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On approximately Breckner s-convex functions[*] [†]

by

Pál Burai[1], Attila Házy[2] and Tibor Juhász[3]

[1]Department of Applied Mathematics and Probability Theory
University of Debrecen
H-4010 Debrecen, Pf. 12, Hungary

[2]Department of Applied Mathematics, University of Miskolc
H-3515 Miskolc-Egyetemváros, Hungary

[3]Institute of Mathematics and Informatics, Eszterházy Károly College, H-3300
Eger, Leányka út 4, Hungary

e-mail: burai.pal@inf.unideb.hu, matha@uni-miskolc.hu, juhaszti@ektf.hu

Abstract: The main goal of this paper is to consider the regularity and convexity properties of a given type of approximately generalized convex functions, namely approximately Breckner s-convex functions (see the origin of the definition in Breckner, 1978).

Our main result is a Bernstein-Doetsch type one. It is proved that the local boundedness of such a type of function from above at a point of its domain implies approximate convexity and stronger regularity properties of the function in question on the whole domain.

Keywords: convexity, approximate convexity, s-convexity, Bernstein–Doetsch theorem, regularity properties of generalized convex functions.

1. Introduction

It is a well known fact that convexity and its generalizations play an important role in different parts of mathematics, mainly in optimization theory. This is one of the topmost motivations for examining the properties of generalized convex functions. On the other hand, generalized convexity or continuity are too strong assumptions from application point of view. So, it is common to discuss the regularity properties of a new class of functions or a perturbed version of the defining inequality. This will be the main point of this work.

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Probably the most significant result of the regularity theory of convex functions is due to Bernstein and Doetsch (1915). They have proved that the local upper boundedness of a Jensen convex function yields its continuity and convexity as well. Breckner has proved the following Bernstein-Doetsch type result in Breckner (1978):

THEOREM 1 *Let $f : D \to \mathbb{R}$ be a rationally s-convex function (see inequality* (1) *below). If it is locally bounded from above at a point of D, then it is continuous and s-convex on D.*

In Burai, Házy and Juhász (2009) the regularity properties of Breckner s-convex functions have been examined, and the previous theorem has been generalized. We carry on this research with the investigation of the regularity and convexity properties of approximately Breckner s-convex functions.

The concepts of s-convexity and rational s-convexity were introduced by Breckner (1978). A real valued function $f : D \to \mathbb{R}$ (where D is a nonempty, convex subset of a real (complex) linear space X) is called *Breckner s-convex* (or simply *s-convex*), if there exists an $s \in]0, 1]$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1}$$

for every $x, y \in D$, and $\lambda \in [0, 1]$. If (1) is fulfilled only for $\lambda \in \mathbb{Q} \cap [0, 1]$, then f is called *rationally Breckner s-convex* (or *rationally s-convex*).

This concept is a generalization of convexity (case $s = 1$). It is an interesting fact that the set of nonnegative s-convex functions strictly flares as s strictly decreases, in addition, s-convex functions have a good relationship with algebraic operations just like the convex ones. For example, the sum of two s-convex functions is s-convex and by multiplying an s-convex function with a non-negative scalar we get an s-convex function again (see Hudzik and Maligranda, 1994, and Burai, Házy and Juhász, 2009) for further information). Moreover, there is a nice correspondence between convex and Breckner s-convex functions, namely, the class of s-convex functions belongs to the class of locally s-Hölder functions (Breckner, 1994; Pycia, 2001). The class of 1-Hölder functions coincides with the class of locally Lipschitz functions. It is a known fact that a convex function is also a locally Lipschitz one if it is locally bounded form above at a point of an open domain (see, e.g., Kuczma, 1985).

Here we would like to examine the regularity and convexity properties of approximately s-convex (more precisely, the so-called Breckner (s, d)-convex) functions. There are several possibilities to define approximate convexity. The two main trends are: defining it with a constant error term (see, e.g., Házy and Páles, 2004; Házy, 2007; Luc, Ngai and Théra, 2000; Páles, 2003; Tabor and Tabor, 2009a,b,c), or defining it with a function error term (see, e.g., Hyers and Ulam, 1952; Ng and Nikodem, 1993).

