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Existence of minimizers for optical flow based optimal control problems under mild regularity assumptions*

by

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Dedicated to Günter Leugering on the occasion of his 65th birthday

Abstract: Optimal control problems governed by a transport equation are investigated that are motivated by optical flow problems. The control is given by the velocity field, corresponding to the optical flow, while the state corresponds to the brightness of image points. The problem is studied in the setting of spatially BV-regular vector fields under very low regularity requirements. Existing stability results for the control-to-state operator are improved and based on this the existence of minimizers for several classes of optimal control problems is proved under mild assumptions on the admissible sets.

Keywords: optimal control, optical flow, transport equation, renormalized solutions, BV vector spaces

1. Introduction

In this paper, we investigate optimal control problems governed by transport equations, where the control is the velocity field. The main focus lies in the analysis of the problem, in particular – existence of optimal controls, under very low regularity requirements on the velocity field and also on the state. The problem class considered is motivated by optical flow based image sequence interpolation. Optical flow basically describes the vector field of velocities of apparent points in the 2D image plane. Assuming that image points of a scene do

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not change their brightness over time while moving, the brightness $u : (0,T)\times\Omega$, with $\Omega \subset \mathbb{R}^2$ denoting the image domain, satisfies a transport equation, where the velocity field is given by the optical flow $b : (0,T)\times\Omega \to \mathbb{R}^2$. The goal of the optical flow problem is to recover b from image data that correspond to snapshots Y_k of $u(t_k,\cdot)$ at time instances t_k. Classical approaches usually compute a steady optical flow between two images. The well-known method by Horn and Schunck (1981), e.g., obtains approximations $\delta_t Y$, $\delta_{x_1} Y$, and $\delta_{x_2} Y$ of $\partial_t u$, $\partial_{x_1} u$ and, $\partial_{x_2} u$, respectively, from two given images via finite differences and then computes $b = (b_1, b_2)^T$—often on a pixel grid—by minimizing

$$J(b) = \int_\Omega (\delta_t Y + b_1\delta_{x_1} Y + b_2\delta_{x_2} Y)^2 \, dx_1 dx_2 + \lambda \int_\Omega (|\nabla b_1|^2 + |\nabla b_2|^2) \, dx_1 dx_2.$$

This function is a weighted sum of a least-squares term, expressing the linearized brightness constancy assumption and an H^1-regularization. Since the 1980s, this and other approaches (e.g. Lucas and Kanade, 1981) were further explored in numerous papers, see Baker et al. (2011) for an overview.

The problem class studied in this paper arises in a different approach, where an unsteady optical flow, as well as the corresponding brightness, are computed from a given sequence of images by solving an optimal control problem of the following form (see Hinterberger and Scherzer, 2001; Borzì, Ito and Kunisch, 2002):

$$\begin{aligned} &\min_{u,b} J(u,b) := \sum_{k=2}^{K} \Upsilon_k \left(\|u(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)} \right) + R(b), && \text{(P)} \\ &\text{s.t. } \partial_t u + \nabla u \cdot b = 0 \qquad \text{in } (0,T)\times\Omega, \\ &u(0,\cdot) = Y_1 \quad \text{in } \Omega. \end{aligned}$$

Formulations of this kind were first studied in Hinterberger and Scherzer (2001) and Borzì, Ito and Kunisch (2002). The optimization variables are the image brightness u, which is the state, and the optical flow b, which is the control. Both are defined on the spatio–temporal domain $(0,T)\times\Omega$ with $\Omega \subset \mathbb{R}^N$. The data Y_k, $k \in \{1,\ldots,K\}$, are a given image sequence, corresponding to time instances $t_k \in [0,T]$. The brightness constancy assumption leads to the transport equation, which constitutes a constraint of the problem. The objective function consists of a term, measuring the misfit between Y_k and u at the time instances, and a regularization term R for b. In this case, a solution u of the transport equation can be seen as a continuous interpolation in time of the image sequence and b is the corresponding optical flow field.

The current paper focuses on the investigation of the optimal control problem (P) for vector fields b with spatial BV-regularity. This low regularity requirement allows for consideration of the practically important situation, in which b contains spatial discontinuities. We will use the results by Ambrosio and followers (Ambrosio, 2004; Crippa, 2007; Crippa, Donadello and Spinolo, 2014a,b); De Lellis, 2006/7) concerning the existence and uniqueness of solutions for the underlying transport equation. All these results build on the concept of renormalized solutions of transport equations, developed and applied by DiPerna and Lions for Sobolev-regular vector fields in DiPerna and Lions (1989). A function u is called a renormalized solution if it satisfies the weak formulation of the transport equation and if every composition $\beta(u)$ of u with a C^1-function β is again a weak solution of the same equation.

DiPerna and Lions proved that any weak solution of the transport equation with Sobolev-regular vector fields is a renormalized solution. This renormalization property then yields uniqueness of weak solutions for the transport equation. In 2004, Ambrosio extended this theory to vector fields with BV-regularity in space and absolutely continuous divergence. Some refinements and extensions were developed in later work by Ambrosio, Crippa, De Lellis and others (Crippa, 2007; De Lellis, 2006/7; Crippa, Donadello and Spinolo; 2014a,b).

A crucial step in the theory of renormalized solutions is the proof of convergence to zero of the so-called commutator

$$r_\varepsilon = b \cdot \nabla(u * \rho_\varepsilon) - (b \cdot \nabla u) * \rho_\varepsilon$$

as $\varepsilon \to 0$, where b denotes some vector field, u the corresponding solution and ρ_ε some mollifier. In contrast to L^1-convergence to zero of the commutator in the Sobolev regular case, the commutator only converges weakly* to some measure σ for general BV-regular vector fields. Therefore, Ambrosio had to develop various new techniques to give an upper bound for σ, which then turns out to be zero. This problem appears again in our second improved theorem of existing stability results for the control-to-state operator: in the proofs to this theorem, a similar term as the commutator appears and we use the same techniques that Ambrosio had developed to prove convergence to zero of this term as $\varepsilon \to 0$. Due to these improvements in the results for stability, we are able to show the existence of minimizing points of the optimization problem (P) under quite mild regularity assumptions.

Borzì, Ito and Kunisch (2002) discussed well-posedness of the transport equation in a setting with Sobolev regularity, but did not study the existence of solutions to the optimal control problem. In 2011, Chen (2011) and Chen and Lorenz (2011) developed further the theory for a specific version of (P). For vector fields b with Sobolev regularity in space and vanishing divergence, they showed existence of minimizing points for their optimal control problem. Their theoretical results are based on results of DiPerna and Lions (1989), related to the well-posedness of solutions for the transport equation with Sobolev regular vector fields.

The goal of this paper is to show the following result about the existence of optimal solutions of (P) in spaces of minimal regularity:

THEOREM 1.1 *Let*

$$b \in \mathrm{V}^2 = \\ = \left\{ b \in L^\infty((0,T)\times\Omega)^N \cap L^2((0,T), BV(\Omega))^N \;\middle|\; \operatorname{div} b \in L^2((0,T), L^\infty(\Omega)) \right\}$$

with

$$b(t) \in \mathrm{W}_{\varepsilon,\delta}(\Omega) := \left\{ w \in L^1(\Omega) \middle| \; |w(x)| \le \delta \operatorname{dist}(x, \partial\Omega) \right. \\ \left. \textit{for almost all } x \in \Omega \textit{ with } \operatorname{dist}(x,\partial\Omega) \le \varepsilon \right\}$$

for almost all $t \in (0,T)$ *and for some fixed chosen* $\varepsilon > 0$ *and* $\delta \ge 0$*. Then, if*

$$\|b\|_{L^\infty((0,T)\times\Omega))^N} + \|\operatorname{div} b\|_{L^2((0,T),L^\infty(\Omega))} \le M$$

holds for some $M > 0$*, there exist optimal solutions for the problem* (P) *in the admissible sets.*

The regularization term R is of the form

$$R(b) = \frac{\alpha}{2} \int_0^T \Gamma_1 \left(\|Db(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}} \right) dt + R_i(b),$$

where we consider the following options for R_i:

(i) $R_1(b) \equiv 0$,

(ii) $R_2(b) = \frac{\beta}{2} \int_0^T \Gamma_2 \left(\|\partial_t b(t,\cdot)\|^2_{L^2(\Omega)^N} \right) dt$,

(iii) $R_3(b) = \frac{\gamma}{2} \int_0^T \Gamma_3 \left(\|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)} \right) dt$,

(iv) $R_4(b) = R_2(b) + R_3(b)$

and we will add some further constraints on b for some of these terms. The precise setting is presented in Section 8.

The paper is structured in the following way: Section 2 summarizes the required existence and uniqueness theory. For later use in stability results for transport equations, it is essential to study the weak limit of products of weakly convergent sequences of functions. Section 3 develops the required result of compensated compactness type. Since the available stability results for transport equations are not sufficient for our purposes, suitable extensions are developed in Sections 4 and 5. As Bochner integrability is not well suited for non-separable image spaces such as BV, Gelfand integrability is used in this case. Hence, Section 6 studies the predual of $BV(\Omega)$ in order to interpret the weak*-topology on $BV(\Omega)$ as the true weak*-topology on dual spaces. Section 7 provides some required prerequisites concerning the closedness properties of certain sets of functions bounded in $L^q((0,T), BV(\Omega)^N)$. The main result of this paper, the existence of solutions to the considered class of optimal control problems, is proven in Section 8.

Notation

Throughout, $T > 0$ denotes the length of the time interval $(0, T)$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$. We distinguish two cases for functions $f : (0, T) \to X$ with values in a Banach space X: If X is separable, we assume that the functions f are Bochner integrable. Otherwise, if $X = Y'$ is a non-separable dual space, we assume that the considered functions are Gelfand integrable, i.e., that the function $t \mapsto \langle f(t), y\rangle$ is Lebesgue integrable for any $y \in Y$. Further information on Bochner and Gelfand integrability can be found in Aliprantis and Border (2006), Emmrich (2004), Okada, Ricker and Pérez (2008), and Schweizer (2013). For the Banach space $BV(\Omega)$ we define the subspace

$$BV_0(\Omega) := \{g \in BV(\Omega)|\ \mathcal{T}g = 0\},$$

where $\mathcal{T}$ denotes the trace operator (see, e.g., Ambrosio, Fusco and Pallara, 2000). Further information on BV-functions and their properties can be found in Ambrosio, Fusco and Pallara (2000) and Attouch, Buttazzo and Michaille (2014). In the following, for any $q \in [1, \infty]$ we set $q' \in [1, \infty]$ as the value such that $\frac{1}{q} + \frac{1}{q'} = 1$ is satisfied.

