Control and Cybernetics

vol. **48** (2019) No. 2

Sufficient conditions for unique global solutions in optimal control of semilinear equations with C^1-nonlinearity*

by

Ahmad Ahmad Ali[1], **Klaus Deckelnick**[2] **and Michael Hinze**[3]

[1]Technische Universität Chemnitz, Fakultät für Mathematik,
09107 Chemnitz, Germany
[2]Institut für Analysis und Numerik, Otto–von–Guericke–Universität Magdeburg,
Universitätsplatz 2, 39106 Magdeburg, Germany
[3]Mathematisches Institut, Universität Koblenz–Landau,
Universitätstrasse 1, 56070 Koblenz, Germany

Dedicated to Günter Leugering on the occasion of his 65th birthday

Abstract: We consider a semilinear elliptic optimal control problem possibly subject to control and/or state constraints. Generalizing previous work, presented in Ahmad Ali, Deckelnick and Hinze (2016) we provide a condition which guarantees that a solution of the necessary first order conditions is a global minimum. A similiar result also holds at the discrete level where the corresponding condition can be evaluated explicitly. Our investigations are motivated by Günter Leugering, who raised the question whether the problem class considered in Ahmad Ali, Deckelnick and Hinze (2016) can be extended to the nonlinearity $\phi(s) = s|s|$. We develop a corresponding analysis and present several numerical test examples demonstrating its usefulness in practice.

Keywords: optimal control, semilinear PDE, uniqueness of global solutions

1. Introduction and problem setting

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded, convex polygonal/polyhedral domain, in which we consider the semilinear elliptic PDE

$$-\Delta y + \phi(\cdot, y) = u \quad \text{in } \Omega, \tag{1}$$

$$y = 0 \quad \text{on } \partial\Omega. \tag{2}$$

*Submitted: February 2019; Accepted: August 2019

We assume that $\phi : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $\phi(x, 0) = 0$ a.e. in Ω and that

$$y \mapsto \phi(x, y) \text{ is of class } C^1 \text{ with } \phi_y(x, y) \geq 0 \text{ for almost all } x \in \Omega; \tag{3}$$
$$\forall L \geq 0\ \exists c_L \geq 0 \quad \phi_y(x, y) \leq c_L \text{ for almost all } x \in \Omega \text{ and all } |y| \leq L. \tag{4}$$

Here and from now onwards, ϕ_y and ϕ_{yy} denote the first and second partial derivative of ϕ with respect to y, respectively. Under the above conditions it can be shown that for every $u \in L^2(\Omega)$ the boundary value problem (1), (2) has a unique solution $y =: \mathcal{G}(u) \in H^2(\Omega) \cap H_0^1(\Omega)$.

Next, let us introduce $U_{ad} := \{v \in L^2(\Omega) : u_a \leq v(x) \leq u_b \text{ a.e. in } \Omega\}$, where $u_a, u_b \in \mathbb{R}$ with $-\infty \leq u_a \leq u_b \leq \infty$. For given $y_0 \in L^2(\Omega)$, $\alpha > 0$ we then consider the optimal control problem

$$(\mathbb{P}) \qquad \begin{array}{l} \min_{u \in U_{ad}} J(u) := \frac{1}{2}\|y - y_0\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to } y = \mathcal{G}(u) \text{ and } y_a(x) \leq y(x) \leq y_b(x) \text{ for all } x \in K. \end{array}$$

Here, $y_a, y_b \in C^0(\bar{\Omega})$ satisfy $y_a(x) < y_b(x)$ for all $x \in K$, where $K \subset \bar{\Omega}$ is compact and either $K \subset \Omega$ or $K = \bar{\Omega}$. In the latter case we suppose, in addition, that $y_a(x) < 0 < y_b(x), x \in \partial\Omega$.

It is well known that $(\mathbb{P})$ has a solution, provided that a feasible point exists (compare Casas, 1993). Under some constraint qualification, such as the linearized Slater condition, a local solution $\bar{u} \in U_{ad}$ of $(\mathbb{P})$ then satisfies the following necessary first order conditions, see Casas (1993), Theorem 5.2: There exist $\bar{p} \in L^2(\Omega)$ and a regular Borel measure $\bar{\mu} \in \mathcal{M}(K)$ such that

$$\int_\Omega \nabla\bar{y} \cdot \nabla v + \phi(\cdot, \bar{y})v\, dx = \int_\Omega \bar{u}v\, dx \quad \forall\, v \in H_0^1(\Omega), \quad y_a \leq \bar{y} \leq y_b \text{ in } K, \tag{5}$$

$$\int_\Omega \bar{p}\,(-\Delta v) + \phi_y(\cdot, \bar{y})\bar{p}\, v\, dx = \int_\Omega (\bar{y} - y_0)v\, dx + \int_K v\, d\bar{\mu} \qquad \forall\, v \in H_0^1(\Omega) \cap H^2(\Omega), \tag{6}$$

$$\int_\Omega (\bar{p} + \alpha\bar{u})(u - \bar{u})\, dx \geq 0 \qquad \forall\, u \in U_{ad}, \tag{7}$$

$$\int_K (z - \bar{y})\, d\bar{\mu} \leq 0 \qquad \forall\, z \in C^0(K),\ y_a \leq z \leq y_b \text{ in } K. \tag{8}$$

In view of the nonlinearity of the state equation, problem $(\mathbb{P})$ is in general nonconvex and hence there may be several solutions of the conditions (5)–(8). The problem we are interested in is whether it is possible to establish sufficient conditions, which guarantee that a solution of (5)–(8) is actually a global minimum of $(\mathbb{P})$. A first result in this direction was obtained by the authors in Ahmad Ali, Deckelnick and Hinze (2016) and holds for a class of nonlinearities, which satisfy a certain growth condition:

Theorem 1 *(Ahmad Ali, Deckelnick and Hinze, 2016, Theorem 3.2) Let $d = 2$; suppose that $y \mapsto \phi(x, y)$ belongs to C^2 for almost all $x \in \Omega$ and that there exist $r > 1$ and $M \geq 0$ such that*

$$|\phi_{yy}(x, y)| \leq M(\phi_y(x, y))^{\frac{1}{r}} \quad \text{for almost all } x \in \Omega \text{ and all } y \in \mathbb{R}. \tag{9}$$

Assume that $(\bar{u}, \bar{y}, \bar{p}, \bar{\mu})$ solves (5)–(8) and that

$$\|\bar{p}\|_{L^q} \leq \big(\frac{r-1}{2r-1}\big)^{\frac{1-r}{r}} M^{-1} C_q^{\frac{2-2r}{r}} \alpha^{\frac{\rho}{2}} q^{1/q} r^{1/r} \rho^{\rho/2} (2-\rho)^{\frac{\rho}{2}-1}, \tag{10}$$

where $q := \frac{3r-2}{r-1}$, $\rho := \frac{r+q}{rq}$ and C_q denotes the constant in (16) below. Then $\bar{u}$ is a global minimum for Problem $(\mathbb{P})$. If the above inequality is strict, then $\bar{u}$ is the unique global minimum.

Assumption (9) is satisfied for $\phi_q(y) := |y|^{q-2}y$, provided that $q > 3$, if we choose $r = \frac{q-2}{q-3}$. Günter Leugering recently raised the question whether our theory can be extended to include the case of $q = 3$. The corresponding nonlinearity $\phi_3(y) = |y|y$ appears, for example, in the mathematical modeling of gas flow through pipes with PDEs (see Hante et al., 2017), so that an extension of Theorem 1 to this case could be helpful in understanding the optimal control of pipe networks. As ϕ_3 is no longer C^2 it does not fit directly into the theory above. However, it turns out that, instead, the analysis can be built on the fact that the partial derivative of ϕ_3 with respect to y satisfies a global Lipschitz condition.