We prefer considering the latter using a "metric like" function perturbing equation (1), so similarly to Házy (2007) we introduce the notion of approximate

Breckner s-convexity in the following way. Let $s \in]0,1]$ be a fixed parameter and let the function $d : X \times X \to \mathbb{R}$ be given. A function $f : D \to \mathbb{R}$ is said to be *Breckner* (s,d)*-convex* (or shortly (s,d)*-convex*) if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) + d(x,y) \tag{2}$$

for all $x, y \in D$ and $\lambda \in [0,1]$.

A real valued function $f : D \to \mathbb{R}$ is called *Breckner rationally* (s,d)*-convex (or rationally* (s,d)*-convex)* (in notation $(\mathbb{Q}, s, d)$-convex) if it fulfills (2) for all $\lambda \in \mathbb{Q} \cap [0,1]$, and it is called *Breckner* (λ, s, d)*-convex (or* (λ, s, d)*-convex)* if it fulfills (2) for a fixed parameter $\lambda \in]0,1[$.

In what follows we make some natural assumptions about the perturbation function $d : X \times X \to \mathbb{R}$, namely

$$\begin{array}{lrcl}
(i) & d(x,y) & \ge & 0, \\
(ii) & d(x,y) & = & d(y,x), \\
(iii) & d(x,y) & \le & d(x,z) + d(z,y), \\
(iv) & d(x+z, y+z) & = & d(x,y), \\
(v) & \text{if } d(x,y) & = & 0, \text{ then } d(ux,uy) = 0 \text{ for all rational } u,
\end{array}$$

for all $x, y, z \in X$ The first three properties declare that d is a semimetric on X, (iv) states the translation invariance of it. If $(i) - (iv)$ hold, then we say that d is a *translation invariant semimetric.*

It is easy to see that every nonnegative constant function satisfies the properties $(i) - (v)$. Another important example is the one parameter family of functions $d(x,y) = \|x-y\|^p$ (where $0 \le p \le 1$). These are also translation invariant semimetrics which also fulfill (v). One can create another examples from the previous. Evidently, the linear combination of translation invariant semimetrics with nonnegative coefficients is a translation invariant semimetric, too.

2. Main results

Henceforward $(X, \|\cdot\|)$ denotes a real normed space, and $D \subset X$ is always a nonempty, open, convex set.

We recall that a function $f : D \to \mathbb{R}$ is called locally bounded from above at a point $w \in D$, if there exist positive real numbers r and K such that, $f(x) \le K$ for every $x \in B(w,r) := \{\, x \in X \mid \|x - w\| < r \,\}$.

The first main result states that, if an approximately rationally Breckner s-convex function is bounded from above at a point of its domain, then it is continuous on the whole domain. The second one states approximate Breckner s-convexity of the function if similar conditions are fulfilled as above.

Theorem 2 *Assume that the function d is a continuous, translation invariant semimetric, which fulfills (v) and $d(x,x) = 0$. If $f : D \to \mathbb{R}$ is Breckner* *rationally (s,d)-convex and locally bounded from above at a point of D, then it is continuous.*

THEOREM 3 *Assume that the function d is a continuous, translation invariant semimetric, which fulfills (v) and $d(x,x)=0$. If $f : D \to \mathbb{R}$ is $(\mathbb{Q},s,d)$-convex and locally bounded from above at a point of D, then it is Breckner (s,d)-convex.*

3. Proofs of the main results

We need the following two theorems.