2. Existence and uniqueness of transport equation

In this section, we consider the transport equation

$$\begin{aligned} \partial_t u + b \cdot \nabla u &= 0 \qquad \text{in } (0, T) \times \Omega, \\ u(0, \cdot) &= u_0 \qquad \text{in } \Omega \end{aligned} \tag{1}$$

for some given initial value $u_0 \in L^\infty(\Omega)$ and $b \in L^1((0, T) \times \Omega)^N$. As mentioned in the introduction, we are interested in vector fields b with spatial BV-regularity. For this vector field regularity, Ambrosio (2004) proved the uniqueness of weak solutions of (1) using the concept of renormalized solutions of DiPerna and Lions (see, e.g., DiPerna and Lions, 1989): a weak solution u of the transport equation (1) is called a renormalized solution if for any $\beta \in C^1(\mathbb{R})$ the composition $\beta \circ u$ is again a weak solution of the same equation with the initial value $\beta(u_0)$. Furthermore, the vector field b of the transport equation has the renormalization property if any solution of the equation is a renormalized solution.

Ambrosio's theory was refined in further works (see, e.g., Crippa, 2007; Crippa, Donadello and Spinolo, 2014a,b; DeLellis, 2006/7) by several authors. We will use these results to obtain a well-defined control-to-state operator for our optimal control problem (P).

Before we start, we first need to clarify what is meant by $b \cdot \nabla u$ when the vector field b is not smooth: if $u \in L^\infty((0, T) \times \Omega)$, $b \in L^1((0, T) \times \Omega)^N$ and $\operatorname{div} b \in L^1((0, T) \times \Omega)$, then we define the distribution $b \cdot \nabla u \in \mathcal{D}'(\mathbb{R} \times \Omega)$ by

$$\langle b \cdot \nabla u, \varphi\rangle = -\langle bu, \nabla\varphi\rangle - \langle u \operatorname{div} b, \varphi\rangle \qquad \forall\ \varphi \in C_c^\infty([0, T) \times \Omega).$$

This leads us to the following general definition of weak solution for the transport equation (1):

Definition 2.1 (Weak solution) *Let* $u_0 \in L^\infty(\Omega)$, $b \in L^1((0,T)\times\Omega)^N$ *with* $\operatorname{div} b \in L^1((0,T)\times\Omega)$. *Then, we call a function* $u \in C([0,T], L^\infty(\Omega) - w^*)$ *a weak solution of* (1), *if the following equation is satisfied*

$$\int_0^T \int_\Omega u \left(\partial_t \varphi + b \cdot \nabla \varphi + \varphi \operatorname{div} b\right) \, dxdt = -\int_\Omega u_0 \varphi(0,\cdot) \, dx$$

for all $\varphi \in C_c^\infty([0,T)\times\Omega)$.

The following theorem states the existence and uniqueness of solutions for the transport equation (1) on bounded spatial domains. This result can be easily concluded from Theorem 1.1 in Crippa, Donadello and Spinolo (2014a), Theorem 1.1 in Crippa, Donadello and Spinolo (2014b) and Remark 2.2.2 in Crippa (2007).

Theorem 2.1 (Existence and uniqueness of solutions) *Let* $u_0 \in L^\infty(\Omega)$ *and* $b \in L^\infty((0,T)\times\Omega)^N \cap L^1((0,T), BV_0(\Omega))^N$ *with* $\operatorname{div} b \in L^1((0,T), L^\infty(\Omega))$. *Then, the transport equation* (1) *has a unique weak renormalized solution* $u \in C([0,T], L^\infty(\Omega) - w^*)$. *Furthermore,*

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}$$

for any $t \in [0,T]$ *and the vector field* b *has the renormalization property.*

For the subsequent sections, we define for $q \in [1,\infty)$ the sets of vector fields

$$\mathrm{V}^q := \left\{ b \in L^q((0,T), BV(\Omega))^N \cap L^\infty((0,T)\times\Omega)^N \middle| \ \operatorname{div} b \in L^q((0,T), L^\infty(\Omega)) \right\} \tag{2}$$

and

$$\mathrm{V}_0^q := \left\{ b \in \mathrm{V}^q \middle| \ b \in L^q((0,T), BV_0(\Omega))^N \right\}.$$

Then, due to Theorem 2.1, the solution operator S, given by

$$\begin{aligned} \mathrm{S} : L^\infty(\Omega) \times \mathrm{V}_0^1 &\to C([0,T], L^\infty(\Omega) - w^*), \\ (u_0, b) &\mapsto S(u_0, b) = u, \end{aligned} \tag{3}$$

is well-defined.

3. A compensated compactness result for weakly convergent sequences

In this section, we prove a result, which is reminiscent of the compensated compactness results of Tartar (1979) and Murat (2005): the product of two weakly convergent sequences converges to the product of their weak limits if the sequences satisfy some regularity assumptions. The theorem we present is a generalization of Proposition 1 in Moussa (2016) to the case, in which one of the sequences has codomain $BV(\Omega)$, instead of Sobolev regularity, as in Moussa (2016). We will use this statement in the proofs for the stability theorems in the subsequent sections, where we will be faced with the situation that we have to specify the limit of the product of weakly convergent vector fields with their weakly convergent solutions. We start with two auxiliary lemmas.

LEMMA 3.1 *Let $q \in [1, \infty]$ and let $(f_n) \subset L^q((0,T), BV_0(\Omega))$ be a bounded sequence. Then*

$$f_n(\cdot, \cdot + h) - f_n \to 0 \quad \text{in } L^q((0,T), L^1(\Omega)) \quad \text{as } h \to 0$$

uniformly in $n \in \mathbb{N}$.

PROOF: We take the standard mollifier ρ_ε for $\varepsilon > 0$ and set $g_{n,k} := f_n * \rho_{1/k}$, where we extend f_n by zero to the entire $\mathbb{R}^N$ in the spatial variable. Then, we estimate for almost all $t \in (0,T)$ and for $h \in \mathbb{R}^N$

$$\begin{aligned}\int\limits_{\mathbb{R}^N} |g_{n,k}(t, x+h) - g_{n,k}(t,x)| \; dx &= \int\limits_{\mathbb{R}^N} \left| \int\limits_0^1 \nabla g_{n,k}(t, x + rh)^\top h \; dr \right| \; dx \\ &\leq |h|_\infty \int\limits_0^1 \int\limits_{\mathbb{R}^N} |\nabla g_{n,k}(t,x)|_1 \; dx dr \\ &\leq |h|_\infty \, \|\nabla f_n(t, \cdot)\|_{\mathcal{M}(\Omega)^N} ,\end{aligned}$$

where we use Theorem 2.1 (b) from Ambrosio, Dusco and Pallara (2000) for the last inequality. Integrating over $(0,T)$ yields

$$\left(\int\limits_0^T \|g_{n,k}(t, \cdot + h) - g_{n,k}(t, \cdot)\|^q_{L^1(\Omega)} \, dt \right)^{1/q} \leq |h|_\infty \, \|f_n\|_{L^q((0,T), BV(\Omega))} \leq C \, |h|_\infty ,$$

where $C > 0$ denotes an upper bound for the sequence (f_n). With the following estimate

$$\begin{aligned}\|f_n(\cdot,\cdot+h)-f_n\|_{L^q((0,T),L^1(\Omega))} &\le \|f_n(\cdot,\cdot+h)-f_n\|_{L^q((0,T),L^1(\mathbb{R}^N))}\\ &\le \|f_n(\cdot,\cdot+h)-g_{n,k}(\cdot,\cdot+h)\|_{L^q((0,T),L^1(\mathbb{R}^N))}\\ &+\|f_n-g_{n,k}\|_{L^q((0,T),L^1(\mathbb{R}^N))}\\ &+\|g_{n,k}(\cdot,\cdot+h)-g_{n,k}\|_{L^q((0,T),L^1(\mathbb{R}^N))}\\ &\le 2\,\|f_n-g_{n,k}\|_{L^q((0,T),L^1(\mathbb{R}^N))}\\ &+\|g_{n,k}(\cdot,\cdot+h)-g_{n,k}\|_{L^q((0,T),L^1(\mathbb{R}^N))}\,,\end{aligned}$$

we deduce the statement: For any given $\varepsilon > 0$, we choose $k(n) \in \mathbb{N}$ for each $n \in \mathbb{N}$ such that

$$\|f_n-g_{n,k}\|_{L^q((0,T,L^1(\mathbb{R}^N))} \le \frac{\varepsilon}{4}$$

for all $k \ge k(n)$ and $\delta = \varepsilon/2C$, where C is the constant in the proof. Then, for $|h|_\infty \le \delta$

$$\|f_n(\cdot,\cdot+h)-f_n\|_{L^q((0,T,L^1(\Omega))} \le \tfrac{\varepsilon}{2}+C\,|h|_\infty \le \varepsilon. \qquad \Box$$

LEMMA 3.2 *Let $q \in [1,\infty]$, $\rho \in C_c^\infty(\mathbb{R}^N)$ be some mollifier for the spatial variable and let $(f_n) \subset L^q((0,T),BV_0(\Omega))$ and $(g_n) \subset L^{q'}((0,T),L^\infty(\Omega))$ be bounded sequences. Then, the commutator*

$$S_{n,k} := f_n(g_n * \rho_{1/k}) - (f_n g_n) * \rho_{1/k}$$

converges uniformly in $n \in \mathbb{N}$ to zero in $L^1((0,T)\times\Omega)$ as $k \to \infty$.

PROOF: For $t \in (0,T)$ and $x \in \Omega$ we have

$$S_{n,k}(t,x) = \int_{\mathbb{R}^N} (f_n(t,x)-f_n(t,x-y))\,g_n(t,x-y)\rho_{1/k}(y)\,dy$$

and thus, integrating over $(0,T)\times\Omega$ yields

$$\begin{aligned}&\int_0^T\int_\Omega |S_{n,k}(t,x)|\ dxdt\\ &\le \|g_n\|_{L^{q'}((0,T),L^\infty(\Omega))}\int_{\mathbb{R}^N}\rho_{1/k}(y)\,\|f_n-f_n(\cdot,\cdot-y)\|_{L^q((0,T),L^1(\Omega))}\ dy\\ &\le C\int_{\{y|\ |y|\le 1/k\}}\rho_{1/k}(y)\,\|f_n-f_n(\cdot,\cdot-y)\|_{L^q((0,T),L^1(\Omega))}\ dy,\end{aligned}$$

where $C > 0$ denotes an upper bound for (g_n) in $L^{q'}((0,T), L^\infty(\Omega))$. Then, Lemma 3.1 yields the statement. □

Now, we turn to the main statement of this section. The proof of this theorem is a reproduction of the proof of Proposition 1 in Moussa (2016), adjusted and extended to functions $f_n, f \in L^q((0,T), BV_0(\Omega))$ and weak convergence in $L^1((0,T)\times\Omega)$.