The purpose of this paper is to generalize Theorem 1 in several directions. To begin, we shall replace (9) by a condition that can be formulated for C^1–nonlinearities ϕ and is satisfied by the functions ϕ_q for every $q \geq 3$, thus including the case suggested by Günter Leugering, see (14). A second generalization concerns the choice of the norm $\|\bar{p}\|_{L^q}$ in condition (10). Even though the integration index $q = \frac{3r-2}{r-1}$ is quite natural (solve $r = \frac{q-2}{q-3}$ for q), it is, nevertheless, possible to formulate a corresponding result not just for one index but for q belonging to a suitable interval, see (19), thus giving additional flexibility in its application. Our arguments are natural extensions of the analysis presented in Ahmad Ali, Deckelnick and Hinze (2016) and will also cover the case of $d = 3$, left out in Theorem 1.

There is a lot of literature available considering the problem $(\mathbb{P})$. For a broad overview, we refer the reader to the references of the respective citations. In Casas (1993) this problem is studied for boundary controls. The regularity of optimal controls of $(\mathbb{P})$ and their associated multipliers is investigated in Casas and Tröltzsch (2010) and Casas, Mateos and Vexler (2014). Sufficient second order conditions are discussed in, e.g., Casas and Mateos (2002), Casas (2008) and Casas, De Los Reyes and Tröltzsch (2008), when the set K contains finitely/infinitely many points. For the role of those conditions in

PDE constrained optimization see, e.g., Casas and Tröltzsch (2015).

The finite element discretization of problem $(\mathbb{P})$ in rather general settings has been studied in Arada, Casas and Tröltzsch (2002), Casas and Mateos (2002) and Hinze and Meyer (2012). Convergence rates for sets K, containing only finitely many points are established in Merino, Tröltzsch and Vexler (2010) for finite dimensional controls, and in Casas (2002) for control functions. Only in Neitzel, Pfefferer and Rösch (2015) and Ahmad Ali, Deckelnick and Hinze (2016) an error analysis is provided for general pointwise state constraints in K. Error analysis for linear-quadratic control problems can be found in, e.g., Casas, Mateos and Vexler (2014), Deckelnick and Hinze (2007a,b), and Meyer (2008). Improved error estimates for the state in the case of weakly active state constraints are provided in Neitzel and Wollner (2016). A detailed discussion of discretization concepts and error analysis in PDE-constrained control problems can be found in Hinze and Rösch (2012), Hinze and Tröltzsch (2010), and in Hinze et al. (2009), Chapter 3.

The organization of the paper is as follows: in Section 2 we shall develop the optimality conditions outlined above. In addition to the criteria based on an L^q–norm of $\bar{p}$, we shall also include a result that uses the sign of $\bar{p}$. The variational discretization of $(\mathbb{P})$ is considered in Section 3 and is based on a finite element approximation of (1), (2) that uses numerical integration for the nonlinear term. We obtain corresponding optimality criteria for discrete stationary points and apply these conditions in a series of numerical tests in Section 4, including the nonlinearity $\phi(y) = y|y|$.

2. Optimality conditions for $(\mathbb{P})$

In what follows we assume that $(\bar{u}, \bar{y}, \bar{p}, \bar{\mu})$ is a solution of (5)–(8). Let $u \in U_{ad}$ be a feasible control, $y = \mathcal{G}(u)$ the associated state such that $y_a \le y \le y_b$ in K. A straightforward calculation shows that

$$\begin{aligned} J(u) - J(\bar{u}) = \frac{1}{2}\|y - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \\ + \alpha \int_\Omega \bar{u}(u - \bar{u})dx + \int_\Omega (\bar{y} - y_0)(y - \bar{y})dx. \end{aligned} \tag{11}$$

By combining (6) for $v := y - \bar{y}$ with (8) and (1) we deduce that

$$\begin{aligned} &\int_\Omega (\bar{y} - y_0)(y - \bar{y})\, dx \\ &= -\int_\Omega \bar{p}\, \Delta(y - \bar{y})\, dx + \int_\Omega \phi_y(\cdot, \bar{y})\, \bar{p}\, (y - \bar{y})\, dx - \int_K (y - \bar{y}) d\bar{\mu} \\ &\ge \int_\Omega (u - \bar{u})\bar{p}\, dx - \int_\Omega \big(\phi(\cdot, y) - \phi(\cdot, \bar{y}) - \phi_y(\cdot, \bar{y})(y - \bar{y})\big)\bar{p}\, dx. \end{aligned}$$

Upon inserting this relation into (11) and recalling (7), we finally obtain

$$J(u) - J(\bar{u}) \geq \frac{1}{2}\|y - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} - R(u), \tag{12}$$

where

$$R(u) = \int_\Omega \big(\phi(\cdot, y) - \phi(\cdot, \bar{y}) - \phi_y(\cdot, \bar{y})(y - \bar{y})\big)\bar{p}\, dx. \tag{13}$$

2.1. Conditions involving the sign of $\bar{p}$

A natural first idea to deduce global optimality from (12) consists in identifying situations in which $R(u) \leq 0$ for all $u \in U_{ad}$. We have the following result:

Theorem 2 *Suppose that there exists an interval $I \subset \mathbb{R}$ such that $y \mapsto \phi(x, y)$ is convex (concave) on I for almost all $x \in \Omega$. Furthermore, assume that for every $u \in U_{ad}$ the solution $y = \mathcal{G}(u)$ with $y_a \leq y \leq y_b$ in K satisfies $y(x) \in I$ for all $x \in \Omega$. If $\bar{p} \leq 0$ ($\bar{p} \geq 0$) a.e. on Ω, then $\bar{u}$ is the unique global minimum of* ($\mathbb{P}$).

Proof Suppose that $y \mapsto \phi(x, y)$ is convex. Then our assumptions imply that

$$\phi(x, y(x)) - \phi(x, \bar{y}(x)) - \phi_y(x, \bar{y}(x))(y(x) - \bar{y}(x)) \geq 0 \quad \text{for almost all } x \in \Omega$$

which yields that $R(u) \leq 0$, since $\bar{p} \leq 0$ a.e. in Ω. Hence, $J(u) > J(\bar{u})$ for $u \neq \bar{u}$ by (12). $\square$

In general, we cannot expect the adjoint variable $\bar{p}$ to have a sign without additional conditions on the data of the problem. The following result is similar in spirit to a sufficient condition involving a suitable bound on y_0, obtained in Mignot (1976), Theorem 5.4 and in Ito and Kunisch (2000), Section 5.2, for the optimal control of the obstacle problem.

Lemma 1 *Suppose that $K = \emptyset$ and that $u_a = 0, u_b < \infty$. Let $y_b \in H^2(\Omega)$ satisfy*

$$-\Delta y_b + \phi(\cdot, y_b) \geq u_b \;\; \text{in } \Omega, \quad y_b \geq 0 \;\; \text{on } \partial\Omega.$$

Then $0 \leq \mathcal{G}(u) \leq y_b$ in $\bar{\Omega}$ for every $u \in U_{ad}$. Also, if $y_0 \geq y_b$ a.e. in Ω, then $\bar{p} \leq 0$ in Ω.

