1 + B_2^T (L_k^2 - F_k^1 Z_k^2)],$$

$$\hat{A}_{22k} = A_{22} - B_2 R_{11}^{-1} B_2^T F_k^1,$$

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$$\begin{bmatrix} \tilde{Q}^3_{11k} & \tilde{Q}^3_{12k} \\ \tilde{Q}^{3T}_{12k} & \tilde{Q}^3_{22k} \end{bmatrix} := \begin{bmatrix} Q^3_{11} & Q^3_{12} \\ Q^{3T}_{12} & Q^3_{22} \end{bmatrix} + \begin{bmatrix} L^{1T}_k & L^{2T}_k - Z^{2T}_k F^{1T*}_k \\ 0 & F^{1T*}_k \end{bmatrix}$$

$$\begin{bmatrix} \hat{S}_1 & \hat{S}_2 \\ \hat{S}^T_2 & \hat{S}_3 \end{bmatrix} \times \begin{bmatrix} L^1_k & 0 \\ L^2_k - F^{1*}_k Z^2_k & F^{1*}_k \end{bmatrix} +$$

$$\begin{bmatrix} L^{1T}_k & L^{2T}_k - Z^{2T}_k F^{2T}_k \\ 0 & F^{2T}_k \end{bmatrix} \begin{bmatrix} \tilde{S}_1 & \tilde{S}_2 \\ \tilde{S}^T_2 & \tilde{S}_3 \end{bmatrix} \begin{bmatrix} L^1_k & 0 \\ L^2_k - F^2_k Z^2_k & F^2_k \end{bmatrix},$$

$\hat{S}_1 = B_1 R_{11}^{-1} R_{31} R_{11}^{-1} B_1^T$, $\hat{S}_2 = B_1 R_{11}^{-1} R_{31} R_{11}^{-1} B_2^T$,

$\hat{S}_3 = B_2 R_{11}^{-1} R_{31} R_{11}^{-1} B_2^T$,

$\tilde{S}_1 = C_1 R_{12}^{-1} R_{32} R_{12}^{-1} C_1^T$, $\tilde{S}_2 = C_1 R_{12}^{-1} R_{32} R_{12}^{-1} C_2^T$,

$\tilde{S}_3 = C_2 R_{12}^{-1} R_{32} R_{12}^{-1} C_2^T$,

$$\begin{aligned} \tilde{A}_{11k} &= A_{11} - B_1 R_{11}^{-1}[B_1^T L_k^1 + B_2^T(L_k^2 - F_k^{1*} Z_k^2)] \\ &\quad - C_1 R_{12}^{-1}[C_1^T L_k^1 + C_2^T(L_k^2 - F_k^2 Z_k^2)], \end{aligned}$$

$$\tilde{A}_{12k} = A_{12} - B_1 R_{11}^{-1} B_2^T F_k^{1*} - C_1 R_{12}^{-1} C_2^T F_k^2,$$

$$\begin{aligned} \tilde{A}_{21k} &= A_{21} - B_2 R_{11}^{-1}[B_1^T L_k^1 + B_2^T(L_k^2 - F_k^{1*} Z_k^2)] \\ &\quad - C_2 R_{12}^{-1}[C_1^T L_k^1 + C_2^T(L_k^2 - F_k^2 Z_k^2)], \end{aligned}$$

$$\tilde{A}_{22k} = A_{22} - B_2 R_{11}^{-1} B_2^T F_k^{1*} - C_2 R_{12}^{-1} C_2^T F_k^2.$$