Theorem 3.1 *Let $q \in (1,\infty]$. Furthermore, let $(g_n) \subset L^{q'}((0,T), L^\infty(\Omega)) \cap L^\infty((0,T)\times\Omega)$ and $(f_n) \subset L^q((0,T), BV_0(\Omega))$ be bounded sequences in each of these spaces, such that*

$$f_n \rightharpoonup f \quad \text{in } L^1((0,T)\times\Omega) \quad \text{and} \quad g_n \rightharpoonup g \quad \text{in } L^{q'}((0,T)\times\Omega),$$

where $f \in L^q((0,T), BV_0(\Omega))$ and $g \in L^{q'}((0,T), L^\infty(\Omega)) \cap L^\infty((0,T)\times\Omega)$. If $(\partial_t g_n)$ is a bounded sequence in $L^1((0,T), (W^{m,2}(\Omega))')$ for some $m \in \mathbb{N}$, then

$$f_n g_n \overset{*}{\rightharpoonup} fg \quad \text{in } \mathcal{M}((0,T)\times\Omega).$$

Proof: We perform the same steps as in the previously mentioned proof. With Lebesgue's dominated convergence theorem we obtain

$$f(g * \rho_{1/k}) \to fg \quad \text{in } L^1((0,T)\times\Omega) \quad \text{as } k \to \infty. \tag{4}$$

Furthermore, since $(g_n) \subset L^{q'}((0,T), L^\infty(\Omega))$ is bounded, we obtain for a fixed $k \in \mathbb{N}$ that

$$(g_n * \rho_{1/k})_n \quad \text{and} \quad \big(\nabla(g_n * \rho_{1/k})\big)_n = \big(g_n * \nabla\rho_{1/k}\big)_n$$

are bounded in $L^1((0,T)\times\Omega)$ and $L^1((0,T)\times\Omega)^N$, respectively. In addition, if we consider $\partial_t g_n(t,\cdot)$ as a distribution on $\mathbb{R}^N$ for almost all $t \in (0,T)$, i.e. if we define its application on $\rho_{1/k} \in C_c^\infty(\mathbb{R}^N)$ as $\partial_t g_n(t,\cdot)(\varphi|_\Omega)$, then the convolution is defined as

$$(\partial_t g_n(t,\cdot) * \rho_{1/k})(x) = \partial_t g_n(t,\cdot)(\rho_{1/k}(x-\cdot)|_\Omega).$$

Hence, we conclude for $\varphi \in C_0((0,T)\times\Omega)$ that

$$\begin{aligned}
&\left| \int_0^T \int_\Omega (\partial_t g_n(t,\cdot) * \rho_{1/k})(x)\varphi(t,x)\; dxdt \right| \\
&\quad \le \|\varphi\|_{C((0,T)\times\Omega)} \int_0^T \int_\Omega \left\|\rho_{1/k}(x-\cdot)\right\|_{W^{m,2}(\Omega)} \|\partial_t g_n(t,\cdot)\|_{(W^{m,2}(\Omega))'}\, dxdt \\
&\quad \le |\Omega|\, \|\varphi\|_{C((0,T)\times\Omega)} \left\|\rho_{1/k}\right\|_{W^{m,2}(\mathbb{R}^N)} \|\partial_t g_n\|_{L^1((0,T),(W^{m,2}(\Omega))')} \\
&\quad \le C_k \|\varphi\|_{C((0,T)\times\Omega)},
\end{aligned}$$

where $C_k > 0$ denotes a bound depending on $k \in \mathbb{N}$. Thus, $\big(\partial_t(g_n * \rho_{1/k})\big)$ is a bounded sequence in $\mathcal{M}((0,T)\times\Omega)$. Summing up, we obtain that $(g_n * \rho_{1/k})_n$ is a bounded sequence in $BV((0,T)\times\Omega)$ for any $k \in \mathbb{N}$. As a consequence, there exists a subsequence $(g_{n_l} * \rho_{1/k})_l$ being convergent to some h_k in $L^1((0,T)\times\Omega)$ for a fixed $k \in \mathbb{N}$. Since $g_n \rightharpoonup g$ in $L^{q'}((0,T)\times\Omega)$, we easily obtain that $g_n * \rho_{1/k} \rightharpoonup g * \rho_{1/k}$ in $L^1((0,T)\times\Omega)$ as $n \to \infty$ and thus $h_k = g * \rho_{1/k}$. With a proof by contradiction we deduce that the whole sequence $g_n * \rho_{1/k} \to g * \rho_{1/k}$ in $L^1((0,T)\times\Omega)$ as $n \to \infty$. Now, using a standard diagonal argument, we can find a subsequence (labeled by n again) such that

$$g_n * \rho_{1/k}(t,x) \to g * \rho_{1/k}(t,x) \quad \text{for almost all } (t,x) \in (0,T)\times\Omega \text{ and for all } k \in \mathbb{N}$$

as $n \to \infty$. In addition, we have that $(g_n * \rho_{1/k})_n$ is a bounded subset of $L^\infty((0,T)\times\Omega)$ for each $k \in \mathbb{N}$, due to the boundedness of (g_n) in $L^\infty((0,T)\times\Omega)$. Thus, $g_n * \rho_{1/k} \to g * \rho_{1/k}$ in $L^p((0,T)\times\Omega)$ for any $p < \infty$. Furthermore, (f_n) is bounded in $L^r((0,T)\times\Omega)$ for $r = \min(q, N/(N-1))$ and we obtain for any $\varphi \in L^\infty((0,T)\times\Omega)$ and $k \in \mathbb{N}$

$$\begin{aligned}\big|\langle f_n(g_n * \rho_{1/k}) - f(g * \rho_{1/k}), \varphi\rangle\big| \le{}& \|\varphi\|_{L^\infty((0,T)\times\Omega)}\,\|f_n\|_{L^r((0,T)\times\Omega)} \\ &\cdot \big\|g_n * \rho_{1/k} - g * \rho_{1/k}\big\|_{L^{r'}((0,T)\times\Omega)} \\ &+ \big|\langle f_n - f, (g * \rho_{1/k})\varphi\rangle\big| \to 0\end{aligned} \tag{5}$$

as $n \to \infty$, i.e. $f_n(g_n * \rho_{1/k}) \rightharpoonup f(g * \rho_{1/k})$ in $L^1((0,T)\times\Omega)$. Since (f_n) is bounded in $L^1((0,T)\times\Omega)$ and (g_n) is bounded in $L^\infty((0,T)\times\Omega)$, we obtain that (f_ng_n) is bounded in $L^1((0,T)\times\Omega)$. Finally, we deduce that for any fixed $\varphi \in C_0((0,T)\times\Omega)$

$$\begin{aligned}\big|\langle (f_ng_n) * \rho_{1/k} - f_ng_n, \varphi\rangle\big| &= \big|\langle f_ng_n, \varphi * \rho_{1/k} - \varphi\rangle\big| \\ &\le \|f_ng_n\|_{L^1((0,T)\times\Omega)}\,\big\|\varphi * \rho_{1/k} - \varphi\big\|_{C((0,T)\times\Omega)} \\ &\le C\,\big\|\varphi * \rho_{1/k} - \varphi\big\|_{C((0,T)\times\Omega)} \to 0\end{aligned} \tag{6}$$

since φ is uniformly continuous in $(0,T)\times\Omega$. Summing up, we conclude, for any $\varphi \in C_0((0,T)\times\Omega)$ that:

$$\begin{aligned}|\langle fg - f_ng_n, \varphi\rangle| \le{}& \big|\langle fg - f(g * \rho_{1/k}), \varphi\rangle\big| \\ &+ \big|\langle f(g * \rho_{1/k}) - f_n(g_n * \rho_{1/k}), \varphi\rangle\big| \\ &+ \big|\langle f_n(g_n * \rho_{1/k}) - (f_ng_n) * \rho_{1/k}, \varphi\rangle\big| \\ &+ \big|\langle (f_ng_n) * \rho_{1/k} - f_ng_n, \varphi\rangle\big|.\end{aligned}$$

Then, the first, third and fourth terms on the right hand side converge uniformly in $n \in \mathbb{N}$ as $k \to \infty$ due to Lemma 3.2 and the estimates (4) and (6). Therefore, for any ε we choose $k(\varepsilon) \in \mathbb{N}$ such that the sum of the first, third and fourth term is smaller than ε for any $k \ge k(\varepsilon)$. Then, for fixed $k(\varepsilon)$, we can choose $n(\varepsilon) \in \mathbb{N}$ such that the second term is smaller than ε for all $n \geq n(\varepsilon)$ due to the estimate (5). Consequently,

$$|\langle fg - f_n g_n, \varphi\rangle| \leq 2\varepsilon \quad \forall\, n \geq n(\varepsilon)$$

which proves the statement. □

4. Stability of the solution operator: first improvement

In Crippa (2007) and DiPerna and Lions (1989) it is mentioned (and proven) that solutions of the transport equation are elements of $C\left([0,T], L^p_{loc}\left(\mathbb{R}^N\right)\right)$ for any $p \in [1,\infty)$. This can be easily deduced from the renormalization property of vector fields. In DiPerna and Lions (1989) it is additionally shown that sequences of solutions are strongly convergent in $C\left([0,T], L^p_{loc}\left(\mathbb{R}^N\right)\right)$ if the sequences of vector fields and initial data satisfy some convergence assumptions. For the proof, arguments of Arzelà-Ascoli type are used. Arzelà-Ascoli is also used by Crippa (2007), but it is just shown that sequences of solutions are convergent in $C\left([0,T], L^p\left(\mathbb{R}^N\right) - w\right)$. In the first stability theorem we present the proof for convergence in $C([0,T], L^p(\Omega) - w)$, based on the theorem of Arzelà-Ascoli in locally convex spaces. In contrast to Crippa, where strong convergence of the vector fields is required, our assumptions only demand weak convergence of the vector fields in $L^1((0,T)\times\Omega)^N$. In DiPerna and Lions (1989) it is shown that weak convergence of the vector fields is sufficient if the uniform convergence of the translation relation appearing in Lemma 3.1 is satisfied by the sequence of vector fields. In addition, it is also mentioned that this condition is fulfilled if the vector fields are a bounded sequence in $L^q((0,T), X)^N$, where X is a Banach space embedding compactly into $L^1(\Omega)$. In Lemma 3.1, we have shown this for the special case of $X = BV_0(\Omega)$. These results were sufficient for DiPerna and Lions to prove weak convergence of $b_n u_n$ to bu in $L^1((0,T)\times\Omega)^N$, which we summed up to the compensated compactness result in the previous section. With the aid of some auxiliary statements building on renormalization arguments, we additionally show strong convergence of solutions in $C([0,T], L^p(\Omega))$ for any $p \in [1,\infty)$. Again, we start this section with two auxiliary lemmas.