Proof Let $u \in U_{ad}$ and set $y = \mathcal{G}(u)$. If we test (5) with $v = y^-$ we have

$$\int_\Omega |\nabla y^-|^2\, dx = -\int_\Omega \phi(\cdot, y^-)\, y^-\, dx + \int_\Omega u\, y^-\, dx \leq 0$$

using (3), the fact that $\phi(\cdot, 0) = 0$, as well as $u \geq 0$. We infer that $y^- \equiv 0$ and hence $y \geq 0$ in $\bar{\Omega}$. Next, $y - y_b$ satisfies

$$-\Delta(y - y_b) + [\phi(\cdot, y) - \phi(\cdot, y_b)] \leq u - u_b \leq 0 \quad \text{a.e. in } \Omega.$$

Testing with $(y-y_b)^+$ then gives $y \le y_b$ in $\bar\Omega$. Finally, since $K=\emptyset$, the adjoint state satisfies

$$-\Delta \bar p + \phi_y(\cdot,\bar y)\bar p = \bar y - y_0 \le y_b - y_0 \le 0 \quad \text{ a.e. in } \Omega$$

since $\bar y \le y_b$ by what we have already shown. We infer that $\bar p \le 0$ in a similar way as above. $\square$

Example 1 *Let $a \in L^\infty(\Omega)$ with $a \ge 0$ a.e. in Ω. Then, the functions $\phi(x,y) = e^{a(x)y}-1$ and $\phi(x,y) = a(x)|y|^{q-2}y$ $(q \ge 3)$ are convex on $\mathbb{R}$ and $[0,\infty)$, respectively. Hence if $K=\emptyset$ and u_a, u_b and y_0 are chosen as in Lemma 1, then Theorem 2 and Lemma 1 imply that a solution of the necessary first order conditions will be the unique global minimum of $(\mathbb{P})$.*

2.2. Conditions involving a bound on $\|\bar p\|_{L^q}$

As mentioned above, it will in general not be possible to establish a sign on the adjoint variable $\bar p$, so that one is left with trying to bound $|R(u)|$ in terms of $\frac{1}{2}\|y-\bar y\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u-\bar u\|^2_{L^2(\Omega)}$. In what follows we shall assume that there exists $\gamma \in [0,1)$ and $M \ge 0$ such that

$$\Big|\frac{\phi_y(x,y_2)-\phi_y(x,y_1)}{y_2-y_1}\Big| \le M\Big(\frac{\phi(x,y_2)-\phi(x,y_1)}{y_2-y_1}\Big)^\gamma \tag{14}$$

for almost all $x \in \Omega$ and for all $y_1, y_2 \in \mathbb{R}, y_1 \ne y_2$. Note that (14) holds with $\gamma = 0$ if $y \mapsto \phi_y(x,y)$ is globally Lipschitz uniformly in $x \in \Omega$. Furthermore, it is not difficult to verify that (14) is satisfied with $\gamma = \frac{1}{r}$, provided that (9) holds.

Example 2 *Let $\phi(x,y) = a(x)|y|^{q-2}y$, where $q \ge 3$ and $a \in L^\infty(\Omega)$ with $a(x) \ge 0$ a.e. in Ω. Then, ϕ satisfies (14) with $\gamma = \frac{q-3}{q-2}$ and $M = (q-2)(q-1)^{\frac{1}{q-2}}\|a\|^{\frac{1}{q-2}}_{L^\infty(\Omega)}$.*

In what follows we shall make use of the elementary inequality (see, e.g., Ahmad Ali, Deckelnick and Hinze, 2016, Lemma 7.1)

$$a^\lambda b^\mu \le \frac{\lambda^\lambda \mu^\mu}{(\lambda+\mu)^{\lambda+\mu}}(a+b)^{\lambda+\mu}, \quad a,b \ge 0, \lambda,\mu > 0, \tag{15}$$

as well as of the Gagliardo–Nirenberg interpolation inequality

$$\|f\|_{L^q} \le C_q \|f\|^{1-\theta}_{L^2}\|\nabla f\|^\theta_{L^2} \tag{16}$$

where $\theta = d(\frac{1}{2}-\frac{1}{q})$ and $2 \le q < \infty$ if $d=2$ and $2 \le q \le 6$ if $d=3$. Explicit values for the constant C_q in (16) can, e.g., be found in Nasibov (1990) and

Veling (2002), see also Ahmad Ali, Deckelnick and Hinze (2016), Theorem 7.3. Before we state our main result we mention that it is well known that $\bar{p} \in W_0^{1,s}(\Omega)$ for all $s \in [1, \frac{d}{d-1})$. In particular we infer, with the help of a standard embedding result, that

$$\bar{p} \in L^q(\Omega) \begin{cases} \text{for every } 1 \leq q < \infty & \text{if } d = 2; \\ \text{for every } 1 \leq q < 3 & \text{if } d = 3. \end{cases} \tag{17}$$

Furthermore, we have that

$$\bar{p} \in L^\infty(\Omega) \text{ if } K = \emptyset \text{ or } K = \bar{\Omega} \text{ with } y_a, y_b \in W^{2,\infty}(\Omega). \tag{18}$$

In order to see (18) we note that $\bar{p} \in H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ by elliptic regularity theory if $K = \emptyset$. On the other hand, if $K = \bar{\Omega}$ with $y_a, y_b \in W^{2,\infty}(\Omega)$ we may apply Theorem 3.1 and Section 4.2 of Casas, Mateos and Vexler (2014) to obtain that $\bar{p} \in L^\infty(\Omega)$.

THEOREM 3 *Assume that ϕ satisfies (14) and let $(\bar{u}, \bar{y}, \bar{p}, \bar{\mu}) \in U_{ad} \times (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times \mathcal{M}(K)$ be a solution of (5)–(8). Furthermore, choose $q > 1$ such that*

$$\frac{1}{1-\gamma} < q < \infty \quad \text{if } d = 2; \qquad \frac{3}{2(1-\gamma)} \leq q < 3 \quad \text{if } d = 3 \tag{19}$$

and define for $t := \frac{2q(1-\gamma)}{q(1-\gamma)-1}$ and $\rho := \frac{d}{2q} + \gamma$ the quantity

$$\eta(\alpha, q, d) := \Big(\frac{1-\gamma}{2-\gamma}\Big)^{\gamma-1} M^{-1} C_t^{2(\gamma-1)} \alpha^{\frac{\rho}{2}} (\frac{d}{2q})^{-\frac{d}{2q}} \gamma^{-\gamma} (2-\rho)^{\frac{\rho}{2}-1} \rho^{\frac{\rho}{2}}, \tag{20}$$

where C_t is the constant in (16). If the inequality

$$\|\bar{p}\|_{L^q} \leq \eta(\alpha, q, d) \tag{21}$$

is satisfied, then $\bar{u}$ is a global minimum for Problem $(\mathbb{P})$. If the inequality (21) is strict, then $\bar{u}$ is the unique global minimum. The assertions hold for $\frac{3}{2(1-\gamma)} \leq q < \infty$ and $d = 3$ provided that $K = \emptyset$ or $K = \bar{\Omega}$ with $y_a, y_b \in W^{2,\infty}(\Omega)$.

PROOF To begin, note that (17) and (18) imply that $\bar{p} \in L^q(\Omega)$ for the cases that we consider. Our starting point is again (12), in which we write the remainder term as

$$R(u) = \int_\Omega \bar{p}(y - \bar{y}) \int_0^1 [\phi_y(\cdot, \bar{y} + t(y - \bar{y})) - \phi_y(\cdot, \bar{y})] dt \, dx. \tag{22}$$

We claim that for all $y_1, y_2 \in \mathbb{R}, y_1 \neq y_2$ we have

$$\begin{aligned} &\Big| \int_0^1 [\phi_y(\cdot, y_1 + t(y_2 - y_1)) - \phi_y(\cdot, y_1)] dt \Big| \\ &\qquad \leq L_\gamma |y_2 - y_1|^{1-2\gamma} \big((\phi(\cdot, y_2) - \phi(\cdot, y_1))(y_2 - y_1) \big)^\gamma, \end{aligned} \tag{23}$$

where $L_\gamma = M\big(\frac{1-\gamma}{2-\gamma}\big)^{1-\gamma}$ and M is given by (14). To see this, let us suppress temporarily the dependence on x and introduce

$$\phi_\epsilon(y) := \int_{\mathbb{R}} \zeta_\epsilon(z)\, \phi(y-z)\, dz, \quad y \in \mathbb{R},$$

where $(\zeta_\epsilon)_{0<\epsilon<1} \subset C_0^\infty(\mathbb{R})$ is a sequence of mollifiers, satisfying

$$\zeta_\epsilon \geq 0, \quad \mathrm{supp}\zeta_\epsilon \subset [-\epsilon, \epsilon], \text{ and } \int_{\mathbb{R}} \zeta_\epsilon(z)\, dz = 1.$$

