

Existence, uniqueness and convergence of simultaneous
distributed-boundary optimal control problems*

by

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Abstract: We consider a steady-state heat conduction problem P for the Poisson equation with mixed boundary conditions in a bounded multidimensional domain Ω . We also consider a family of problems P_α for the same Poisson equation with mixed boundary conditions, $\alpha > 0$ being the heat transfer coefficient defined on a portion Γ_1 of the boundary. We formulate simultaneous *distributed and Neumann boundary* optimal control problems on the internal energy g within Ω and the heat flux q , defined on the complementary portion Γ_2 of the boundary of Ω for quadratic cost functional. Here, the control variable is the vector (g, q) . We prove existence and uniqueness of the optimal control (\bar{g}, \bar{q}) for the system state of P , and $(\bar{g}_\alpha, \bar{q}_\alpha)$ for the system state of P_α , for each $\alpha > 0$, and we give the corresponding optimality conditions. We prove strong convergence, in suitable Sobolev spaces, of the vectorial optimal controls, system and adjoint states governed by the problems P_α to the corresponding vectorial optimal control, system and adjoint states governed by the problem P , when the parameter α goes to infinity. We also obtain estimations between the solutions of these vectorial optimal control problems and the solution of two scalar optimal control problems characterized by fixed g (with boundary optimal control \bar{q}) and fixed q (with distributed optimal control \bar{g}), respectively, for cases both of $\alpha > 0$ and $\alpha = \infty$.

Keywords: simultaneous optimal control problems, mixed elliptic problems, optimality condition, elliptic variational equalities, vectorial optimal control problems

1. Introduction

We consider a bounded domain Ω in \mathbb{R}^n , whose regular boundary Γ consist of the union of the two disjoint portions Γ_1 and Γ_2 with $med(\Gamma_i) > 0$ for $i = 1, 2$. We consider the following steady-state heat conduction problems P and P_α (for each parameter value $\alpha > 0$) respectively, with mixed boundary conditions:

$$-\Delta u = g \text{ in } \Omega \quad u|_{\Gamma_1} = b \quad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \quad (1)$$

$$-\Delta u = g \text{ in } \Omega \quad -\frac{\partial u}{\partial n}|_{\Gamma_1} = \alpha(u - b) \quad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \quad (2)$$

where g is the internal energy in Ω , $b > 0$ is the temperature on Γ_1 for (1) and the temperature of the external neighborhood of Γ_1 for (2), q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient on Γ_1 (Newton law or Robin condition on Γ_1). The following hypothesis: $g \in L^2(\Omega)$, $q \in L^2(\Gamma_2)$, and $b \in H^{\frac{1}{2}}(\Gamma_1)$ is assumed to hold. Problems (1) and (2) can be considered as the steady-state Stefan problem for suitable data q , g and b , see Tarzia (1979), or Tabacman and Tarzia (1989).

We denote by $u_{(g,q)}$ and $u_{(\alpha,g,q)}$ the unique solutions of the elliptic problems (1) and (2), respectively, whose variational formulations are given, as in Kinderlehrer and Stampacchia (1980), by:

$$a(u_{(g,q)}, v) = L_{(g,q)}(v), \quad \forall v \in V_0, \quad u_{(g,q)} \in K \quad (3)$$

$$a_\alpha(u_{(\alpha,g,q)}, v) = L_{(\alpha,g,q)}(v), \quad \forall v \in V, \quad u_{(\alpha,g,q)} \in V \quad (4)$$

where

$$V = H^1(\Omega), \quad V_0 = \{v \in V / v|_{\Gamma_1} = 0\}, \quad K = v_0 + V_0,$$

$$R = L^2(\Gamma_1), \quad H = L^2(\Omega), \quad Q = L^2(\Gamma_2)$$

for $v_0 \in V$ given, with $v_0|_{\Gamma_1} = b$ and

$$(g, h)_H = \int_{\Omega} gh \, dx; \quad (q, \eta)_Q = \int_{\Gamma_2} q\eta \, d\gamma, \quad (b, v)_R = \int_{\Gamma_1} bv \, d\gamma$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx; \quad a_\alpha(u, v) = a(u, v) + \alpha (u, v)_R$$

$$L_{(g,q)}(v) = (g, v)_H - (q, v)_Q; \quad L_{(\alpha,g,q)}(v) = L_{(g,q)}(v) + \alpha (b, v)_R.$$

The bilinear form a is coercive on V_0 , with coerciveness constant $\lambda > 0$ and the bilinear form a_α is coercive on V with coerciveness constant $\lambda_\alpha = \lambda_1 \min(1, \alpha)$, where $\lambda_1 > 0$ is the coerciveness constant for the bilinear form a_1 , see Kinderlehrer and Stampacchia (1980), or Tabacman and Tarzia (1989).