Lemma 4.1 *Let $g, g^2 \in C([0,T], L^2(\Omega) - w)$. Then $g \in C([0,T], L^2(\Omega))$.*

Proof: For $\varphi \equiv 1 \in L^2(\Omega)$ we deduce that

$$\|g(t,\cdot)\|^2_{L^2(\Omega)} = \int_\Omega g^2(t,x)\varphi\, dx \to \int_\Omega g^2(s,x)\varphi\, dx = \|g(s,\cdot)\|^2_{L^2(\Omega)} \quad \text{as } t \to s \text{ in } [0,T].$$

Since, in addition, $g(t,\cdot) \rightharpoonup g(s,\cdot)$ in $L^2(\Omega)$ as $t \to s$, the statement is proven. □

Lemma 4.2 *Let $(g_n), (g_n^2) \subset C([0,T], L^2(\Omega) - w)$ be two sequences such that*

$$g_n \to g \quad \text{and} \quad g_n^2 \to g^2 \quad \text{in } C([0,T], L^2(\Omega) - w),$$

with limits $g, g^2 \in C([0,T], L^2(\Omega) - w)$. Then,

$$g_n, g \in C([0,T], L^2(\Omega)) \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad g_n \to g \quad \text{in } C([0,T], L^2(\Omega)).$$

Proof: Due to Lemma 4.1 we know that $g_n, g \in C([0,T], L^2(\Omega))$ for all $n \in \mathbb{N}$. Furthermore, considering that $g_n^2 \to g^2$ in $C([0,T], L^2(\Omega) - w)$ and choosing $\varphi \equiv 1 \in L^2(\Omega)$, we conclude that

$$\sup_{t\in[0,T]} \left| \|g_n(t,\cdot)\|_{L^2(\Omega)}^2 - \|g(t,\cdot)\|_{L^2(\Omega)}^2 \right| \to 0 \quad \text{as } n \to \infty. \tag{7}$$

In addition, we estimate

$$\sup_{t\in[0,T]} \left| \int_\Omega (g_n(t,x) - g(t,x))^2 \ dx \right| \le \sup_{t\in[0,T]} \left| \int_\Omega \left(g_n(t,x)^2 - g(t,x)^2\right) dx \right| \tag{8}$$

$$+ 2 \sup_{t\in[0,T]} \left| \int_\Omega g(t,x)(g(t,x) - g_n(t,x)) dx \right|. \tag{9}$$

Obviously, term (8) tends to zero as $n \to \infty$. For the second term, (9), we introduce the functions

$$L_n : L^2(\Omega) \to \mathbb{R},\ \varphi \mapsto \sup_{t\in[0,T]} |h_{n,\varphi}(t)|$$
$$\text{with } h_{n,\varphi}(t) := \int_\Omega \varphi(x)(g(t,x) - g_n(t,x)) \ dx.$$

These functions are Lipschitz continuous: obviously, $h_{n,\varphi} \in C([0,T])$ for any $\varphi \in L^2(\Omega)$ and $n \in \mathbb{N}$ and we estimate

$$|L_n(\varphi) - L_n(\psi)| = \left| \|h_{n,\varphi}\|_{C([0,T])} - \|h_{n,\psi}\|_{C([0,T])} \right| \le \|h_{n,\varphi} - h_{n,\psi}\|_{C([0,T])}$$
$$\le C \|\varphi - \psi\|_{L^2(\Omega)}.$$

The constant $C > 0$ is independent of $n \in \mathbb{N}$ due to the uniform boundedness of $\sup_{t\in[0,T]} \|g_n(t,\cdot)\|_{L^2(\Omega)}$ with respect to $n \in \mathbb{N}$, shown in (7). We define the set $A := \{g(t,\cdot) | t \in [0,T]\} \subset L^2(\Omega)$. This set is compact, since it is the image of a compact set under a continuous function. Hence, for each function L_n, there exists an element $\varphi_n \in A$ such that

$$L_n(\varphi_n) = \max_{\psi\in A} L_n(\psi).$$

Since $(\varphi_n) \subset A$, there exists a subsequence (φ_{n_k}), converging to some $\varphi \in A$ in $L^2(\Omega)$. Furthermore, for any $n \in \mathbb{N}$, we have the estimate $\left|h_{n,g(t,\cdot)}(t)\right| \le \sup_s \left|h_{n,g(t,\cdot)}(s)\right| \le L_n(\varphi_n)$. Thus, we conclude that

$$\begin{aligned}\sup_{t\in[0,T]} \left|h_{n_k,g(t,\cdot)}(t)\right| &\le \sup_{t\in[0,T]} \left|h_{n_k,\varphi_{n_k}-\varphi}(t)\right| + \sup_{t\in[0,T]} \left|h_{n_k,\varphi}(t)\right| \\ &\le C \left\|\varphi_{n_k} - \varphi\right\|_{L^2(\Omega)} + \sup_{t\in[0,T]} \left|h_{n_k,\varphi}(t)\right|.\end{aligned}$$

Both terms on the right hand side tend to zero as $k \to \infty$. Summing up, the term in (9) converges to 0 for $n = n_k$, $k \to \infty$ and, therefore, $g_{n_k} \to g$ in $C([0,T], L^2(\Omega))$. Now, a standard proof by contradiction yields that the whole sequence (g_n) converges to g in $C([0,T], L^2(\Omega))$. □

With the aid of these two lemmas we can prove the first (improved) stability theorem for the solution operator S.

Theorem 4.1 (First stability theorem) *Let $b \in \mathrm{V}_0^1$ and let the initial value satisfy $u_0 \in L^\infty(\Omega)$. Furthermore, let $(b_n) \subset \mathrm{V}_0^1$ and $(u_{0,n}) \subset L^\infty(\Omega)$ be two sequences with the following properties:*

(i) $(u_{0,n})$ is bounded in $L^\infty(\Omega)$ and converges to u_0 in $L^1(\Omega)$,

(ii) (a) (b_n) converges strongly to b in $L^1((0,T)\times\Omega)^N$ or

(b) (b_n) is bounded in $L^q((0,T), BV_0(\Omega))^N$ for some $q > 1$ and $b_n \rightharpoonup b$ in $L^1((0,T)\times\Omega)^N$.

(iii) $(\operatorname{div} b_n)$ converges strongly to $\operatorname{div} b$ in $L^1((0,T)\times\Omega)$.

Then, for any $1 \le p < \infty$, the sequence of unique solutions $(u_n) \subset C([0,T], L^\infty(\Omega) - w^*)$ of (1) with vector fields b_n and initial data $u_{0,n}$ is a subset of $C([0,T], L^p(\Omega))$ and converges in $C([0,T], L^p(\Omega))$ to the unique solution $u \in C([0,T], L^p(\Omega))$ of (1) with vector field b and initial value u_0.

Proof: We first prove the theorem for the special case of $p = 2$ and then derive the general statement from this.
Let $(b_n) \subset \mathrm{V}_0^1$ and $(u_{0,n})$ be sequences with limits $b \in \mathrm{V}_0^1$ and $u_0 \in L^\infty(\Omega)$ as assumed in the theorem. Then, $\|u_n(t,\cdot)\|_{L^\infty(\Omega)} \le C_1 < \infty$ for any $t \in [0,T]$ and any $n \in \mathbb{N}$, due to Theorem 2.1. Therefore, $(u_n(t,\cdot)) \subset L^2(\Omega)$ represents a relatively compact subset with respect to the weak topology in $L^2(\Omega)$ for all $t \in [0,T]$. In addition, we set $g_{n,\varphi} := \langle u_n(t,\cdot), \varphi\rangle$ for $\varphi \in C_c^\infty(\Omega)$ and we conclude with $\psi \in C_c^\infty((0,T))$ that

$$\int_0^T \psi(t) \frac{d}{dt} \langle u_n(t,\cdot), \varphi\rangle dt = -\int_0^T \psi'(t) \langle u_n(t,\cdot), \varphi\rangle dt$$

$$= \int_0^T \psi(t) \left[\langle u_n(t,\cdot) b_n(t,\cdot), \nabla\varphi\rangle + \langle u_n(t,\cdot) \operatorname{div} b_n(t,\cdot), \varphi\rangle\right] dt,$$

i.e. $(g_{n,\varphi})$ is weakly differentiable with derivative

$$g'_{n,\varphi}(t) = \langle u_n(t,\cdot)b_n(t,\cdot), \nabla\varphi\rangle + \langle u_n(t,\cdot)\operatorname{div} b_n(t,\cdot), \varphi\rangle.$$

We estimate for $r, s \in [0,T]$ with $s < r$

$$\int_s^r \left|g'_{n,\varphi}(t)\right| dt \le \int_s^r h_n(t) dt,$$

where $h_n(t) = C_1 \cdot C(\varphi)\left[\|b_n(t,\cdot)\|_{L^1(\Omega)^N} + \|\operatorname{div} b_n(t,\cdot)\|_{L^1(\Omega)}\right]$ and $C(\varphi) > 0$ is a bound, depending on φ. The set of functions (h_n) form a uniformly integrable set in both cases: due to the strong convergence of $(\operatorname{div} b_n)$ in $L^1((0,T)\times\Omega)$ and in case (a) due to the strong convergence of (b_n) to b in $L^1((0,T)\times\Omega)^N$ and in case (b) due to the estimate

$$\int_s^r \|b_n(t,\cdot)\|_{L^1(\Omega)^N}\, dt \le \|b_n\|_{L^q((0,T),L^1(\Omega))^N} |r-s|^{1/q'} \le C_2 |r-s|^{1/q'}.$$

Hence, the set of functions $\left(\left|g'_{n,\varphi}\right|\right)$ is also uniformly integrable for fixed $\varphi \in C_c^\infty(\Omega)$ and thus, we deduce equicontinuity for the sequence $(g_{n,\varphi})$ for any $\varphi \in L^2(\Omega)$ in the following: let $(\varphi_k) \subset C_c^\infty(\Omega)$ be a sequence converging to φ in $L^2(\Omega)$ and let $0 \le s < r \le T$. Then, we obtain

$$\begin{aligned} &|g_{n,\varphi}(r) - g_{n,\varphi}(s)| \\ &\le \left(\|u_n(r,\cdot)\|_{L^2(\Omega)} + \|u_n(s,\cdot)\|_{L^2(\Omega)}\right)\|\varphi_k - \varphi\|_{L^2(\Omega)} + \int_s^r \left|g'_{n,\varphi_k}(t)\right| dt. \end{aligned}$$