Since $\phi'_\epsilon(y) = \int_{\mathbb{R}} \zeta_\epsilon(z)\phi'(y-z)\, dz$, we have that

$$\phi''_\epsilon(y) = \lim_{h\to 0} \int_{\mathbb{R}} \zeta_\epsilon(z) \frac{\phi'(y+h-z) - \phi'(y-z)}{h}\, dz$$

so that we obtain with the help of (14) and Hölder's inequality

$$\begin{aligned} |\phi''_\epsilon(y)| &\leq M \int_{\mathbb{R}} \zeta_\epsilon(z)(\phi'(y-z))^\gamma\, dz = M \int_{\mathbb{R}} (\zeta_\epsilon(z))^{1-\gamma} \big(\zeta_\epsilon(z)\, \phi'(y-z)\big)^\gamma\, dz \\ &\leq M\Big(\int_{\mathbb{R}} \zeta_\epsilon(z)\phi'(y-z)\, dz\Big)^\gamma = M(\phi'_\epsilon(y))^\gamma. \end{aligned}$$

We may, therefore, apply Lemma 7.2 from Ahmad Ali, Deckelnick and Hinze (2016) for $\gamma \in (0,1)$ to deduce that

$$\Big| \int_0^1 [\phi'_\epsilon(y_1 + t(y_2 - y_1)) - \phi'_\epsilon(y_1)]dt \Big| \leq L_\gamma |y_2 - y_1| \Big(\int_0^1 \phi'_\epsilon(y_1 + t(y_2 - y_1))\, dt\Big)^\gamma,$$

but the above estimate easily extends to the case of $\gamma = 0$. The bound (23) now follows by sending $\epsilon \to 0$. If we insert (23) into (22), we find that

$$\begin{aligned} |R(u)| &\leq L_\gamma \int_\Omega |\bar{p}|\, |y-\bar{y}|^{2-2\gamma} \big((\phi(\cdot,y) - \phi(\cdot,\bar{y}))(y-\bar{y})\big)^\gamma\, dx \qquad (24) \\ &\leq L_\gamma\, \|\bar{p}\|_{L^q} \|y-\bar{y}\|^{2(1-\gamma)}_{L^{2s(1-\gamma)}} \Big(\int_\Omega (\phi(\cdot,y) - \phi(\cdot,\bar{y}))(y-\bar{y})dx\Big)^\gamma, \end{aligned}$$

where we have used Hölder's inequality with exponents $q, r = \frac{1}{\gamma}$ and $s = \frac{q}{q(1-\gamma)-1}$. Note that

$$2s(1-\gamma) = \frac{2q(1-\gamma)}{q(1-\gamma)-1} = t \in \begin{cases} (2,\infty), & \text{if } d = 2; \\ (2,6], & \text{if } d = 3 \end{cases}.$$

in view of our assumptions on q. We may therefore use (16) in order to estimate $\|y-\bar{y}\|_{L^t}$, and obtain with

$$\theta = d\big(\frac{1}{2} - \frac{1}{t}\big) = \frac{d}{2q(1-\gamma)} \quad \text{and hence} \quad 2(1-\gamma)\theta = \frac{d}{q}$$

that

$$|R(u)| \leq$$
$$L_\gamma C_t^{2(1-\gamma)} \|\bar{p}\|_{L^q} \|y-\bar{y}\|_{L^2}^{2(1-\gamma)-\frac{d}{q}} \|\nabla(y-\bar{y})\|_{L^2}^{\frac{d}{q}} \Big(\int_\Omega (\phi(\cdot,y)-\phi(\cdot,\bar{y}))(y-\bar{y})dx\Big)^\gamma.$$

Applying (15) with $\lambda = \frac{d}{2q}$ and $\mu = \gamma$ and recalling that $\rho = \frac{d}{2q} + \gamma$, we may continue

$$\begin{aligned} |R(u)| \quad &\leq \quad L_\gamma C_t^{2(1-\gamma)} \|\bar{p}\|_{L^q} \|y-\bar{y}\|_{L^2}^{2(1-\gamma)-\frac{d}{q}} \\ &\qquad \times \frac{(\frac{d}{2q})^{\frac{d}{2q}} \gamma^\gamma}{\rho^\rho} \Big(\|\nabla(y-\bar{y})\|_{L^2}^2 + \int_\Omega (\phi(\cdot,y)-\phi(\cdot,\bar{y}))(y-\bar{y})dx\Big)^\rho. \end{aligned}$$

If we take the difference of the PDEs, satisfied by $\bar{y}$ and y, and test it with $y-\bar{y}$ we easily deduce that

$$\|\nabla(y-\bar{y})\|_{L^2}^2 + \int_\Omega \big(\phi(\cdot,y)-\phi(\cdot,\bar{y})\big)(y-\bar{y})\,dx \leq \|y-\bar{y}\|_{L^2}\|u-\bar{u}\|_{L^2},$$

which yields

$$\begin{aligned} |R(u)| \quad &\leq \quad L_\gamma C_t^{2(1-\gamma)} \frac{(\frac{d}{2q})^{\frac{d}{2q}} \gamma^\gamma}{\rho^\rho} \|\bar{p}\|_{L^q} \|y-\bar{y}\|_{L^2}^{2(1-\gamma)-\frac{d}{q}+\rho} \|u-\bar{u}\|_{L^2}^\rho \\ &= \quad 2L_\gamma C_t^{2(1-\gamma)} \alpha^{-\frac{\rho}{2}} \frac{(\frac{d}{2q})^{\frac{d}{2q}} \gamma^\gamma}{\rho^\rho} \|\bar{p}\|_{L^q} \big(\frac{1}{2}\|y-\bar{y}\|_{L^2}^2\big)^{1-\frac{\rho}{2}} \big(\frac{\alpha}{2}\|u-\bar{u}\|_{L^2}^2\big)^{\frac{\rho}{2}}. \end{aligned}$$

Using once more (15), this time with $\lambda = 1-\frac{\rho}{2}, \mu = \frac{\rho}{2}$, we finally deduce that

$$|R(u)| \leq$$
$$2L_\gamma C_t^{2(1-\gamma)} \alpha^{-\frac{\rho}{2}} \frac{(\frac{d}{2q})^{\frac{d}{2q}} \gamma^\gamma}{\rho^\rho} \big(1-\frac{\rho}{2}\big)^{1-\frac{\rho}{2}} \big(\frac{\rho}{2}\big)^{\frac{\rho}{2}} \|\bar{p}\|_{L^q} \big(\frac{1}{2}\|y-\bar{y}\|_{L^2}^2 + \frac{\alpha}{2}\|u-\bar{u}\|_{L^2}^2\big)$$
$$= L_\gamma C_t^{2(1-\gamma)} \alpha^{-\frac{\rho}{2}} \big(\frac{d}{2q}\big)^{\frac{d}{2q}} \gamma^\gamma (2-\rho)^{1-\frac{\rho}{2}} \rho^{-\frac{\rho}{2}} \|\bar{p}\|_{L^q} \big(\frac{1}{2}\|y-\bar{y}\|_{L^2}^2 + \frac{\alpha}{2}\|u-\bar{u}\|_{L^2}^2\big).$$

If we use this estimate in (12) and recall (20), as well as $L_\gamma = M\big(\frac{1-\gamma}{2-\gamma}\big)^{1-\gamma}$, we infer that $J(u) - J(\bar{u}) \geq 0$, provided that (21) holds, so that $\bar{u}$ is a global solution of problem ($\mathbb{P}$). If the inequality in (21) is strict, then $\bar{u}$ is the unique global minimum of problem ($\mathbb{P}$). □

Remark 1 *Suppose that $d = 2$ and that ϕ satisfies (9) for some $r > 1, M \geq 0$, so that (14) holds with $\gamma = \frac{1}{r}$. If we set $q := \frac{3r-2}{r-1}$, then q satisfies (19) while $t = q$ and $\rho = \frac{1}{q} + \frac{1}{r} = \frac{r+q}{rq}$, so that Theorem 1 is a special case of Theorem 3.*