We formulate the following simultaneous *distributed and Neumann boundary* optimal control problems, see Lions (1968) or Tröltzsch (2010):

$$\text{Find } (\bar{g}, \bar{q}) \in H \times U_{ad} \quad \text{such that} \quad J(\bar{g}, \bar{q}) = \min_{g \in H, q \in U_{ad}} J(g, q) \quad (5)$$

$$\text{Find } (\bar{g}_\alpha, \bar{q}_\alpha) \in H \times U_{ad} \quad \text{such that} \quad J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) = \min_{g \in H, q \in U_{ad}} J_\alpha(g, q) \quad (6)$$

with $U_{ad} = \{q \in Q : q \geq 0 \text{ on } \Gamma_2\}$ and the cost functionals $J : H \times Q \rightarrow \mathbb{R}_0^+$ and $J_\alpha : H \times Q \rightarrow \mathbb{R}_0^+$ being given by:

$$J(g, q) = \frac{1}{2} \|u_{(g,q)} - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2 \quad (7)$$

$$J_\alpha(g, q) = \frac{1}{2} \|u_{(\alpha,g,q)} - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2 \quad (8)$$

where $z_d \in H$, $u_{(g,q)}$ and $u_{(\alpha,g,q)}$ are the unique solutions of the elliptic variational equalities (3) and (4) respectively, and the positive constants M_1 and M_2 are given. We remark that we denote the control variables by g and q , these two variables corresponding usually to the internal energy and the heat flux respectively, in heat transfer problems.

The use of the variational equality theory in connection with optimization and optimal control problems was done in Belgacem, El Fekih and Metoui (2003), Bensoussan (1974), Casas (1986), Casas and Raymond (2006), Kirchner, Meidner and Vexler (2011), Mignot and Puel (1984).

In Section 2, we obtain the existence and uniqueness of the vectorial optimal control (\bar{g}, \bar{q}) of the problem (5) and of the vectorial optimal control $(\bar{g}_\alpha, \bar{q}_\alpha)$ of the problem (6), for each $\alpha > 0$. We also give the optimality conditions in relation to the adjoint state $p_{(\bar{g}, \bar{q})}$ for (5) and $p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$ for (6).

In Section 3, we obtain estimations between the first component of the simultaneous optimal control \bar{g} and the scalar optimal control \bar{g} studied in Gariboldi and Tarzia (2003) (see the optimization problem (18)), and the second component of the simultaneous optimal control \bar{q} and the scalar optimal control \bar{q} studied in Gariboldi and Tarzia (2008) (see the optimization problem (19)). In the optimal control problems (5) and (6) we have considered two control variables simultaneously, that is, the solution is a vectorial optimal control, while in the optimal control problems, given in Gariboldi and Tarzia (2003) and Gariboldi and Tarzia (2008), respectively, we have considered only one control variable, namely the solutions are scalar optimal controls. Moreover, we characterize the optimal control (\bar{g}, \bar{q}) as a fixed point on $H \times Q$ for a suitable operator W . In a similar way, we obtain estimations for the optimal controls of the problems P_α , for each $\alpha > 0$, and we characterize the optimal control $(\bar{g}_\alpha, \bar{q}_\alpha)$ as a fixed point on $H \times Q$ for a suitable operator W_α .

In Section 4, we prove the strong convergence, in suitable Sobolev spaces, of the optimal controls $(\bar{g}_\alpha, \bar{q}_\alpha)$ of the problems (6) to the optimal control (\bar{g}, \bar{q}) of the problem (5), of the system states $u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$ to the system state $u_{(\bar{g}, \bar{q})}$, and

of the adjoint states $p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$ to the adjoint state $p_{(\bar{g}, \bar{q})}$, when the parameter α goes to infinity. We also prove the convergence of the corresponding cost functional when α goes to infinity.

This asymptotic behavior can be considered to be very important in the optimal control of heat transfer problems, because the Dirichlet boundary condition, given in (1), can be approximated by the relevant physical condition given by the Newton law or the Robin boundary condition given in (2), see Carslaw and Jaeger (1959). Therefore, the goal of this paper is to approximate a Dirichlet boundary condition in a vectorial optimal control problem, governed by an elliptic variational equality, by a Robin boundary condition in a family of vectorial optimal control problems, governed also by elliptic variational equalities, for a large positive coefficient α . As the particular cases of our results can be considered the ones given in Gariboldi and Tarzia (2003) when the scalar control variable is the internal energy g for both state systems (1) and (2), and in Gariboldi and Tarzia (2008) when the scalar control variable is the heat flux q on the boundary Γ_2 for both state systems (1) and (2). In Belgacem, El Fekih and Metoui (2003) the control variable is the temperature b on the boundary Γ_1 for the state system (1), and the temperature of the external neighborhood b on the boundary Γ_1 for the state systems (2), this being essentially different with respect to the present vectorial optimal control problems.