Now, for $\varepsilon > 0$, we find $k_\varepsilon \in \mathbb{N}$ and $\delta(\varepsilon) > 0$ such that $\|\varphi_{k_\varepsilon} - \varphi\|_{L^2(\Omega)} \le \varepsilon$ and $\int_s^r \left|g'_{n,\varphi_{k_\varepsilon}}(t)\right| dt \le \varepsilon$ hold if $|r-s| \le \delta(\varepsilon)$. Then, $|g_{n,\varphi}(r) - g_{n,\varphi}(s)| \le (C_3+1)\varepsilon$, where $C_3 = 2|\Omega|^{1/2} C_1$. Consequently, Arzelà-Ascoli yields that there exists a subsequence (u_{n_k}) and some $v \in C([0,T], L^2(\Omega)-w)$ such that $u_{n_k} \to v$ in $C([0,T], L^2(\Omega)-w)$. Using Lebesgue's dominated convergence theorem and some simple calculations yield in case (a) that v satisfies the weak formulation with vector field b and initial data u_0. Hence, v is a weak solution of the transport equation with vector field b and the initial value u_0, and thus is unique, i.e. $u = v$. In case (b), the same calculations yield that for any $\psi \in C_c^\infty([0,T)\times\Omega)$

$$\begin{aligned} &\int_\Omega u_{0,n}\psi(0,\cdot)\, dx + \int_0^T \int_\Omega u_n\partial_t\psi + u_n\psi \operatorname{div} b_n\, dxdt \\ &\qquad \to \int_\Omega u_0\psi(0,\cdot)\, dx \int_0^T \int_\Omega v\partial_t\psi + v\psi\operatorname{div} b\, dxdt. \end{aligned}$$

It remains to show that

$$\int_0^T \int_\Omega u_n b_n \cdot \nabla\psi \; dxdt \to \int_0^T \int_\Omega vb \cdot \nabla\psi \; dxdt$$

is satisfied. Our aim is to use Theorem 3.1. Therefore, we have to show that $(\partial_t u_n)$ is a bounded subset of $L^1((0,T),(W^{m,2}(\Omega))')$. We choose m so large that $W^{m,2}(\Omega) \hookrightarrow C^1(\Omega)$. We know from above that for any $\varphi \in W^{m,2}(\Omega)$ and for almost all $t \in (0,T)$

$$\langle \partial_t u_n(t,\cdot), \varphi\rangle = \langle u_n(t,\cdot) b_n(t,\cdot), \nabla\varphi\rangle + \langle u_n(t,\cdot) \operatorname{div} b_n(t,\cdot), \varphi\rangle,$$

i.e. $\partial_t u_n(t,\cdot) \in (W^{m,2}(\Omega))'$ and thus, we estimate for $\vartheta \in L^\infty((0,T),W^{m,2}(\Omega))$

$$|\langle \partial_t u_n, \vartheta\rangle| \leq C_4 \, \|\vartheta\|_{L^\infty((0,T),W^{m,2}(\Omega))}$$

for some $C_4 > 0$ independent of $n \in \mathbb{N}$. The principle of uniform boundedness now yields that $(\partial_t u_n)$ is a bounded sequence in $L^1((0,T),(W^{m,2}(\Omega))')$, and we can apply Theorem 3.1, leading to

$$\int_0^T \int_\Omega u_n b_n \cdot \nabla\psi \; dxdt \to \int_0^T \int_\Omega vb \cdot \nabla\psi \; dxdt$$

for any $\psi \in C_c^\infty((0,T) \times \Omega)$. The general case, i.e. for test functions in $C_c^\infty([0,T) \times \Omega)$, can be deduced using smooth cut-off functions in time, i.e. $(\eta_k) \subset C_c^\infty((0,T))$ with $0 \leq \eta_k(t) \leq 1$, $\eta_k(t) \to \chi_{(0,T)}(t)$ and $\eta_k' \overset{*}{\rightharpoonup} \delta_0 - \delta_T$ for all $t \in (0,T)$, $k \in \mathbb{N}$ as $k \to \infty$. Thus, v satisfies the weak formulation and, as above, we deduce that $v = u$. Finally, by a standard proof of contradiction, we obtain that the whole sequence (u_n) converges to u in $C([0,T], L^2(\Omega) - w)$.

Furthermore, following the previous argumentation, we obtain that $(u_n)^2$ converges to u^2 in $C([0,T], L^2(\Omega) - w)$ due to the renormalization property of b. Then, Lemma 4.2 yields that $u_n, u \in C([0,T], L^2(\Omega))$ for all $n \in \mathbb{N}$ and $u_n \to u$ in $C([0,T], L^2(\Omega))$.

It remains to show the result for general $p < \infty$. The case $1 \leq p \leq 2$ is obviously satisfied, due to the continuous embedding of $C([0,T], L^2(\Omega))$ into $C([0,T], L^p(\Omega))$ for $p \leq 2$. Therefore, we only have to show the statement for the case of $2 < p < \infty$. So, let $2 < p < \infty$ and let $t, s \in [0,T]$. Then, we estimate

$$\|u_n(t,\cdot) - u_n(s,\cdot)\|_{L^p(\Omega)}^p \leq C_5^{p-2} \, \|u_n(t,\cdot) - u_n(s,\cdot)\|_{L^2(\Omega)}^2 \to 0$$

as $t \to s$. Obviously, the estimate also works for u. In the same way we estimate for $t \in [0,T]$

$$\|u_n(t,\cdot) - u(t,\cdot)\|_{L^p(\Omega)}^p \leq C_6^{p-2} \, \|u_n(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)}^2$$

and taking the supremum over $[0,T]$ yields the statement. □

5. Stability of the solution operator: second improvement

In this section, we improve over the previous stability result. The improvement consists in replacing the strong convergence of $(\operatorname{div} b_n)$ to some $\operatorname{div} b$ in $L^1((0,T)\times\Omega)$ with boundedness of $(\operatorname{div} b_n)$ in $L^1((0,T),L^\infty(\Omega))$. This refined result will be needed in the proof of existence of minimizing points for the optimal control problems in the last section. In DiPerna and Lions (1989), this result is shown in Theorem II.5 for vector fields with spatial Sobolev regularity under stronger assumptions on the convergence of the vector fields than we require. The idea of DiPerna and Lions' proof is the following: they convolve the unique solution u, corresponding to the vector field b, with some mollifier ρ_ε and obtain $u_\varepsilon := u * \rho_\varepsilon$. Then, they show that the function u_ε satisfies the same transport equation, but with some inhomogeneity r_ε. This inhomogeneity converges strongly to zero in some Lebesgue space as $\varepsilon \to 0$ (Theorem II.1 in DiPerna and Lions, 1989). As a next step, they consider the difference $u_n - u_\varepsilon$ of the unique weak solutions u_n, corresponding to the vector fields b_n and the smoothed u_ε. For this difference, they can show that it is uniformly bounded in n by two terms: by the L^1-norm of the difference $u - u_\varepsilon$ and by the L^1-norm of r_ε. Taking the limit in ε yields their statement in the end.

We take the same route to show our results for vector fields with spatial BV-regularity. Unfortunately, the proof is much more complicated and we are confronted with the same problem as Ambrosio had with the commutator $r_\varepsilon = (\operatorname{div}(bu)) * \rho_\varepsilon - \operatorname{div}(b(u * \rho_\varepsilon))$: DiPerna and Lions had the case that their commutator converged strongly to zero in some Lebesgue space as $\varepsilon \to 0$, whereas Ambrosio's commutator can only be split into a strongly convergent part $r_{1,\varepsilon}$ and some weakly*-convergent part $r_{2,\varepsilon}$. Then, Ambrosio had to show carefully that this second term also vanishes as $\varepsilon \to 0$. The same problem appears here with the inhomogeneity r_ε, appearing in the transport equation, satisfied by the convolved solution u_ε. This inhomogeneity can only be split into a „good" part $r_{1,\varepsilon}$, being convergent in some Lebesgue space, and a „bad" part, for which we have some estimate for the limit as $\varepsilon \to 0$. Therefore, most of this section resembles the approach of Crippa in his thesis Crippa (2007) and we use the same techniques to tackle the problems. We start with some lemma that is a reproduction, with some modifications, of Proposition 3.2 in DeLellis (2006/7). An incomplete proof of the statement is given in DeLellis (2006/7) and a complete, but longer proof is given in Lemma 3.1.11 in Jarde (2018).

Lemma 5.1 *Let $1 \le q < \infty$, let $g \in L^q\left((0,T), BV(\mathbb{R}^N)\right)^N$ and let $z, w \in \mathbb{R}^N$. Then, the difference quotient*

$$\frac{w^\top (g(t, x+\delta z) - g(t,x))}{\delta}$$

can be written down as $w^\top g_{1,\delta,z} + w^\top g_{2,\delta,z}$, where

(i) $w^\top g_{1,\delta,z} \to w^\top J_g z$ in $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$ as $\delta \to 0$, where J_g denotes the Radon-Nikodym derivative of the absolutely continuous part $D^a g$ of Dg with respect to $\mathcal{L}^N$.

(ii) *For any compact set $K \subset \mathbb{R}^N$ and for almost all $t \in (0,T)$ we have*

$$\limsup_{\delta\to 0} \int\limits_K \left|w^\top g_{2,\delta,z}(t,x)\right| \ dx \le \left|(w^\top D^s g z)(t,\cdot)\right|(K)$$

where $D^s g$ denotes the singular part of the measure Dg with respect to $\mathcal{L}^N$. Furthermore, for any measurable set $I \subset (0,T)$ we have

$$\limsup_{\delta\to 0} \int\limits_I \left(\int_K \left|w^\top g_{2,\delta,z}(t,x)\right| \ dx\right)^q \ dt \le \int\limits_I \left(\left|(w^\top D^s g z)(t,\cdot)\right|(K)\right)^q \ dt.$$

(iii) *For every compact set $K \subset \mathbb{R}^N$, for almost all $t \in (0,T)$ and $\varepsilon > 0$, we have*

$$\sup_{\delta\in(0,\varepsilon)} \int\limits_K \left(\left|w^\top g_{1,\delta,z}(t,x)\right| + \left|w^\top g_{2,\delta,z}(t,x)\right|\right) \ dx \le |w||z||Dg(t,\cdot)|(K_\varepsilon),$$

where $K_\varepsilon = \{x \in \mathbb{R}^N \,|\, \operatorname{dist}(x,K) \le \varepsilon\}$. Furthermore, for any measurable set $I \subset (0,T)$ we have

$$\sup_{\delta\in(0,\varepsilon)} \int\limits_I \left(\int_K \left(\left|w^\top g_{1,\delta,z}(t,x)\right| + \left|w^\top g_{2,\delta,z}(t,x)\right|\right) \ dx\right)^q \ dt$$
$$\le \int\limits_I \left(|w||z||Dg(t,\cdot)|(K_\varepsilon)\right)^q \ dt.$$

The next theorem is an adaptation of Theorem II.1 in DiPerna and Lions (1989) for vector fields with spatial BV-regularity instead of Sobolev regularity. It plays an important role in the proof for the second (improved) stability theorem. Before we present the theorem, we first need to introduce some definition.