3. Variational discretization

In this section, we consider the case of $d = 2$ and we let $\mathcal{T}_h$ be an admissible triangulation of $\Omega \subset \mathbb{R}^2$. We introduce the following spaces of linear finite elements:

$$X_h := \{v_h \in C^0(\bar{\Omega}) : v_{h|T} \text{ is a linear polynomial on each } T \in \mathcal{T}_h\},$$
$$X_{h0} := \{v_h \in X_h : v_{h|\partial\Omega} = 0\}.$$

The Lagrange interpolation operator I_h is defined by

$$I_h : C^0(\bar{\Omega}) \to X_h, \qquad I_h y := \sum_{i=1}^{n} y(x_i)\phi_i,$$

where $x_1, \dots, x_n$ denote the nodes in the triangulation $\mathcal{T}_h$ and $\{\phi_1, \dots, \phi_n\}$ is the set of basis functions of the space X_h, which satisfy $\phi_i(x_j) = \delta_{ij}$. We discretize (1), (2), using numerical integration for the nonlinear part: for a given $u \in L^2(\Omega)$, find $y_h \in X_{h0}$ such that

$$\int_\Omega \nabla y_h \cdot \nabla v_h + I_h[\phi(\cdot, y_h)v_h]\, dx = \int_\Omega u v_h\, dx \quad \forall\, v_h \in X_{h0}. \tag{25}$$

Using the monotonicity of $y \mapsto \phi(\cdot, y)$ and the Brouwer fixed-point theorem one can show that (25) admits a unique solution $y_h =: \mathcal{G}_h(u) \in X_{h0}$. The variational discretization (see Hinze, 2005) of Problem ($\mathbb{P}$) then reads:

$$(\mathbb{P}_h) \qquad \begin{array}{l} \min_{u \in U_{ad}} J_h(u) := \frac{1}{2}\|y_h - y_0\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to } y_h = \mathcal{G}_h(u),\ y_a(x_j) \le y_h(x_j) \le y_b(x_j),\ x_j \in \mathcal{N}_h, \end{array}$$

where $\mathcal{N}_h := \{x_j \,|\, x_j \text{ is a node of } T \in \mathcal{T}_h, \text{ such that } T \cap K \neq \emptyset\}$. It can be shown that ($\mathbb{P}_h$) has a solution, provided that a feasible point exists. In practice, candidates for solutions are calculated by solving the system of necessary first order conditions, which reads: find $\bar{u}_h \in U_{ad}, \bar{y}_h \in X_{h0}, \bar{p}_h \in X_{h0}, \bar{\mu}_j \in \mathbb{R}, x_j \in \mathcal{N}_h$ such that $y_a(x_j) \le \bar{y}_h(x_j) \le y_b(x_j), x_j \in \mathcal{N}_h$ and

$$\int_\Omega \nabla \bar{y}_h \cdot \nabla v_h + I_h[\phi(\cdot, \bar{y}_h)v_h]\, dx = \int_\Omega \bar{u}_h v_h\, dx \quad \forall\, v_h \in X_{h0}, \tag{26}$$

$$\begin{aligned} &\int_\Omega \nabla \bar{p}_h \cdot \nabla v_h + I_h[\phi_y(\cdot, \bar{y}_h)\bar{p}_h v_h]\, dx \\ &= \int_\Omega (\bar{y}_h - y_0)v_h\, dx + \sum_{x_j \in \mathcal{N}_h} \bar{\mu}_j v_h(x_j)\, \forall\, v_h \in X_{h0}, \end{aligned} \tag{27}$$

$$\int_\Omega (\bar{p}_h + \alpha\bar{u}_h)(u - \bar{u}_h)\, dx \ge 0 \qquad \forall\, u \in U_{ad}, \tag{28}$$

$$\sum_{x_j \in \mathcal{N}_h} \bar{\mu}_j(y_j - \bar{y}_h(x_j)) \le 0 \qquad \forall\, (y_j)_{x_j \in \mathcal{N}_h}, y_a(x_j) \le y_j \le y_b(x_j), x_j \in \mathcal{N}_h. \tag{29}$$

In order to formulate the analogue of Theorem 3 we introduce the following h–dependent norm on X_h:

$$\|v_h\|_{h,q} := \Big(\int_\Omega I_h[|v_h|^q]dx\Big)^{\frac{1}{q}}, \quad v_h \in X_h,\ 1 \le q < \infty.$$

THEOREM 4 *Suppose that ϕ and $q > 1$ satisfy the conditions (14) and (19) respectively and let $\bar u_h \in U_{ad}$, $\bar y_h \in X_{h0}$, $\bar p_h \in X_{h0}$, $(\bar\mu_j)_{x_j\in\mathcal{N}_h}$ be a solution of (26)–(29). If*

$$\|\bar p_h\|_{h,q} \le (\frac{1}{4})^{1-\gamma-\frac{1}{q}}\,\eta(\alpha,q,2), \tag{30}$$

then $\bar u_h$ is a global minimum for Problem $(\mathbb{P}_h)$. If the inequality (30) is strict, then $\bar u_h$ is the unique global minimum. Here, $\eta(\alpha,q,2)$ is defined in (20).

PROOF Just as in the continuous case, we obtain for $u \in U_{ad}$ with $y_h = \mathcal{G}_h(u)$

$$J_h(u) - J_h(\bar u_h) \ge \frac{1}{2}\|y_h - \bar y_h\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u - \bar u_h\|^2_{L^2(\Omega)} - R_h(u), \tag{31}$$

where

$$\begin{aligned} R_h(u) &= \int_\Omega I_h\left[\big(\phi(\cdot,y_h) - \phi(\cdot,\bar y_h) - \phi_y(\cdot,\bar y_h)(y_h-\bar y_h)\big)\bar p_h\right]\,dx \\ &= \int_\Omega I_h\left[\bar p_h(y_h-\bar y_h)\int_0^1(\phi_y(\cdot,\bar y_h + t(y_h-\bar y_h)) - \phi_y(\cdot,\bar y_h))dt\right]dx. \end{aligned} \tag{32}$$

If we use (23) then we obtain, as above, with the help of Hölder's inequality

$$\begin{aligned} |R_h(u)| &\le L_\gamma \int_\Omega I_h\left[|\bar p_h|\,|y_h-\bar y_h|^{2-2\gamma}\big((\phi(\cdot,y_h)-\phi(\cdot,\bar y_h))(y_h-\bar y_h)\big)^\gamma\right]\,dx \\ &\le L_\gamma\|\bar p_h\|_{h,q}\|y_h-\bar y_h\|^{2(1-\gamma)}_{h,2s(1-\gamma)}\Big(\int_\Omega I_h\left[(\phi(\cdot,y_h)-\phi(\cdot,\bar y_h))(y_h-\bar y_h)\right]dx\Big)^\gamma, \end{aligned}$$

where $s = \frac{q}{q(1-\gamma)-1}$. Applying Lemma 2 from the Appendix, we derive

$$\begin{aligned} &|R_h(u)| \\ &\le L_\gamma 4^{\frac{1}{s}}\|\bar p_h\|_{h,q}\|y_h-\bar y_h\|^{2(1-\gamma)}_{L^{2s(1-\gamma)}}\Big(\int_\Omega I_h\left[(\phi(\cdot,y_h)-\phi(\cdot,\bar y_h))(y_h-\bar y_h)\right]dx\Big)^\gamma, \end{aligned}$$

which is the analogue of (24) up to the factor $4^{1/s}$. The rest of the proof now follows in the same way as in Theorem 3, where we use (25) instead of the PDEs. $\square$

We shall investigate condition (30) for different choices of ϕ and q in the numerics section. From the numerical analysis point of view, it is also possible to examine the convergence of a sequence of solutions $(\bar{u}_h, \bar{y}_h, \bar{p}_h, (\bar{\mu}_j)_{x_j \in \mathcal{N}_h})_{0<h<h_0}$ of (26)–(29) that satisfy (30) uniformly in h. Based on Theorem 1, convergence in $L^2(\Omega)$ of $(\bar{u}_h)_{0<h<h_0}$ to a solution $\bar{u}$ of $(\mathbb{P})$ has been obtained in Ahmad Ali, Deckelnick and Hinze (2016), Theorem 4.2, while an error estimate is proven in Ahmad Ali (2017), Ahmad Ali, Deckelnick and Hinze (2018), Theorem 3.1. We expect that these results carry over to the generalized framework, considered in this paper. In this context we also refer to Neitzel, Pfefferer and Rösch (2015) as a further contribution to the error analysis for optimal control of semilinear equations with pointwise bounds on the state. Contrary to our approach, this work is based on second order sufficient optimality conditions for a local solution of the control problem and requires, in particular, a C^2–nonlinearity ϕ.