2. Existence and uniqueness of optimal controls

2.1. Problem P and its optimal control problem

Let $C : H \times Q \rightarrow V_0$ be the application defined by $C(g, q) = u_{(g, q)} - u_{(0, 0)}$, where $u_{(0, 0)}$ is the solution of the problem (1) for $g = 0$ and $q = 0$. We define, in the way similar to that in Gariboldi and Tarzia (2003, 2008), and Lions (1968), the applications $\Pi : (H \times Q) \times (H \times Q) \rightarrow \mathbb{R}$, and $L : H \times Q \rightarrow \mathbb{R}$ as follows:

$$\Pi((g, q), (h, \eta)) = (C(g, q), C(h, \eta))_H + M_1(g, h)_H + M_2(q, \eta)_Q$$

$$L((g, q)) = (C(g, q), z_d - u_{(0, 0)})_H, \quad \forall (g, q), (h, \eta) \in H \times Q.$$

For each $(g, q) \in H \times Q$, we define the adjoint state $p_{(g, q)}$ corresponding to the problem (1), as the unique solution of the mixed elliptic problem

$$-\Delta p = u - z_d \text{ in } \Omega, \quad p|_{\Gamma_1} = 0, \quad \frac{\partial p}{\partial n}|_{\Gamma_2} = 0 \quad (9)$$

whose variational formulation is given by

$$a(p_{(g, q)}, v) = (u_{(g, q)} - z_d, v)_H, \quad \forall v \in V_0, \quad p_{(g, q)} \in V_0 \quad (10)$$

and we have the following properties.

THEOREM 1 *a) J is a coercive and strictly convex functional on $H \times Q$.*

b) The adjoint state $p_{(g,q)}$ satisfy, $\forall (h, \eta) \in H \times Q$:

$$a(p_{(g,q)}, C(h, \eta)) = (C(h, \eta), u_{(g,q)} - z_d)_H = (h, p_{(g,q)})_H - (\eta, p_{(g,q)})_Q$$

c) J is Gâteaux differentiable and J' is given by, $\forall (h, \eta) \in H \times Q$:

$$J'(g, q)(h - g, \eta - q) = \Pi((g, q), (h - g, \eta - q)) - L(h - g, \eta - q) \quad (11)$$

d) There exists a unique solution $(\bar{g}, \bar{q}) \in H \times U_{ad}$ of the vectorial optimal control problem (5) and its optimality condition is given by, $\forall (h, \eta) \in H \times U_{ad}$:

$$(h - \bar{g}, p_{(\bar{g}, \bar{q})} + M_1 \bar{g})_H + (\eta - \bar{q}, M_2 \bar{q} - p_{(\bar{g}, \bar{q})})_Q \geq 0.$$

PROOF (a) It is sufficient to prove that, $\forall (g_2, q_2), (g_1, q_1) \in H \times Q$ and $\forall t \in [0, 1]$, we have, see Lions (1968), Boukrouche and Tarzia (2007), or Tröltzsch (2010):

$$u_{((1-t)g_2+tg_1, (1-t)q_2+tq_1)} = (1-t)u_{(g_2, q_2)} + tu_{(g_1, q_1)} \quad (12)$$

and

$$\begin{aligned} & (1-t)J(g_2, q_2) + tJ(g_1, q_1) - J((1-t)(g_2, q_2) + t(g_1, q_1)) = \\ &= \frac{t(1-t)}{2} [\|u_{(g_2, q_2)} - u_{(g_1, q_1)}\|_H^2 + M_1 \|g_2 - g_1\|_H^2 + M_2 \|q_2 - q_1\|_Q^2] \geq \\ &\geq \frac{Mt(1-t)}{2} \|(g_2 - g_1, q_2 - q_1)\|_{H \times Q}^2, \end{aligned} \quad (13)$$

and

$$\begin{aligned} a(p_{(g,q)}, C(h, \eta)) &= (-\Delta p_{(g,q)}, u_{(g,q)} - u_{(0,0)})_H = \\ &= (h, p_{(g,q)})_H - (\eta, p_{(g,q)})_Q \end{aligned} \quad (14)$$

where

$$\|(g, q)\|_{H \times Q}^2 = \|g\|_H^2 + \|q\|_Q^2, \quad M = \min(M_1, M_2) > 0. \quad \square$$