Definition 5.1 *For any $\rho \in C_c^\infty\left(\mathbb{R}^N\right)$ and any $N \times N$-matrix M we define*

$$\Lambda(M,\rho) = \int\limits_{\mathbb{R}^N} |(\nabla\rho(z))^\top M z| \ dz.$$

Theorem 5.1 *Let $1 \le q < \infty$ and $b \in L^q((0,T), BV_0(\Omega))^N$ with $\operatorname{div} b \in L^q((0,T), L^\infty(\Omega))$ and denote by u the unique weak solution of the transport equation with initial data $u_0 \in L^\infty(\Omega)$. We set $u_\varepsilon := u * \rho_\varepsilon$, where ρ denotes an even mollifier for the spatial variable with $\operatorname{supp}(\rho) \subset \overline{B_1(0)}$ and where we extended u (by zero) to $(0,T) \times \mathbb{R}^N$. Then, u_ε satisfies*

$$\partial_t u_\varepsilon + \operatorname{div}(b u_\varepsilon) - u_\varepsilon \operatorname{div} b = r_\varepsilon \qquad \text{in } (0,T) \times \mathbb{R}^N,$$
$$u_\varepsilon(0,\cdot) = u_0 * \rho_\varepsilon \qquad \text{on } \mathbb{R}^N,$$

where

$$r_\varepsilon = r_{1,\varepsilon} + r_{2,\varepsilon} \qquad \text{with } r_{1,\varepsilon}, r_{2,\varepsilon} \in L^q\left((0,T), L^1(\mathbb{R}^N)\right)$$

and $r_{1,\varepsilon}, r_{2,\varepsilon}$ having the following properties:

(i) There exists some compact set $K \subset \mathbb{R}^N$ independent of ρ, such that

$$r_{1,\varepsilon}|_{(0,T)\times(\mathbb{R}^N\setminus K)} \equiv 0 \quad \text{and} \quad r_{2,\varepsilon}|_{(0,T)\times(\mathbb{R}^N\setminus K)} \equiv 0$$

for any $1 \geq \varepsilon > 0$.

(ii) $r_{1,\varepsilon} \to 0$ in $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$ as $\varepsilon \to 0$ and

(iii) for any measurable set $I \subset (0,T)$ and any compact set $W \subset \mathbb{R}^N$ we have

$$\limsup_{\varepsilon\to 0} \int\limits_I \left(\int_W |r_{2,\varepsilon}(t,x)| \ dx\right)^q dt \leq C \int\limits_I \left(\int_W \Lambda(M_b(t,x),\rho) \ d\left|D^s b(t,\cdot)\right|(x)\right)^q dt.$$

Here, M_b denotes the matrix valued Borel function such that $D^s b = M_b |D^s b|$ and $C > 0$ is a constant depending only on u.

Proof: We have

$$\begin{aligned} 0 &= [\partial_t u + \operatorname{div}(bu) - u \operatorname{div} b] * \rho_\varepsilon \\ &= \partial_t(u * \rho_\varepsilon) + \operatorname{div}(b(u*\rho_\varepsilon)) - u * \rho_\varepsilon \operatorname{div} b + \operatorname{div}(bu) * \rho_\varepsilon \\ &- (u \operatorname{div} b) * \rho_\varepsilon - \operatorname{div}(b(u*\rho_\varepsilon)) + u * \rho_\varepsilon \operatorname{div} b \end{aligned}$$

and thus

$$\partial_t(u_\varepsilon) + \operatorname{div}(b(u_\varepsilon)) - u_\varepsilon \operatorname{div} b = r_\varepsilon,$$

where r_ε is given by

$$r_\varepsilon = (u \operatorname{div} b) * \rho_\varepsilon - u * \rho_\varepsilon \operatorname{div} b + \operatorname{div}(b(u*\rho_\varepsilon)) - \operatorname{div}(bu) * \rho_\varepsilon.$$

Obviously, the term $(u \operatorname{div} b) * \rho_\varepsilon - u * \rho_\varepsilon \operatorname{div} b$ converges to zero in $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$. Thus, we have a closer look at the commutator

$$R_\varepsilon := \operatorname{div}(bu) * \rho_\varepsilon - \operatorname{div}(b(u*\rho_\varepsilon)).$$

We can rewrite R_ε using Lemma 5.1 as

$$\begin{aligned} &R_\varepsilon(t,x) \\ &= -\int\limits_{\mathbb{R}^N} u(t,x+\varepsilon z) b_{1,\varepsilon,z}(t,x)^\top \nabla\rho(z) \ dz - (u*\rho_\varepsilon)(t,x) \operatorname{div} b(t,x) \qquad (10) \\ &\quad - \int\limits_{\mathbb{R}^N} u(t,x+\varepsilon z) b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z) \ dz. \qquad (11) \end{aligned}$$

Then, we define $s_{1,\varepsilon}$ as the function given in (5) and $s_{2,\varepsilon}$ as the function given in (11). We set

$$K := \left\{x \in \mathbb{R}^N \middle| \ \operatorname{dist}(x,\Omega) \leq 2\right\}.$$

Then, since u is zero outside of Ω, we immediately obtain that

$$r_{1,\varepsilon}|_{(0,T)\times(\mathbb{R}^N\setminus K)} \equiv 0 \quad \text{and} \quad r_{2,\varepsilon}|_{(0,T)\times(\mathbb{R}^N\setminus K)} \equiv 0,$$

where we define $r_{1,\varepsilon} := (u \operatorname{div} b) * \rho_\varepsilon - u * \rho_\varepsilon \operatorname{div} b - s_{1,\varepsilon}$ and $r_{2,\varepsilon} = -s_{2,\varepsilon}$. The functions $s_{1,\varepsilon}$ and $s_{2,\varepsilon}$ are elements of $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$, due to the following reason: we set $i = 1, 2$ and estimate

$$\begin{aligned}
&\int_0^T \left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} u(t, x+\varepsilon z) b_{i,\varepsilon,z}(t,x)^\top \nabla\rho(z) \; dz \right| \; dx \right)^q dt \\
&\le \|u\|_{L^\infty((0,T)\times\Omega)} \int_0^T \left(\int_{B_1(0)} \int_K \left| b_{i,\varepsilon,z}(t,x)^\top \nabla\rho(z) \right| \; dxdz \right)^q dt \\
&\le \|u\|_{L^\infty((0,T)\times\Omega)} |B_1(0)|^{q-1} \int_{B_1(0)} \int_0^T \left(|\nabla\rho(z)| \, |z| \, |Db(t,\cdot)| \, (K_\varepsilon) \right)^q dtdz < \infty,
\end{aligned}$$

where we used point (iii) of Lemma 5.1. To finish the proof of point (ii) it remains to show that $s_{1,\varepsilon} \to 0$ in $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$. For almost all $t \in (0,T)$ we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u(t, x+\varepsilon z) b_{1,\varepsilon,z}(t,x)^\top \nabla\rho(z) \; dzdx \\
&\to \int_{\mathbb{R}^N} u(t,x) \sum_{i,j=1}^N e_i^\top J_b(t,x) e_j \int_{\mathbb{R}^N} z_j \partial_{z_i} \rho(z) \; dzdx \\
&= -\int_{\mathbb{R}^N} u(t,x) \operatorname{div} b(t,x) \; dx
\end{aligned}$$

as $\varepsilon \to 0$. Using Lebesgue's dominated convergence theorem and point (iii) of Lemma 5.1 we then obtain that

$$s_{1,\varepsilon} \to 0 \quad \text{in } L^q\left((0,T), L^1(R^N)\right)$$

as $\varepsilon \to 0$. It remains to show the property of $s_{2,\varepsilon}$. Due to point (ii) in Lemma 5.1 we know that for almost all $t \in (0,T)$ and for any compact set $W \subset \mathbb{R}^N$

$$\limsup_{\varepsilon\to 0} \int_W \left| b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z) \right| \; dx \le \left| (\nabla\rho(z))^\top D^s b(t,\cdot) z \right| (W).$$

Moreover, since the support of ρ is a subset of $\overline{B_1(0)}$, we obtain with Fatou's

Lemma for a measurable set $I \subset (0,T)$

$$\limsup_{\varepsilon\to 0} \int_I \left(\int_{\mathbb{R}^N} \int_W \left| b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z) \right| \, dxdz \right)^q dt$$
$$\leq \int_I \left(\int_{\mathbb{R}^N} \left| (\nabla\rho(z))^\top D^s b(t,\cdot) z \right| (W) \, dz \right)^q dt.$$

The last term can be rewritten as

$$\int_I \left(\int_{\mathbb{R}^N} \left| (\nabla\rho(z))^\top D^s b(t,\cdot) z \right| (W) \, dz \right)^q dt$$
$$= \int_I \left(\int_W \Lambda(M_b(t,x),\rho) \, d\left| D^s b(t,\cdot) \right| (x) \right)^q dt.$$

Thus, we conclude that

$$\limsup_{\varepsilon\to 0} \int_I \left(\int_W \left| s_{2,\varepsilon}(t,x) \right| \, dx \right)^q dt$$
$$\leq \limsup_{\varepsilon\to 0} \int_I \left(\int_W \int_{\mathbb{R}^N} \left| u(t,x+\varepsilon z) b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z) \right| \, dzdx \right)^q dt$$
$$\leq \|u\|^q_{L^\infty((0,T)\times\mathbb{R}^N)} \int_I \left(\int_W \Lambda(M_b(t,x),\rho) \, d\left| D^s b(t,\cdot) \right| (x) \right)^q dt.$$

□

Now, we are prepared for the main result of this section, which is a generalization of Theorem II.5 from DiPerna and Lions (1989) to vector fields with spatial BV-regularity.

THEOREM 5.2 (SECOND STABILITY THEOREM) *Let $q \in (1,\infty)$, $u_0 \in L^\infty(\Omega)$ and let $b \in L^\infty((0,T)\times\Omega)^N \cap L^q((0,T), BV_0(\Omega))^N$ with $\operatorname{div} b \in L^q((0,T), L^\infty(\Omega))$. Furthermore, let $(b_n) \subset \mathrm{V}_0^1$ and $(u_{0,n}) \subset L^\infty(\Omega)$ be two sequences with the following properties:*

(i) $(u_{0,n})$ is bounded in $L^\infty(\Omega)$ and converges to u_0 in $L^1(\Omega)$,

(ii) $(b_n) \subset L^q((0,T), BV_0(\Omega))^N$ is bounded and converges weakly to b in $L^1((0,T)\times\Omega)^N$,

(iii) $(\operatorname{div} b_n) \subset L^q((0,T), L^\infty(\Omega))$ and is bounded in $L^1((0,T), L^\infty(\Omega))$.