4. Numerical experiments

In this section we present several numerical experiments, related to Theorem 4. We consider $(\mathbb{P})$ with different choices for the nonlinearity ϕ. For each choice we fix $\Omega := (0,1) \times (0,1)$, while for the desired state y_0 we consider the following two scenarios:

A1: (Reachable desired state) $y_0(x) := 2\sin(2\pi x_1)\sin(2\pi x_2)$.
A2: (Not reachable desired state) $y_0(x) := 60 + 160(x_1(x_1 - 1) + x_2(x_2 - 1))$.

For the control and state bounds we consider the three cases as follows:

Case 1: (Unconstrained problem) $u_b = -u_a = \infty$, $K = \emptyset$.
Case 2: (Control constrained problem) $u_b = -u_a = 5$, $K = \emptyset$.
Case 3: (State constrained problem) $u_b = -u_a = \infty$, $K = \bar{\Omega}, y_b \equiv -y_a \equiv 1$.

For α, we report numerical results for the values $\alpha = 10^i$, $i = -6, -5, \ldots, 3$. The domain Ω is partitioned using a uniform triangulation with mesh size $h = 2^{-5}\sqrt{2}$, generated with the POIMESH command from MATLAB, and the discrete counterpart of the problem is as in Section 3. The resulting discrete optimality system (26)–(29) is solved using the semismooth Newton method.

EXAMPLE 3 *We consider* $\phi(y) := y|y|$. *Then,* $\gamma = 0$ *with* $M = 2$. *Taking* $q = 2$, *the condition reads*

$$\|\bar{p}_h\|_{h,2} \le \frac{1}{2}\,\eta(\alpha, 2, 2)$$

with

$$C_4^{-2} \approx 2.381297723376159.$$

The results are reported in Fig. 1. We see that in the light of Theorem 4, the unique global solution of the considered control problem has been computed for all given values of α, except for Case 2, when $\alpha \leq 10^{-3}$. There, no conclusion can be derived. However, with the coefficient $a(x) := \frac{1}{8}$ we obtain a global unique solution for the whole considered parameter range, see Fig. 2.

EXAMPLE 4 *We consider* $\phi(y) := y^3$. *Then,* $\gamma = 0.5$ *with* $M = 2\sqrt{3}$. *Taking* $q = 3$*, the condition reads*

$$\|\bar{p}_h\|_{L^3(\Omega)} \leq \eta(\alpha, 3, 2)$$

with

$$C_6^{-1} \approx 1.616080082127768.$$

The choice of $q = 3$ is motivated by fact that among the possible choices of the Gagliardo-Nirenberg constant, the value of C_6 is among the smallest possible ones, see Ahmad Ali, Deckelnick and Hinze (2016), Fig. 4. The integrals involving ϕ, and the norm $\|\bar{p}_h\|_{L^3(\Omega)}$ are computed exactly. The results are reported in Fig. 3. We also include, for comparison, the results for $q = 4$, which correspond to the findings of Ahmad Ali, Deckelnick and Hinze (2016), Example 2. As one can see, this choice, in some situations, delivers larger uniqueness intervals for α. Overall, uniqueness of the global solution can be deduced for certain ranges of the parameter α, where uniqueness is more likely in the case of a reachable desired state y_0.

EXAMPLE 5 *We consider* $\phi(y) := y^5$. *Then,* $\gamma = 3/4$ *with* $M = 4 \times (5)^{1/4}$. *Taking* $q = 6$*, the condition reads*

$$\|\bar{p}_h\|_{L^6(\Omega)} \leq \eta(\alpha, 6, 2)$$

with

$$C_6^{-1/2} \approx 1.271251384316953.$$

The choice of $q = 6$ is motivated as in the previous example. This, then, is the situation of Ahmad Ali, Deckelnick and Hinze (2016), Example 3. For comparison, we also include the results obtained with quadrature based on the estimate (30). As one can see, the difference in both approaches (exact integration versus quadrature) is negligible. The results are reported in Fig. 4.

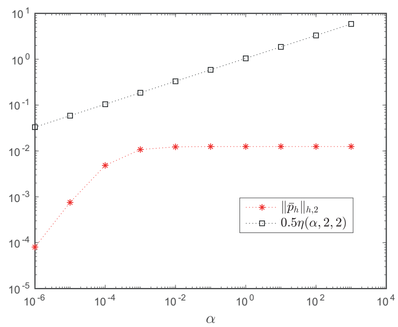
5. Appendix

LEMMA 2 *Let* $d = 2$ *and* $2 \leq q < \infty$. *Then*

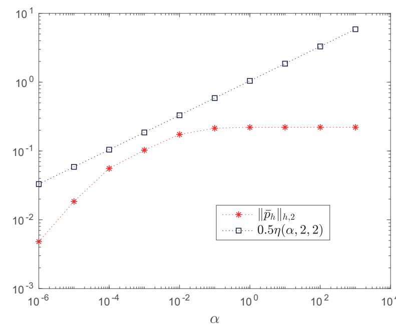
$$\|v_h\|_{L^q} \leq \|v_h\|_{h,q} \leq 4^{\frac{1}{q}} \|v_h\|_{L^q} \quad \textit{for all } v_h \in X_h.$$

PROOF Let us denote by $\hat{T} \subset \mathbb{R}^2$ the unit simplex with vertices $\hat{a}_0 = (0, 0), \hat{a}_1 = (1, 0)$, and $\hat{a}_2 = (0, 1)$. Using a scaling argument it is sufficient to show that

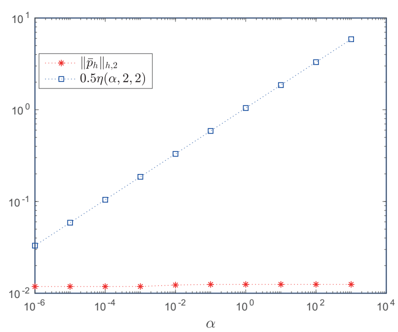
$$\int_{\hat{T}} |p|^q \, d\hat{x} \leq \int_{\hat{T}} \hat{I}_h[|p|^q] d\hat{x} \leq 4 \int_{\hat{T}} |p|^q \, d\hat{x} \quad \text{for all } p \in P_1(\hat{T}), \tag{33}$$