2.2. Problem P_α and its optimal control problem

Let $C_\alpha : H \times Q \rightarrow V$ be the application defined by $C_\alpha(g, q) = u_{(\alpha, g, q)} - u_{(\alpha, 0, 0)}$, where $u_{(\alpha, 0, 0)}$ is the solution of the problem (2) for $g = 0$ and $q = 0$. We define the applications $\Pi_\alpha : (H \times Q) \times (H \times Q) \rightarrow \mathbb{R}$ and $L_\alpha : H \times Q \rightarrow \mathbb{R}$ by the following expressions, $\forall (g, q), (h, \eta) \in H \times Q$:

$$\Pi_\alpha((g, q), (h, \eta)) = (C_\alpha(g, q), C_\alpha(h, \eta))_H + M_1(g, h)_H + M_2(q, \eta)_Q,$$

$$L_\alpha(q) = (C_\alpha(g, q), z_d - u_{(\alpha, 0, 0)})_H.$$

For each $(g, q) \in H \times Q$ and $\alpha > 0$, we define the adjoint state $p_{(\alpha, g, q)}$, corresponding to the problem (2), as the unique solution of the mixed elliptic problem

$$-\Delta p = u - z_d \text{ in } \Omega, \quad -\frac{\partial p}{\partial n}\Big|_{\Gamma_1} = \alpha p, \quad \frac{\partial p}{\partial n}\Big|_{\Gamma_2} = 0, \quad (15)$$

whose variational formulation is given by

$$a_\alpha(p_{(\alpha, g, q)}, v) = (u_{(\alpha, g, q)} - z_d, v)_H, \quad \forall v \in V, \quad p_{(\alpha, g, q)} \in V. \quad (16)$$

We can obtain similar properties to Theorem 2.1, following Boukrouche and Tarzia (2007), Kinderlehrer and Stampacchia (1980), Lions (1968), or Tröltzsch (2010).

THEOREM 2 *We have, for each $\alpha > 0$, the following properties:*

- a) J_α is a coercive and strictly convex functional on $H \times Q$.
- b) The adjoint state $p_{(\alpha, g, q)}$ satisfies, $\forall (h, \eta) \in H \times Q$:

$$a_\alpha(p_{(\alpha, g, q)}, C_\alpha(h, \eta)) = (C_\alpha(h, \eta), u_{(\alpha, g, q)} - z_d)_H = (h, p_{(\alpha, g, q)})_H - (\eta, p_{(\alpha, g, q)})_Q.$$

- c) J_α is Gâteaux differentiable and J'_α is given by, $\forall (h, \eta) \in H \times Q$:

$$J'_\alpha(g, q)(h - g, \eta - q) = \Pi_\alpha((g, q), (h - g, \eta - q)) - L_\alpha(h - g, \eta - q). \quad (17)$$

- d) There exists a unique solution $(\bar{g}_\alpha, \bar{q}_\alpha) \in H \times U_{ad}$ of the vectorial optimal control problem (6) and its optimality condition is given by, $\forall (h, \eta) \in H \times U_{ad}$:

$$(h - \bar{g}_\alpha, p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} + M_1 \bar{g}_\alpha)_H + (\eta - \bar{q}_\alpha, M_2 \bar{q}_\alpha - p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)})_Q \geq 0.$$

3. Estimations

3.1. Estimations with respect to the problem P

We consider the scalar distributed optimal control problem:

$$\text{Find } \bar{g} \in H \quad \text{such that} \quad J_1(\bar{g}) = \min_{g \in H} J_1(g), \quad \text{for fixed } q \in Q, \quad (18)$$

and the scalar Neumann boundary optimal control problem:

$$\text{Find } \bar{q} \in U_{ad} \quad \text{such that} \quad J_2(\bar{q}) = \min_{q \in U_{ad}} J_2(q), \quad \text{for fixed } g \in H, \quad (19)$$

where J_1 is the cost functional given in Gariboldi and Tarzia (2003) plus the constant $\frac{M_2}{2} \|q\|_Q^2$, and J_2 is the functional given in Gariboldi and Tarzia (2008) plus the constant $\frac{M_1}{2} \|g\|_H^2$, that is, $J_1 : H \rightarrow \mathbb{R}_0^+$ and $J_2 : Q \rightarrow \mathbb{R}_0^+$, are given by:

$$J_1(g) = \frac{1}{2} \|u_g - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2, \quad (\text{fixed } q \in Q) \quad (20)$$

$$J_2(q) = \frac{1}{2} \|u_q - z_d\|_H^2 + \frac{M_2}{2} \|q\|_Q^2 + \frac{M_1}{2} \|g\|_H^2, \quad (\text{fixed } g \in H) \quad (21)$$

where u_g and u_q are the unique solutions of the problem (1) for fixed q and g data, respectively.