Then, for any $1 \leq p < \infty$, the sequence of unique solutions $(u_n) \subset C([0,T], L^\infty(\Omega)-w^*)$ of (1) with vector fields b_n and initial data $u_{0,n}$ is a subset of $C([0,T], L^p(\Omega))$ and converges in $C([0,T], L^p(\Omega))$ to the unique solution $u \in C([0,T], L^p(\Omega))$ of (1) with vector field b and initial value u_0.

We prepare the proof of the Theorem in several steps (Lemmas 5.2–5.5). In the following, if some Lebesgue function is just defined on a proper subset of $\mathbb{R}^N$

in the spatial variable, then we extend this function by zero to the whole $\mathbb{R}^N$ if we consider the function as some function defined on $\mathbb{R}^N$ in our calculations.

We take some even mollifier $\rho \in C_c^\infty(B_1(0))$ and we set $u_\varepsilon := u * \rho_\varepsilon$ for the unique solution u of the transport equation with vector field b and initial value u_0. We will prove the theorem in several consecutive lemmas. In the first lemma we obtain an expression for the difference of $u_n - u_\varepsilon$.

LEMMA 5.2 *Under the assumptions of Theorem 5.2 the following expression for the difference $u_n - u_\varepsilon$ holds:*

$$\partial_t \int_K (u_n - u_\varepsilon)^2 \, dx - \int_K (u_n - u_\varepsilon)^2 \operatorname{div} b_n \, dx$$
$$= 2 \int_K (u_n - u_\varepsilon)\,(-r_{1,\varepsilon} - r_{2,\varepsilon} + (b - b_n)\cdot \nabla u_\varepsilon)\, dx, \tag{12}$$

where $K \subset \mathbb{R}^N$ denotes the compact set of Theorem 5.1.

PROOF Due to Theorem 5.1 we deduce that u_ε satisfies

$$\begin{aligned} \partial_t u_\varepsilon + \operatorname{div}(b u_\varepsilon) - u_\varepsilon \operatorname{div} b = r_{1,\varepsilon} + r_{2,\varepsilon} \qquad & \text{in } (0,T)\times\mathbb{R}^N, \\ u_\varepsilon(0,\cdot) = u_0 * \rho_\varepsilon \qquad & \text{on } \mathbb{R}^N. \end{aligned}$$

We first assume that $u_{0,l} \in C_c^\infty(\Omega)$ and $b_l \in C_c^\infty((0,T)\times\Omega)$. Then, the corresponding solution u_l of the transport equation is also smooth with zero spatial boundary value. These functions can be obviously extended in a smooth way to $\mathbb{R}^N$ in the spatial domain. We take $\beta \in C^1(\mathbb{R})$ such that $\beta(0) = 0$. Then, we write

$$\partial_t \beta(u_l - u_\varepsilon) + \operatorname{div}(b_l \beta(u_l - u_\varepsilon)) - \beta(u_l - u_\varepsilon) \operatorname{div} b_l \tag{13}$$
$$= \beta'(u_l - u_\varepsilon)\,(\partial_t(u_l - u_\varepsilon) + \operatorname{div}(b_l(u_l - u_\varepsilon)) - (u_l - u_\varepsilon)\operatorname{div} b_l)$$
$$= \beta'(u_l - u_\varepsilon)\,(-r_{1,\varepsilon} - r_{2,\varepsilon} + (b - b_l)\cdot\nabla u_\varepsilon)\,. \tag{14}$$

For the initial value we have that $\beta(u_l(0,\cdot) - u_\varepsilon(0,\cdot)) = \beta(u_{0,l} - u_0 * \rho_\varepsilon)$. In the following, we denote by K the compact set given in point (i) in Theorem 5.1 and we know that $\Omega \subset K$. Now, integrating over K yields

$$\partial_t \int_K \beta(u_l - u_\varepsilon)\, dx - \int_K \beta(u_l - u_\varepsilon) \operatorname{div} b_l \, dx$$
$$= \int_K \beta'(u_l - u_\varepsilon)\,(-r_{1,\varepsilon} - r_{2,\varepsilon} + (b - b_l)\cdot\nabla u_\varepsilon)\, dx.$$

The choice of $\beta(t) = t^2$ for $t \in \mathbb{R}$ yields that

$$\partial_t \int_K (u_l - u_\varepsilon)^2 \, dx - \int_K (u_l - u_\varepsilon)^2 \operatorname{div} b_l \, dx$$
$$= 2 \int_K (u_l - u_\varepsilon)\,(-r_{1,\varepsilon} - r_{2,\varepsilon} + (b - b_l)\cdot\nabla u_\varepsilon)\, dx.$$

Our first assumption was that u_l, b_l and $u_{0,l}$ are smooth functions. Therefore, we take a sequence of smooth functions $(b_{n,k})_k$ such that

$$b_{n,k} \to b_n \quad \text{in } L^1((0,T)\times\Omega)^N \quad \text{and} \quad \operatorname{div} b_{n,k} \to \operatorname{div} b_n \quad \text{in } L^1((0,T)\times\Omega) \quad \text{as } k\to\infty.$$

In addition, we take a sequence of smooth and bounded functions $(u_{0,n,k})_k \subset C_c^\infty(\Omega)$, converging to $u_{0,n}$ in $L^1(\Omega)$. Then, the above equation is valid for $b_{n,k}$ and $u_{n,k}$ and Theorem 4.1 yields for $k\to\infty$

$$\partial_t \int_K (u_n-u_\varepsilon)^2\,dx - \int_K (u_n-u_\varepsilon)^2 \operatorname{div} b_n\,dx = 2\int_K (u_n-u_\varepsilon)\left(-r_{1,\varepsilon}-r_{2,\varepsilon}+(b-b_n)\cdot\nabla u_\varepsilon\right)\,dx.$$

□

LEMMA 5.3 *Under the assumptions of Theorem 5.2 the following estimate holds:*

$$\begin{aligned}
&\int_K ((u_n-u_\varepsilon)(t,\cdot))^2\,dx \\
&\leq (C_2+1)\cdot\left(C_1\int_0^T\int_K |r_{1,\varepsilon}|\,dxds + \int_K ((u_{0,n}-u_{0,\varepsilon})^2\,dx\right) \\
&+2\left|\int_0^t\int_K (u_n-u_\varepsilon)r_{2,\varepsilon}\,dxds\right| \\
&+2C_2\max_{s\in[0,T]}\left|\int_0^s\int_K (u_n-u_\varepsilon)r_{2,\varepsilon}\,dxdr\right| + 2\left|\int_0^t\int_K (u_n-u_\varepsilon)(b-b_n)\cdot\nabla u_\varepsilon\,dxds\right| \\
&+2C_3\int_0^t \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)}\left|\int_0^s\int_K (u_n-u_\varepsilon)(b-b_n)\cdot\nabla u_\varepsilon\,dxdr\right|\,ds \qquad (15)
\end{aligned}$$

for some constants $C_3, C_2, C_1 > 0$ and any $t\in[0,T]$.

Proof: We use expression (12) of Lemma 5.2 and estimate:

$$\partial_t \int_K ((u_n - u_\varepsilon))^2 \, dx$$
$$\leq \|\operatorname{div} b_n(t,\cdot)\|_{L^\infty(\Omega)} \int_K (u_n - u_\varepsilon)^2 \, dx + C_1 \int_K |r_{1,\varepsilon}| \, dx$$
$$- 2 \int_K (u_n - u_\varepsilon) r_{2,\varepsilon} \, dx$$
$$+ 2 \int_K (u_n - u_\varepsilon)(b - b_n) \cdot \nabla u_\varepsilon \, dx,$$

where $C_1 > 0$ can be chosen as $C_1 := 2 \sup_n \|u_{0,n}\|_{L^\infty(\Omega)} + 2\|u_0\|_{L^\infty(\Omega)}$. By integrating in time, we get

$$\int_K ((u_n - u_\varepsilon)(t,\cdot))^2 \, dx$$
$$\leq \int_0^t \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)} \int_K ((u_n - u_\varepsilon))^2 \, dxds$$
$$+ 2 \left| \int_0^t \int_K (u_n - u_\varepsilon)(b - b_n) \cdot \nabla u_\varepsilon \, dxds \right|$$
$$+ C_1 \int_0^T \int_K |r_{1,\varepsilon}| \, dxds + 2 \left| \int_0^t \int_K (u_n - u_\varepsilon) r_{2,\varepsilon} \, dxds \right| + \int_K ((u_{0,n} - u_{0,\varepsilon})^2 \, dx.$$

Using Grönwall's Lemma we obtain

$$\int_K ((u_n - u_\varepsilon)(t,\cdot))^2 \, dx \leq$$

$$\left(C_1 \int_0^T \int_K |r_{1,\varepsilon}| \, dxds + \int_K ((u_{0,n} - u_{0,\varepsilon})^2 \, dx \right)$$

$$\cdot \left(1 + \int_0^t \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)} \, e^{\int_s^t \|\operatorname{div} b_n(r,\cdot)\|_{L^\infty(\Omega)} dr} \, ds \right) +$$

$$+2\left|\int_0^t\int_K (u_n-u_\varepsilon)r_{2,\varepsilon}\ dxds\right| + 2\left|\int_0^t\int_K (u_n-u_\varepsilon)(b-b_n)\cdot\nabla u_\varepsilon\ dxds\right|$$

$$+2\int_0^t\left|\int_0^s\int_K (u_n-u_\varepsilon)r_{2,\varepsilon}\ dxdr\right| \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)}\, e^{\int_s^t\|\operatorname{div} b_n(r,\cdot)\|_{L^\infty(\Omega)}dr}\ ds$$

$$+2\int_0^t\left|\int_0^s\int_K (u_n-u_\varepsilon)(b-b_n)\cdot\nabla u_\varepsilon\ dxdr\right|\times$$
$$\times\|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)}\, e^{\int_s^t\|\operatorname{div} b_n(r,\cdot)\|_{L^\infty(\Omega)}dr}\ ds.$$

Setting

$$C_2 := e^{\sup_n\int_0^T\|\operatorname{div} b_n(t,\cdot)\|_{L^\infty(\Omega)}dt}\sup_n\int_0^T\|\operatorname{div} b_n(t,\cdot)\|_{L^\infty(\Omega)}\,dt$$
$$\text{and}\quad C_3 := e^{\sup_n\int_0^T\|\operatorname{div} b_n(t,\cdot)\|_{L^\infty(\Omega)}dt}$$

yields the statement of the lemma. □

LEMMA 5.4 *Under the assumptions of Theorem 5.2 we have*

$$\begin{aligned}
&\limsup_{n\to\infty}\left(\int_K |u_n(t,\cdot)-u(t,\cdot)|\ dx\right)^2\\
&\le C_5\int_K |u_\varepsilon(t,\cdot)-u(t,\cdot)|\ dx + C_4\int_K (u_\varepsilon(t,\cdot)-u(t,\cdot))^2\ dx + 2CC_1R_\varepsilon(s^*)\\
&+ C_4C_1(C_1+1)\int_0^T\int_K |r_{1,\varepsilon}|\ dxds + C_4(C_2+1)\int_K ((u_0-u_{0,\varepsilon})^2\ dx\\
&+ 2C_4\left|\int_0^t\int_K (w_1-u_\varepsilon)r_{2,\varepsilon}\ dxds\right|
\end{aligned}\tag{16}$$

for some specific $w_1\in L^\infty((0,T)\times\Omega)$, $s^*\in[0,T]$ *and some function* $R_\varepsilon\in C([0,T])$.