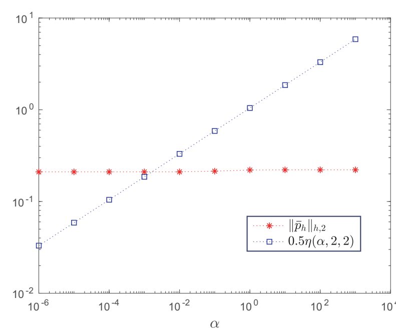
(a) Case 1 with A1



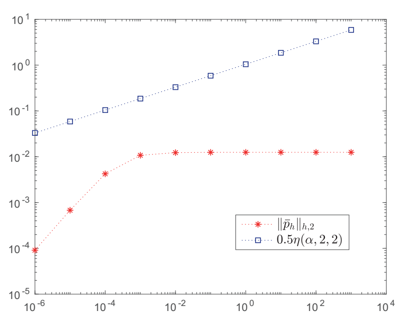
(b) Case 1 with A2



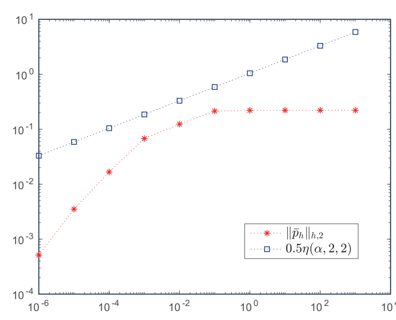
(c) Case 2 with A1



(d) Case 2 with A2



(e) Case 3 with A1



(f) Case 3 with A2

Figure 1: Results for $\phi(s) = s|s|$

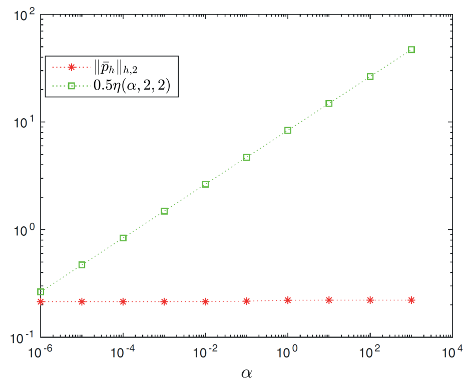


Figure 2: Case 2 with A2 for $\phi(s) = \frac{1}{8}s|s|$

where $\hat{I}_h f = \sum_{j=0}^{2} f(\hat{a}_j)\hat{\phi}_j$ and $\hat{\phi}_j(\hat{a}_i) = \delta_{ij}$. In order to see the first inequality in (33) we observe that

$$\int_{\hat{T}} |p|^q \, d\hat{x} = \int_{\hat{T}} |\sum_{j=0}^{2} p(\hat{a}_j)\hat{\phi}_j|^q d\hat{x} \leq \int_{\hat{T}} \sum_{j=0}^{2} |p(\hat{a}_j)|^q \hat{\phi}_j \, d\hat{x} = \int_{\hat{T}} \hat{I}_h[|p|^q] d\hat{x}$$

in view of the convexity of $t \mapsto |t|^q$ and the properties of $\hat{\phi}_j, j = 0,1,2$. Let us next consider the remaining estimate and first focus on the case of $q = 2$. A straightforward calculation shows that

$$\int_{\hat{T}} \hat{I}_h[|p|^2] d\hat{x} = \frac{1}{6} \sum_{j=0}^{2} |p(\hat{a}_j)|^2,$$

$$\int_{\hat{T}} |p|^2 \, d\hat{x} = \frac{1}{24} \sum_{j=0}^{2} |p(\hat{a}_j)|^2 + \frac{1}{24} |p(\frac{\hat{a}_0 + \hat{a}_1 + \hat{a}_2}{3})|^2,$$

which implies that

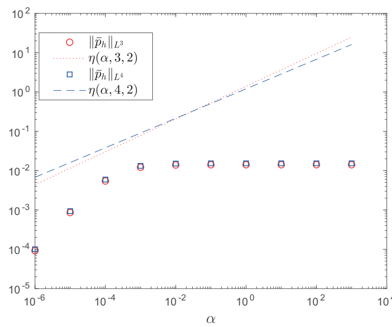
$$\int_{\hat{T}} \hat{I}_h[|p|^2] d\hat{x} \leq 4 \int_{\hat{T}} |p|^2 \, d\hat{x}. \tag{34}$$

Let us introduce the measure $\mu := \sum_{j=0}^{2} m_j \delta_{\hat{a}_j}$ with $m_j = \int_{\hat{T}} \hat{\phi}_j d\hat{x} = \frac{1}{6}, j = 0,1,2$. Clearly,

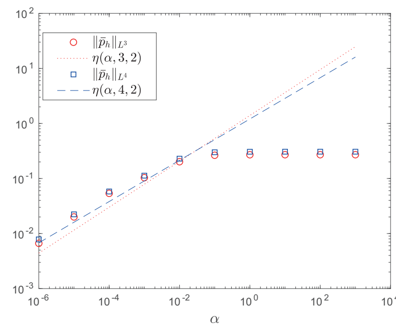
$$\|p\|^q_{L^q(\mu)} := \int_{\hat{T}} |p|^q d\mu = \sum_{j=0}^{2} |p(\hat{a}_j)|^q m_j = \int_{\hat{T}} \hat{I}_h[|p|^q] d\hat{x}.$$

Now, (34) yields that $\|p\|_{L^2(\mu)} \leq 2\|p\|_{L^2(d\hat{x})}$, while $\|p\|_{L^\infty(\mu)} \leq \|p\|_{L^\infty(d\hat{x})}$, so that the Riesz–Thorin convexity theorem implies that

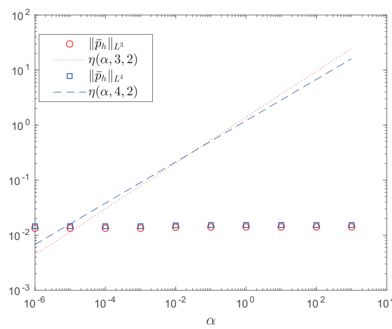
$$\|p\|_{L^q(\mu)} \leq 2^{\frac{2}{q}} \|p\|_{L^q(d\hat{x})} \quad \text{for all } p \in P_1(\hat{T}), \text{ which is (33).} \qquad \square$$