REMARK 1 *The functionals J , J_1 and J_2 satisfy the elemental estimations*

$$J(\bar{g}, \bar{q}) \leq J_1(\bar{g}), \quad \forall q \in Q \quad \text{and} \quad J(\bar{g}, \bar{q}) \leq J_2(\bar{q}), \quad \forall g \in H.$$

In the next theorem we will obtain estimations between the solution of the scalar distributed optimal control problem (18) with the first component of the solution of the vectorial distributed and Neumann boundary optimal control problem (5), and between the solution of the scalar Neumann boundary optimal control problem (19) with the second component of the solution of the vectorial distributed and Neumann boundary optimal control problem (5).

THEOREM 3 *If $(\bar{g}, \bar{q}) \in H \times U_{ad}$ is the unique solution of the vectorial optimal control problem (5), and \bar{g} and \bar{q} are the unique solutions of the scalar optimal control problems (18) and (19) respectively, then:*

$$\|\bar{q} - \bar{q}\|_Q \leq \frac{\|\gamma_0\|}{\lambda M_2} \|u_{(\bar{g}, \bar{q})} - u_{(\bar{g}, \bar{q})}\|_H \quad (22)$$

$$\|\bar{g} - \bar{g}\|_H \leq \frac{1}{\lambda M_1} \|u_{(\bar{g}, \bar{q})} - u_{(\bar{g}, \bar{q})}\|_H, \quad (23)$$

where γ_0 is the trace operator.

PROOF For $g = \bar{g}$, from the optimality condition for \bar{q} , see Gariboldi and Tarzia (2008), we have

$$(M_2 \bar{q} - p_{(\bar{g}, \bar{q})}, \eta - \bar{q})_Q \geq 0, \quad \forall \eta \in U_{ad}. \quad (24)$$

If we take $h = \bar{g} \in H$ in the optimality condition for (\bar{g}, \bar{q}) , we obtain

$$(M_2 \bar{q} - p_{(\bar{g}, \bar{q})}, \eta - \bar{q})_Q \geq 0, \quad \forall \eta \in U_{ad}. \quad (25)$$

Now, taking $\eta = \bar{q} \in U_{ad}$ in (24) and $\eta = \bar{q} \in U_{ad}$ in (25), we obtain

$$(M_2(\bar{q} - \bar{q}) + (p_{(\bar{g}, \bar{q})} - p_{(\bar{g}, \bar{q})}), \bar{q} - \bar{q})_Q \geq 0,$$

and by using $\|p_{(\bar{g}, \bar{q})} - p_{(\bar{g}, \bar{q})}\|_V \leq \frac{1}{\lambda} \|u_{(\bar{g}, \bar{q})} - u_{(\bar{g}, \bar{q})}\|_H$ we deduce

$$\|\bar{q} - \bar{q}\|_Q \leq \frac{\|\gamma_0\|}{M_2} \|p_{(\bar{g}, \bar{q})} - p_{(\bar{g}, \bar{q})}\|_V \leq \frac{\|\gamma_0\|}{\lambda M_2} \|u_{(\bar{g}, \bar{q})} - u_{(\bar{g}, \bar{q})}\|_H;$$

therefore the estimation (22) holds. Similarly, the estimation (23) holds. \square

When we consider the vectorial distributed and Neumann boundary optimal control problem (5) without restrictions, i.e. $U_{ad} = Q$, then we can characterize the solution of (5) by using the fixed point theory.

Let $W : H \times Q \rightarrow H \times Q$ be the operator defined by

$$W(g, q) = \left(-\frac{1}{M_1} p_{(g, q)}, \frac{1}{M_2} p_{(g, q)}\right). \quad (26)$$

THEOREM 4 *There exists a positive constant $C_0 = C_0(\lambda, \gamma_0, M_1, M_2)$ such that, $\forall (g_1, q_1), (g_2, q_2) \in H \times Q$:*

$$\|W(g_2, q_2) - W(g_1, q_1)\|_{H \times Q} \leq C_0 \|(g_2, q_2) - (g_1, q_1)\|_{H \times Q} \quad (27)$$

and W is a contraction operator if and only if data satisfy the following condition:

$$C_0 = \frac{\sqrt{2}}{\lambda^2} \sqrt{\frac{1}{M_1^2} + \frac{\|\gamma_0\|^2}{M_2^2}} (1 + \|\gamma_0\|) < 1. \quad (28)$$