THEOREM 4 *Let $\lambda \in]0,1[$ be a given number. Assume that the function d is a continuous, translation invariant semimetric and there exists a constant $C>0$ such that $d\left(\frac{1}{1-\lambda}x, \frac{1}{1-\lambda}y\right) \le Cd(x,y)$ for every $x,y \in X$. If $f : D \to \mathbb{R}$ is Breckner (λ,s,d)-convex, and locally bounded from above at a point $w \in D$, then f is locally bounded on D.*

Proof. First we prove that f is locally bounded from above on D. Define the sequence of sets D_n by

$$D_0 := \{w\}, \qquad D_{n+1} := \lambda D_n + (1-\lambda)D.$$

Then, it follows by induction that

$$D_n = \lambda^n w + (1-\lambda^n)D.$$

Using induction on n, we prove that f is locally upper bounded at each point of D_n. By assumption, f is locally upper bounded at $w \in D_0$. Assume that f is locally upper bounded at each point of D_n. For an arbitrary $x \in D_{n+1}$, there exist $x_0 \in D_n$ and $y_0 \in D$ such that $x = \lambda x_0 + (1-\lambda)y_0$. By the inductive assumption, there exist an $r>0$ and a constant $M_0 \ge 0$ such that $f(x') \le M_0$ and $d(x',x_0) \le M_0$ for all $x' \in B(x_0,r)$. Because of the continuity of d, we can choose r such that the previous is true. Then, by the (λ,s,d)-convexity of f, we have

$$\begin{aligned} f(\lambda x' + (1-\lambda)y_0) &\le \lambda^s f(x') + (1-\lambda)^s f(y_0) + d(x',y_0) \\ &\le \lambda^s M_0 + (1-\lambda)^s f(y_0) + d(x',x_0) + d(x_0,y_0) \\ &\le \lambda^s M_0 + (1-\lambda)^s f(y_0) + M_0 + d(x_0,y_0) =: M. \end{aligned}$$

Therefore, for $y \in U := \lambda B(x_0,r) + (1-\lambda)y_0 = B(\lambda x_0 + (1-\lambda)y_0, \lambda r) = B(x,\lambda r)$, we get that $f(y) \le M$. Thus, f is locally bounded above at $x \in D_{n+1}$, so f is locally bounded above on D_{n+1} .

On the other hand, one can easily see that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

Indeed, for fixed $x \in D$, define the sequence x_n by

$$x_n := \frac{x - \lambda^n w}{1 - \lambda^n}.$$

Then $x_n \to x$ if $n \to \infty$. As the set is open, there exists an $n_0 \in \mathbb{N}$, such that $x_n \in D$ if $n \geq n_0$. Therefore

$$x = \lambda^n w + (1 - \lambda^n) x_n \in \lambda^n w + (1 - \lambda^n) D = D_n.$$

Thus, f is locally bounded from above on D.

We prove now that f is locally bounded from below. Let $q \in D$ be arbitrary. Since f is locally bounded from above at the point q, there exist $\varrho > 0$ and $M > 0$ such that $f(x) \leq M$ and $d(x, q) \leq M$ if $x \in B(q, \varrho)$. (Just like in the first part of the proof, we can find such a ϱ, using the continuity of d.) Let $x \in B(q, (1 - \lambda)\varrho)$ and $y := \frac{1}{1-\lambda} q - \frac{\lambda}{1-\lambda} x$. Then y is in $B(q, \lambda\varrho) \subset B(q, \varrho)$. Then, by (λ, s, d)-convexity of f and by properties of d, we get

$$\begin{aligned} f(q) &= f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) + d(x, y) \\ &= \lambda^s f(x) + (1-\lambda)^s f(y) + d\left(\tfrac{1}{1-\lambda} x - \tfrac{\lambda}{1-\lambda} x, \tfrac{1}{1-\lambda} q - \tfrac{\lambda}{1-\lambda} x\right) \\ &= \lambda^s f(x) + (1-\lambda)^s f(y) + d\left(\tfrac{1}{1-\lambda} x, \tfrac{1}{1-\lambda} q\right) \\ &\leq \lambda^s f(x) + (1-\lambda)^s f(y) + C d(x, q) \end{aligned}$$

which implies

$$\begin{aligned} f(x) &\geq \frac{1}{\lambda^s} f(q) - \frac{(1-\lambda)^s}{\lambda^s} f(y) - \frac{1}{\lambda^s} C d(x, q) \\ &\geq \frac{1}{\lambda^s} f(q) - \frac{(1-\lambda)^s}{\lambda^s} M - \frac{1}{\lambda^s} C d(x, q) =: M^*. \end{aligned}$$

Therefore, f is locally bounded from below at any point of D. ■

The next result states that the local upper boundedness of a rationally (s, d)-convex function at a point of D yields its continuity at this point as well.