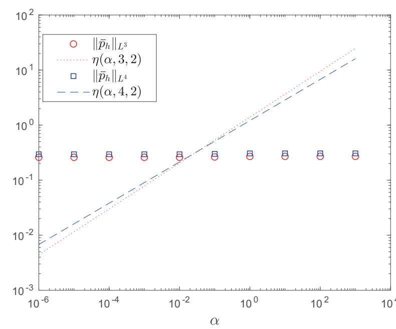
(a) Case 1 with A1



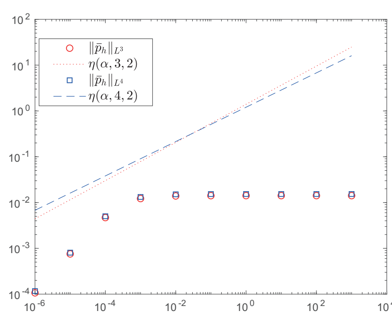
(b) Case 1 with A2



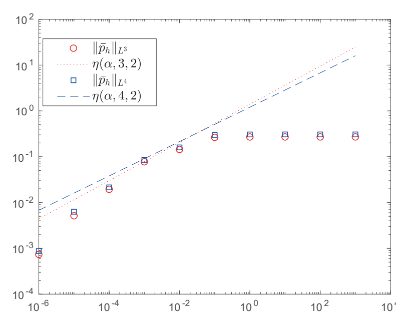
(c) Case 2 with A1



(d) Case 2 with A2

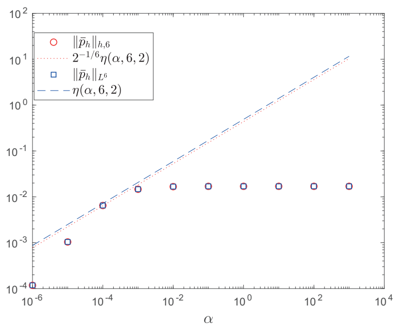


(e) Case 3 with A1

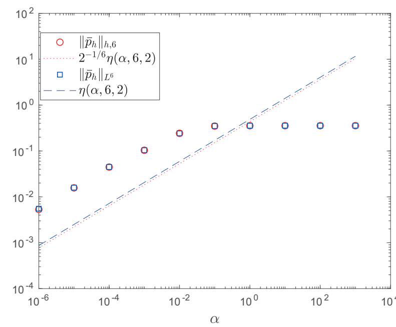


(f) Case 3 with A2

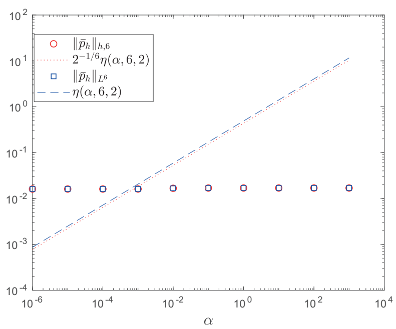
Figure 3: Results for $\phi(s) = s^3$



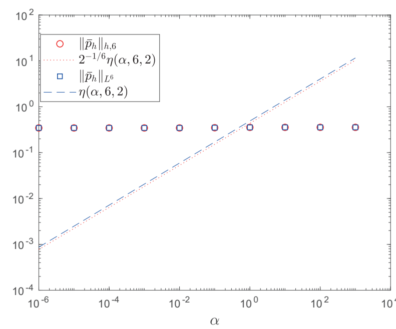
(a) Case 1 with A1



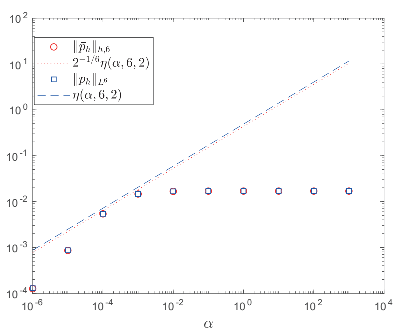
(b) Case 1 with A2



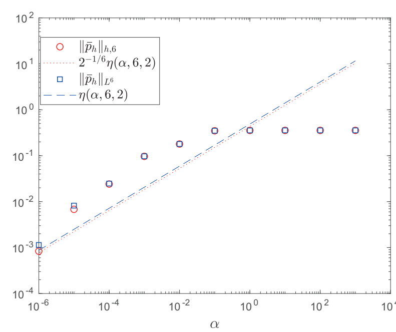
(c) Case 2 with A1



(d) Case 2 with A2



(e) Case 3 with A1



(f) Case 3 with A2

Figure 4: Results for $\phi(s) = s^5$

References

Ahmad Ali, A. (2017) *Optimal Control of Semilinear Elliptic PDEs with State Constraints - Numerical Analysis and Implementation*, PhD thesis. Dissertation. Hamburg, Universität Hamburg.

Ahmad Ali, A., Deckelnick, K. and Hinze, M. (2016) Global minima for semilinear optimal control problems. *Computational Optimization and Applications*, **65**, 261–288.

Ahmad Ali, A., Deckelnick, K. and Hinze, M. (2018) Error analysis for global minima of semilinear optimal control problems. *Mathematical Control and Related Fields (MCRF)* **8**.

Arada, N., Casas, E. and Tröltzsch, F. (2002) Error estimates for the numerical approximation of a semilinear elliptic control problem. *Computational Optimization and Applications*, **23**, 201–229.

Casas, E. (1993) Boundary control of semilinear elliptic equations with pointwise state constraints. *SIAM Journal on Control and Optimization*, **31**, 993–1006.

Casas, E. (2002) Error estimates for the numerical approximation of semilinear elliptic control problems with finitely many state constraints. *ESAIM: Control, Optimisation and Calculus of Variations*, **8**, 345–374.

Casas, E. (2008) Necessary and sufficient optimality conditions for elliptic control problems with finitely many pointwise state constraints. *ESAIM: Control, Optimisation and Calculus of Variations*, **14**, 575–589.

Casas, E., De Los Reyes, J. C. and Tröltzsch, F. (2008) Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints. *SIAM Journal on Optimization*, **19**, 616–643.

Casas, E. and Mateos, M. (2002a) Second order optimality conditions for semilinear elliptic control problems with finitely many state constraints. *SIAM Journal on Control and Optimization*, **40**, 1431–1454.

Casas, E. and Mateos, M. (2002b) Uniform convergence of the FEM. Applications to state constrained control problems. *Comput. Appl. Math.*, **21** , 67–100, *Special Issue in Memory of Jacques-Louis Lions.*

Casas, E., Mateos, M. and Vexler, B. (2014) New regularity results and improved error estimates for optimal control problems with state constraints. *ESAIM. Control, Optimisation and Calculus of Variations*, **20**, 803–822.

Casas, E. and Tröltzsch, F. (2010) Recent advances in the analysis of pointwise state-constrained elliptic optimal control problems. *ESAIM: Control, Optimisation and Calculus of Variations*, **16**, 581–600.

Casas, E. and Tröltzsch, F. (2015) Second order optimality conditions and their role in pde control. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **117**, 3–44.

Deckelnick, K. and Hinze, M. (2007a) Convergence of a finite element approximation to a state-constrained elliptic control problem. *SIAM Journal on Numerical Analysis*, **45**, 1937–1953.

Deckelnick, K. and Hinze, M. (2007b) A finite element approximation to elliptic control problems in the presence of control and state constraints. *Hamburger Beiträge zur Angewandten Mathematik*.

Hante, F.M., Leugering, G., Martin, A., Schewe, L. and Schmidt, M. (2017) Challenges in Optimal Control Problems for Gas and Fluid Flow in Networks of Pipes and Canals: From Modeling to Industrial Applications. In: P. Manchanda, R. Lozi and A. Siddiqi (eds) *Industrial Mathematics and Complex Systems. Industrial and Applied Mathematics*. Springer, Singapore.

Hinze, M., Pinnau, R., Ulbrich M. and Ulbrich, S. (2009) *Optimization with PDE Constraints. Mathematical Modelling: Theory and Applications*, **23**, Springer, New York.

Hinze, M. (2005) A variational discretization concept in control constrained optimization: The linear-quadratic case. *Computational Optimization and Applications*, **30**, 45–61.

Hinze, M. and Meyer, C. (2012) Stability of semilinear elliptic optimal control problems with pointwise state constraints. *Computational Optimization and Applications*, **52**, 87–114.

Hinze, M. and Rösch, A. (2012) Discretization of optimal control problems. In: *Constrained Optimization and Optimal Control for Partial Differential Equations*, Springer, **160**, 391–430.

Hinze, M. and Tröltzsch, F. (2010) Discrete concepts versus error analysis in pde-constrained optimization. *GAMM-Mitteilungen*, **33**, 148–162.

Ito, K. and Kunisch, K. (2000) Optimal control of elliptic variational inequalities. *Appl. Math. Optim.* **41**, 343–364.

Merino, P., Tröltzsch, F. and Vexler, B. (2010) Error estimates for the finite element approximation of a semilinear elliptic control problem with state constraints and finite dimensional control space. *ESAIM: Mathematical Modelling and Numerical Analysis*, **44**, 167–188.

Meyer, C. (2008) Error estimates for the finite-element approximation of an elliptic control problem with pointwise state and control constraints. *Control and Cybernetics*, **37**, 51–83.

Mignot, F. (1976) Contrôle dans les inéquations variationelles elliptiques. *J. Funct. Anal.* **22**, 130–185.

Nasibov, S.M. (1990) On optimal constants in some Sobolev inequalities and their application to a nonlinear Schrödinger equation. *Soviet. Math. Dokl.* **40**, 110–115, translation of *Dokl. Akad. Nauk SSSR* 307:538-542 (1989).

Neitzel, I., Pfefferer J. and Rösch, A. (2015) Finite element discretization of state-constrained elliptic optimal control problems with semilinear state equation. *SIAM Journal on Control and Optimization*, **53** , 874–904.

Neitzel, I. and Wollner, W. (2017) A priori L^2-discretization error estimates for the state in elliptic optimization problems with pointwise inequality state constraints. *Numer. Math., online first.*

Veling, E.J.M. (2002) Lower Bounds for the Infimum of the Spectrum of the

Schrödinger Operator in $\mathbb{R}^N$ and the Sobolev Inequalities. *JIPAM. Journal of Inequalities in Pure & Applied Mathematics* **3**, Article 63 [electronic only].