PROOF By using the estimations, $\forall (g_1, q_1), (g_2, q_2) \in H \times Q$:

$$\|u_{(g_1, q_1)} - u_{(g_2, q_2)}\|_V \leq \frac{1}{\lambda} (\|g_2 - g_1\|_H + \|\gamma_0\| \|q_2 - q_1\|_Q), \quad (29)$$

$$\|p_{(g_1, q_1)} - p_{(g_2, q_2)}\|_V \leq \frac{1}{\lambda} \|u_{(g_1, q_1)} - u_{(g_2, q_2)}\|_H \quad (30)$$

we obtain

$$\|W(g_2, q_2) - W(g_1, q_1)\|_{H \times Q}^2 \leq \left(\frac{1}{M_1^2} + \frac{\|\gamma_0\|^2}{M_2^2} \right) \frac{1}{\lambda^4} [\|g_2 - g_1\|_H + \|\gamma_0\| \|q_2 - q_1\|_Q]^2.$$

Then (27) holds and the operator W is a contraction if and only if data satisfy inequality (28). \square

COROLLARY 1 *If data satisfy inequality (28) then the unique solution $(\bar{g}, \bar{q}) \in H \times Q$ of the vectorial optimal control problem (5) can be obtained as the unique fixed point of the operator W , that is:*

$$W(\bar{g}, \bar{q}) = \left(-\frac{1}{M_1} p_{(\bar{g}, \bar{q})}, \frac{1}{M_2} p_{(\bar{g}, \bar{q})} \right) = (\bar{g}, \bar{q}).$$

3.2. Estimations with respect to the problem P_α

For each $\alpha > 0$, we consider the scalar optimal control problems:

$$\text{Find } \bar{g}_\alpha \in H \quad \text{such that} \quad J_{1\alpha}(\bar{g}_\alpha) = \min_{g \in H} J_{1\alpha}(g), \quad (31)$$

$$\text{Find } \bar{q}_\alpha \in U_{ad} \quad \text{such that} \quad J_{2\alpha}(\bar{q}_\alpha) = \min_{q \in U_{ad}} J_{2\alpha}(q), \quad (32)$$

where $J_{1\alpha} : H \rightarrow \mathbb{R}_0^+$ and $J_{2\alpha} : Q \rightarrow \mathbb{R}_0^+$ are given by:

$$J_{1\alpha}(g) = \frac{1}{2} \|u_{\alpha g} - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2, \quad (\text{fixed } q \in Q) \quad (33)$$

$$J_{2\alpha}(q) = \frac{1}{2} \|u_{\alpha q} - z_d\|_H^2 + \frac{M_2}{2} \|q\|_Q^2 + \frac{M_1}{2} \|g\|_H^2, \quad (\text{fixed } g \in H) \quad (34)$$

where $J_{1\alpha}$ is the functional studied in Gariboldi and Tarzia (2003) plus the constant $\frac{M_2}{2} \|q\|_Q^2$, $J_{2\alpha}$ is the functional studied in Gariboldi and Tarzia (2008) plus the constant $\frac{M_1}{2} \|g\|_H^2$, and the system states $u_{\alpha g}$ and $u_{\alpha q}$ are the unique solutions of the problem (2) for fixed data q and g , respectively.

REMARK 2 *The functionals J_α , $J_{1\alpha}$ and $J_{2\alpha}$ satisfy the estimations*

$$J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) \leq J_{1\alpha}(\bar{g}_\alpha), \forall q \in Q \text{ and } J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) \leq J_{2\alpha}(\bar{q}_\alpha), \forall g \in H.$$

Estimations between the solution of the scalar distributed optimal control problem (31) with respect to the first component of the solution of the vectorial distributed and Neumann boundary optimal control problem (6), and estimations between the solution of the scalar Neumann boundary optimal control problem (32) with respect to the second component of the solution of the vectorial distributed and Neumann boundary optimal control problem (6) are given in the next theorem whose proof is omitted.

THEOREM 5 *If $(\bar{g}_\alpha, \bar{q}_\alpha) \in H \times U_{ad}$ is the unique solution of the vectorial optimal control problem (6), and \bar{g}_α and \bar{q}_α are the unique solutions of the scalar optimal control problems (31) and (32), respectively, then we have the following estimations*

$$\begin{aligned} \|\bar{q}_\alpha - \bar{q}_\alpha\|_Q &\leq \frac{\|\gamma_0\|}{\lambda M_2} \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}\|_H \\ \|\bar{g}_\alpha - \bar{g}_\alpha\|_H &\leq \frac{1}{\lambda M_1} \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}\|_H. \end{aligned}$$