THEOREM 5 *Assume that the function d is a continuous, translation invariant semimetric and $d(x, x) = 0$. If $f : D \to \mathbb{R}$ is $(\mathbb{Q}, s, d)$-convex and locally bounded from above at a point $w \in D$, then it is continuous at w.*

Proof. Because of f being locally bounded from above at the point $w \in D$, there exist constants $r > 0$ and $K \geq 0$ such that $f(x) \leq K$ for every $x \in B(w, r)$. Let ε be an arbitrary positive constant. Then there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ is an arbitrarily fixed positive integer, then

$$\left(\frac{1}{n}\right)^s K + \left[\left(1 - \frac{1}{n}\right)^s - 1\right] f(w) < \frac{\varepsilon}{4}, \tag{3}$$

$$\left(\frac{1}{n-1}\right)^s K + \left[1 - \frac{1}{(1-\frac{1}{n})^s}\right] f(w) < \frac{\varepsilon}{4} \tag{4}$$

and

$$\frac{2}{(1-\frac{1}{n})^s} < 3. \tag{5}$$

Let $r_1 = \min\{r, \frac{\varepsilon}{4}\}$. Using the continuity of d and the assumption $d(w,w) = 0$, there exists $r_1' < r_1$ such that $d(x,w) < r_1$ if $x \in B(w, r_1')$, and let $\delta < \frac{r_1'}{n}$. We prove that

$$|f(x) - f(w)| < \varepsilon \qquad (x \in B(w,\delta)).$$

For $x \in B(w,\delta)$ there exist $y, z \in B(w, r_1')$ such that

$$\begin{aligned} x &= \frac{1}{n}y + \left(1-\frac{1}{n}\right)w, \qquad \text{so} \qquad y = nx - (n-1)w \\ w &= \frac{1}{n}z + \left(1-\frac{1}{n}\right)x, \qquad \text{so} \qquad z = nw - (n-1)x. \end{aligned}$$

Indeed,

$$\|y - w\| = \|nx - nw\| = n\|x - w\| \le n\delta < r_1',$$

and similarly

$$\|z - w\| = \|(n-1)(x-w)\| = (n-1)\|x-w\| \le (n-1)\delta < r_1';$$

that is $y, z \in B(w, r_1')$.

According to the $(\mathbb{Q}, s, d)$-convexity of f,

$$\begin{aligned} f(x) &\le \left(\frac{1}{n}\right)^s f(y) + \left(1-\frac{1}{n}\right)^s f(w) + d(y,w) \\ &\le \left(\frac{1}{n}\right)^s K + \left(1-\frac{1}{n}\right)^s f(w) + r_1, \end{aligned} \tag{6}$$

and

$$\begin{aligned} f(w) &\le \left(\frac{1}{n}\right)^s f(z) + \left(1-\frac{1}{n}\right)^s f(x) + d(z,x) \\ &\le \left(\frac{1}{n}\right)^s K + \left(1-\frac{1}{n}\right)^s f(x) + d(z,w) + d(w,x) \\ &\le \left(\frac{1}{n}\right)^s K + \left(1-\frac{1}{n}\right)^s f(x) + 2r_1. \end{aligned} \tag{7}$$

Using (6) and (3) we get

$$\begin{aligned} f(x) - f(w) &\leq \left(\frac{1}{n}\right)^s K + \left[\left(1 - \frac{1}{n}\right)^s - 1\right] f(w) + r_1 \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \end{aligned} \tag{8}$$