In the way similar to Theorem 4, we can now characterize the solution of the vectorial distributed and Neumann boundary optimal control problem (6), without restrictions, proving that a suitable operator W_α is a contraction. It is presented in the next theorem and the proof is omitted. We define the operator $W_\alpha : H \times Q \rightarrow H \times Q$, for each $\alpha > 0$, by the expression

$$W_\alpha(g, q) = \left(-\frac{1}{M_1} p(\alpha, g, q), \frac{1}{M_2} p(\alpha, g, q)\right). \quad (35)$$

THEOREM 6 *W_α is a Lipschitz operator over $H \times Q$, that is, there exists a positive constant $C_{0\alpha} = C_{0\alpha}(\lambda_\alpha, \gamma_0, M_1, M_2)$, such that:*

$$\|W_\alpha(g_2, q_2) - W_\alpha(g_1, q_1)\|_{H \times Q} \leq C_{0\alpha} \|(g_2 - g_1, q_2 - q_1)\|_{H \times Q} \quad (36)$$

and W_α is a contraction operator if and only if data satisfy the following inequality:

$$C_{0\alpha} = \frac{\sqrt{2}}{\lambda_\alpha^2} \sqrt{\frac{1}{M_1^2} + \frac{\|\gamma_0\|^2}{M_2^2}} (1 + \|\gamma_0\|) < 1. \quad (37)$$

COROLLARY 2 *If data satisfy inequality $C_{0\alpha} < 1$, then the unique solution $(\bar{g}_\alpha, \bar{q}_\alpha) \in H \times Q$ of the vectorial optimal control problem (6) can be obtained as the unique fixed point of the operator W_α , that is:*

$$W_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) = \left(-\frac{1}{M_1} p(\alpha, \bar{g}_\alpha, \bar{q}_\alpha), \frac{1}{M_2} p(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)\right) = (\bar{g}_\alpha, \bar{q}_\alpha).$$

4. Convergence when $\alpha \rightarrow +\infty$

LEMMA 1 *For each $\alpha > 0$, $(g, q) \in H \times Q$, $b \in H^{1/2}(\Gamma_1)$, we have the following limits:*

$$i) \lim_{\alpha \rightarrow \infty} \|u_{(\alpha, g, q)} - u_{(g, q)}\|_V = 0 \quad ii) \lim_{\alpha \rightarrow \infty} \|p_{(\alpha, g, q)} - p_{(g, q)}\|_V = 0. \quad (38)$$

PROOF We proceed in a similar way to the one given in Gariboldi and Tarzia (2003) and in Gariboldi and Tarzia (2008).

THEOREM 7 *i) If $u_{(\bar{g}, \bar{q})}$ and $u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$ are the unique system states, corresponding of the vectorial optimal control problems (5) and (6), respectively, then:*

$$\lim_{\alpha \rightarrow \infty} \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})}\|_V = 0. \quad (39)$$

ii) If $p_{(\bar{g}, \bar{q})}$ and $p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$ are the unique adjoint states, corresponding to the vectorial optimal control problems (5) and (6), respectively, then:

$$\lim_{\alpha \rightarrow \infty} \|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}\|_V = 0. \quad (40)$$

iii) If (\bar{g}, \bar{q}) and $(\bar{g}_\alpha, \bar{q}_\alpha)$ are the unique solutions of the simultaneous distributed and Neumann boundary optimal control problems (5) and (6), respectively, then:

$$\lim_{\alpha \rightarrow \infty} \|(\bar{g}_\alpha, \bar{q}_\alpha) - (\bar{g}, \bar{q})\|_{H \times Q} = 0. \quad (41)$$

PROOF The proof is given in two steps:

Step 1. From the optimal control problem (6) we deduce that there exist positive constants C_1 , C_2 , and C_3 , independent of α , such that

$$\|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - z_d\|_H \leq C_1, \quad \|\bar{g}_\alpha\|_H \leq C_2, \quad \|\bar{q}_\alpha\|_Q \leq C_3. \quad (42)$$

Now, if we take $v = u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})} \in V$ in the variational equality (4), following Gariboldi and Tarzia (2003) or Gariboldi and Tarzia (2008), we obtain, for $\alpha > 1$, $\|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}\|_V \leq C_4$ where $C_4 = C_4(C_2, C_3, \gamma_0, u_{(\bar{g}, \bar{q})}, \lambda_1)$ is independent of α . Therefore,

$$\exists \mu \in K \text{ such that } u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} \rightharpoonup \mu \text{ weakly in } V, \text{ when } \alpha \rightarrow +\infty. \quad (43)$$