Using the inequality (7) we have

$$f(x) \geq \frac{f(w) - \left(\frac{1}{n}\right)^s K - 2r_1}{(1 - \frac{1}{n})^s},$$

which together with (4) imply that

$$\begin{aligned} f(x) - f(w) &\geq \left[\frac{1}{(1-\frac{1}{n})^s} - 1\right] f(w) - \left(\frac{1}{n-1}\right)^s K - \frac{2r_1}{(1-\frac{1}{n})^s} \\ &> -\left(\frac{\varepsilon}{4} + \frac{2}{(1-\frac{1}{n})^s}\frac{\varepsilon}{4}\right). \end{aligned}$$

According to (5)

$$f(x) - f(w) > -\left(\frac{\varepsilon}{4} + 3\frac{\varepsilon}{4}\right) = -\varepsilon. \tag{9}$$

The inequalities (8) and (9) show that $|f(x) - f(w)| < \varepsilon$. Consequently, f is continuous at w, so the proof is complete. ∎

Remark 1 *This result is not true for Breckner* (λ, s, d)*-convex functions. In the case, when* $d(x, y) = 0$ *for all* $x, y \in D$*, an example was given in Burai, Házy and Juhász (2009), showing that the Breckner* $(1/2, s, 0)$*-convexity and locally upper boundedness do not imply the continuity of the function. For the reader's convenience we repeat this example: Let*

$$f(x) := \begin{cases} x^s, & \text{if } x \in](2^s - 1)^{1/s}, 1[\setminus \mathbb{Q}; \\ 1, & \text{if } x \in](2^s - 1)^{1/s}, 1[\cap \mathbb{Q}. \end{cases}$$

This function is Breckner $(1/2, s, 0)$*-convex, bounded and nowhere continuous.*

3.1. Proof of Theorem 2

It follows from property (v) that for every $\lambda \in]0, 1[$ there exists a constant C_λ such that $d\left(\frac{1}{1-\lambda}x, \frac{1}{1-\lambda}y\right) \leq C_\lambda d(x, y)$. So, according to Theorem 4, f is locally bounded at every point of D. So, we can use the previous theorem, which implies the continuity of f at every point of D.

3.2. Proof of Theorem 3

We prove that the function f is (λ, s, d)-convex for all $\lambda \in [0,1]$. Let $\lambda \in [0,1]$ be arbitrary. Then there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \in \mathbb{Q}$ and $\lambda_n \to \lambda$ (when n tends to ∞). Applying $(\mathbb{Q}, s, d)$-convexity of f, we get

$$f\left(\lambda_n x + (1-\lambda_n)y\right) \leq \lambda_n^s f(x) + (1-\lambda_n)^s f(y) + d(x,y). \tag{10}$$

The local upper boundedness of f implies the continuity of f (according to Theorem 2). Therefore, taking the limit $n \to \infty$ in (10), we get

$$f\left(\lambda x + (1-\lambda)y\right) \leq \lambda^s f(x) + (1-\lambda)^s f(y) + d(x,y),$$

which proves the Breckner (s,d)-convexity of f.

4. Applications

Corollary 1 *If $f : D \to \mathbb{R}$ is Breckner $(\mathbb{Q}, s, d)$-convex and locally bounded from above at a point of D (where d is a continuous, translation invariant semimetric which fulfills (v)), then f is locally bounded on D.*

The next statement is an immediate consequence of the previous theorem and Steinhaus' and Piccard's theorems (see Steinhaus, 1920; Piccard, 1942).

Corollary 2 *Let D be an open convex subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ be a (λ, s, d)-convex (or $(\mathbb{Q}, s, d)$-convex) function. Assume that the function d is a continuous, translation invariant semimetric and there exists a constant $C > 0$ such that $d\left(\frac{1}{1-\lambda}x, \frac{1}{1-\lambda}y\right) \leq Cd(x,y)$ (or fulfills (v)) and there exist a Lebesgue-measurable set of positive measure (or a Baire-measurable set of second category) $S \subseteq D$ and a Lebesgue-measurable (respectively Baire-measurable) function $g : S \to \mathbb{R}$ such that $f \leq g$ on S. Then f is locally bounded on D.*

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