Taking $v = p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})} \in V$ in the variational equality (16), we obtain that there exists a positive constant $C_5 = C_5(C_1, p_{(\bar{g}, \bar{q})}, \lambda_1)$, such that $\|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}\|_V \leq C_5$ and next

$$\exists \xi \in V_0 \text{ such that } p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} \rightharpoonup \xi \text{ weakly in } V, \text{ when } \alpha \rightarrow +\infty. \quad (44)$$

Moreover, from (42), we deduce that there exist $f \in Q$ and $h \in H$ such that

$$\bar{q}_\alpha \rightharpoonup f \text{ weakly in } Q, \text{ when } \alpha \rightarrow +\infty \quad (45)$$

$$\bar{g}_\alpha \rightharpoonup h \text{ weakly in } H, \text{ when } \alpha \rightarrow +\infty. \quad (46)$$

For $v \in V_0$, taking into account (43), (45), (46) and taking the limit as α goes to infinity, we have that

$$a(\mu, v) = (h, v)_H - (f, v)_Q, \quad \forall v \in V_0, \mu \in K \quad (47)$$

and by the uniqueness of the solution of (3), we get $\mu = u_{hf}$.

Now, for $v \in V_0$, taking into account (44), with the parameter α going to infinity in the variational equality (16), we have that

$$a(\xi, v) = (u_{hf} - z_d, v)_H, \quad \forall v \in V_0, \xi \in V_0 \quad (48)$$

and from the uniqueness of the solution of (10), we get $\xi = p_{hf}$. Next,

$$\begin{aligned} J(h, f) &\leq \liminf_{\alpha \rightarrow \infty} J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) \leq \liminf_{\alpha \rightarrow \infty} J_\alpha(h', f') = \\ &= \lim_{\alpha \rightarrow \infty} J_\alpha(h', f') = J(h', f'), \quad \forall (h', f') \in H \times Q, \end{aligned}$$

and from the uniqueness of the solution to the problem (5), we have that $h = \bar{g}$ and $f = \bar{q}$. Therefore, we have proved that

$$u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} \rightharpoonup u_{(\bar{g}, \bar{q})} \text{ weakly in } V, \text{ when } \alpha \rightarrow +\infty \quad (49)$$

$$p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} \rightharpoonup p_{(\bar{g}, \bar{q})} \text{ weakly in } V, \text{ when } \alpha \rightarrow +\infty. \quad (50)$$

Step 2. Taking $h = 0$ and $\eta = \bar{q}$ in the optimality condition for the vectorial optimal control problem (6), $h = 0$ and $\eta = \bar{q}_\alpha$ in the optimality condition for the vectorial optimal control problem (5), we have

$$(\bar{q}_\alpha - \bar{q}, M_2(\bar{q} - \bar{q}_\alpha) + (p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}))_Q \geq 0;$$

then, we deduce that

$$\|\bar{q} - \bar{q}_\alpha\|_Q \leq \frac{\|\gamma_0\|}{M_2} \|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}\|_V. \quad (51)$$

Next, in a similar way, taking $h = \bar{g}$ and $\eta = 0$ in the optimality condition for the problem (6) and $h = \bar{g}_\alpha$ and $\eta = 0$ in the optimality condition for the problem (5), we prove that

$$\|\bar{g} - \bar{g}_\alpha\|_H \leq \frac{1}{M_2} \|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}\|_V. \quad (52)$$

Now, from (49) and the following inequalities, for $\alpha > 1$,

$$\begin{aligned}
& \lambda_1 \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})}\|_V^2 + (\alpha - 1) \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})}\|_R^2 \leq \\
& \leq (g, u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})})_H - (q, u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})})_Q \\
& \quad - a(u_{(\bar{g}, \bar{q})}, u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})})
\end{aligned}$$

we get that the result (39) holds. In a similar way, from (50) and the inequality

$$\begin{aligned}
& \lambda_1 \|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}\|_V^2 \leq \\
& \leq (u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - z_d, p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})})_H - \\
& \quad - a(p_{(\bar{g}, \bar{q})}, p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}) - \alpha (p_{(\bar{g}, \bar{q})}, p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})})_R
\end{aligned}$$

we obtain the limit (40). Finally, from (39), (40), and the estimations (51) and (52), we deduce that the limit (41) holds. \square

COROLLARY 3 *If (\bar{g}, \bar{q}) and $(\bar{g}_\alpha, \bar{q}_\alpha)$ are the unique solutions of the simultaneous distributed and Neumann boundary optimal control problems (5) and (6), respectively, then we have:*

$$\lim_{\alpha \rightarrow \infty} |J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) - J(\bar{g}, \bar{q})| = 0.$$

PROOF The conclusion follows from the definition of J , J_α and the last theorem. \square

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