PROOF: The proof of Theorem 4.1 shows that there are subsequences $(u_n),(u_n^2)\in C([0,T],L^\infty(\Omega)\text{–}w^*)$ and $(u_n\operatorname{div} b_n),(u_n^2\operatorname{div} b_n)\in L^1((0,T),L^\infty(\Omega))$

(labeled by n again) and $w_1, w_2 \in L^\infty((0,T)\times\Omega)$ and $w_3, w_4 \in L^1((0,T)\times\Omega)$ such that $u_n \overset{*}{\rightharpoonup} w_1$ in $L^\infty((0,T)\times\Omega)$ and

$$\begin{aligned} u_n &\rightharpoonup w_1 &\quad \text{and} \quad& u_n^2 \rightharpoonup w_2 &\quad \text{in } C([0,T], L^2(\Omega)-w),\\ u_n \operatorname{div} b_n &\rightharpoonup w_3 &\quad \text{and} \quad& u_n^2 \operatorname{div} b_n \rightharpoonup w_4 &\quad \text{in } L^1((0,T)\times\Omega). \end{aligned}$$

In particular, $w_1(0,\cdot) = u_0$ and $w_2(0,\cdot) = u_0^2$. We restrict ourselves to these subsequences. Furthermore, the mappings $R_{n,\varepsilon} : [0,T] \to \mathbb{R}$, defined by $s \mapsto R_{n,\varepsilon}(s) := \left|\int_0^s \int_K (u_n - u_\varepsilon) r_{2,\varepsilon}\ dxdr\right|$, are equicontinuous in n: for $0 \le s \le t \le T$ we obtain that

$$|R_{n,\varepsilon}(t) - R_{n,\varepsilon}(s)| \le \left| \int_s^t \int_K (u_n - u_\varepsilon) r_{2,\varepsilon}\ dxdr \right| \le C_1 \int_s^t \int_K |r_{2,\varepsilon}|\ dxdr.$$

We set $R_\varepsilon : [0,T] \to \mathbb{R}$, $s \mapsto R_\varepsilon(s) := \left|\int_0^s \int_K (w_1 - u_\varepsilon) r_{2,\varepsilon}\ dxdr\right|$ and obtain that $R_{n,\varepsilon}(s) \to R_\varepsilon(s)$ for all $s \in [0,T]$. As $R_{n,\varepsilon}$ are continuous functions for all $n \in \mathbb{N}$, we find $s_n \in [0,T]$ such that

$$R_{n,\varepsilon}(s_n) := \max_{s\in[0,T]} R_{n,\varepsilon}(s).$$

Then, (s_n) represents a bounded sequence and thus, there is a convergent subsequence (s_n) (which is labeled by n again) with limit $s^* \in [0,T]$. We restrict our considerations to this subsequence. We conclude for the subsequence that

$$|R_{n,\varepsilon}(s_n) - R_\varepsilon(s^*)| \le |R_{n,\varepsilon}(s_n) - R_{n,\varepsilon}(s^*)| + |R_{n,\varepsilon}(s^*) - R_\varepsilon(s^*)| \to 0 \quad (17)$$

as $n \to \infty$, since $R_{n,\varepsilon}$ are equicontinuous. Now, we estimate

$$\begin{aligned} &\left(\int_K |u_n - u|\ dx \right)^2 \\ &\le \left(\int_K |u_n - u_\varepsilon|\ dx \right)^2 \\ &\qquad + \left(\int_K |u_\varepsilon - u|\ dx \right)^2 + 2 \int_K |u_n - u_\varepsilon|\ dx \int_K |u_\varepsilon - u|\ dx \\ &\le C_4 \int_K (u_n - u_\varepsilon)^2\ dx + C_4 \int_K (u_\varepsilon - u)^2\ dx + C_5 \int_K |u_\varepsilon - u|\ dx \end{aligned} \quad (18)$$

with $C_4 = |K|^{1/2}$. As in the proof of Theorem 4.1, we obtain as a consequence of Theorem 3.1 that

$$u_n b_n \overset{*}{\rightharpoonup} w_1 b \quad \text{in } \mathcal{M}((0,T)\times\Omega)^N. \quad (19)$$

Since (u_n) is bounded in $L^\infty((0,T)\times\Omega)$ and (b_n) is bounded in $L^p((0,T)\times\Omega)^N$ for $p=\min(q,N/(N-1))$, we obtain that $(u_n b_n)$ is bounded in $L^p((0,T)\times\Omega)^N$ and thus with (19) we deduce that $u_n b_n \rightharpoonup w_1 b$ in $L^p((0,T)\times\Omega)^N$. Consequently, we obtain that

$$\left|\int_0^s\int_K (u_n-u_\varepsilon)(b-b_n)\cdot\nabla u_\varepsilon\ dxdr\right| \to 0 \quad \text{as } n\to 0$$

for any $s\in[0,T]$ and with Lebesgue's dominated convergence theorem we conclude that

$$\int_0^t \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)}\left|\int_0^s\int_K (u_n-u_\varepsilon)(b-b_n)\cdot\nabla u_\varepsilon\ dxdr\right|\ ds\to 0$$

as $n\to\infty$ for any $t\in[0,T]$. Taking the limes superior over n and using estimates (5.3), (18), as well as relation (17), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\left(\int_K |u_n(t,\cdot)-u(t,\cdot)|\ dx\right)^2\\
&\le C_5\int_K |u_\varepsilon(t,\cdot)-u(t,\cdot)|\ dx + C_4\int_K (u_\varepsilon(t,\cdot)-u(t,\cdot))^2\ dx\\
&\qquad +2C_4C_2R_\varepsilon(s^*)+C_4C_1(C_2+1)\int_0^T\int_K |r_{1,\varepsilon}|\ dxds\\
&\qquad +C_4(C_2+1)\int_K ((u_0-u_{0,\varepsilon})^2\ dx + 2C_4\left|\int_0^t\int_K (w_1-u_\varepsilon)r_{2,\varepsilon}\ dxds\right|.
\end{aligned}$$

□

Lemma 5.5 *Under the assumptions of Theorem 5.2 there exists a sequence* (ε_m) *with* $0<\varepsilon_m\le 1$ *for all* $m\in\mathbb{N}$ *and* $\varepsilon_m\to 0$ *as* $m\to\infty$ *such that*

$$2(w_1-u_{\varepsilon_m})r_{2,\varepsilon_m}\overset{*}{\rightharpoonup}\sigma \quad \text{in } \mathcal{M}([0,T]\times K) \quad \text{as } m\to\infty.$$

The measure $\sigma\in\mathcal{M}([0,T]\times K)$ *is independent of the mollifier* ρ.

Proof: We know that

$$2\sup_{0<\varepsilon\le 1}\int_0^T\int_K |w_1(t,x)-u_\varepsilon(t,x)|\,|r_{2,\varepsilon}(t,x)|\ dxdt<\infty$$

and thus, there exists a sequence (ε_m) with $0 < \varepsilon_m \leq 1$ for all $m \in \mathbb{N}$ and $\varepsilon_m \to 0$ such that $2(w_1 - u_{\varepsilon_m})r_{2,\varepsilon_m}$ converges to some $\sigma_\rho \in \mathcal{M}([0,T] \times K)$. This limit measure σ_ρ is not depending on ρ:
for $t \in (0,T)$ we take the following sequence $(\eta_{t,k}) \subset C_c^\infty([0,T))$, such that

$$0 \leq \eta_{t,k}(s) \leq 1 \,\forall\, s \in (0,T), \quad \eta_{t,k}(s) \to \chi_{[0,t]}(s) \,\forall\, s \in [0,T) \quad \text{and} \quad \eta'_{t,k} \to \delta_0 - \delta_t$$

in the distributional sense. Lebesgue's dominated convergence theorem then yields that $\eta_{t,k} \to \chi_{[0,t]}$ in $L^r((0,T))$ for all $1 \leq r < \infty$ and for any $t \in [0,T)$. Hence, from the equation given by lines (13) and (14) we deduce, by setting $\beta(t) = t^2$ for all $t \in \mathbb{R}$ and integrating over $[0,T] \times K$ with test functions $\varphi \in C_c^\infty([0,T] \times K)$ and fixed $s \in [0,T)$, that

$$\begin{aligned} 0 = \int_0^T \eta'_{s,k} \int_K (u_n - u_{\varepsilon_m})^2 \varphi \, dxdt + \int_K \eta_{s,k}(0)\varphi(0,\cdot)(u_n(0,\cdot) - u_{\varepsilon_m}(0,\cdot))^2 \, dx \\ + \int_0^T \int_K (u_n - u_{\varepsilon_m})^2 \eta_{s,k}(\partial_t \varphi + b_n \cdot \nabla\varphi + \varphi \operatorname{div} b_n) \\ + 2(u_n - u_{\varepsilon_m})\varphi\eta_{s,k}(-r_{1,\varepsilon_m} - r_{2,\varepsilon_m} + (b - b_n)\cdot \nabla u_{\varepsilon_m}) \, dxdt. \end{aligned}$$

where u_n and b_n denote the above solutions and vector fields. Now, taking the limit in n yields, with the same argument as in the proof of the previous lemma for products of weakly convergent sequences,

$$\begin{aligned} 0 = & \int_0^T \int_K (w_2 - 2w_1 u_{\varepsilon_m} + u_{\varepsilon_m}^2)(\varphi\eta'_{s,k} + \eta_{s,k}(\partial_t\varphi + b\cdot\nabla\varphi))dxdt \\ & + \int_0^T \int_K \varphi\eta_{s,k}(w_4 - 2w_3 u_{\varepsilon_m} + u_{\varepsilon_m}^2 \operatorname{div} b) \, dxdt \\ & + \int_K \eta_{s,k}(0)\varphi(0,\cdot)\left(u_0^2 - 2u_{\varepsilon_m}(0,\cdot)u_0 + (u_{\varepsilon_m}(0,\cdot))^2\right) \, dx \\ & - 2\int_0^T \int_K (w_1 - u_{\varepsilon_m})\varphi\eta_{s,k}(r_{1,\varepsilon_m} + r_{2,\varepsilon_m}) \, dxdt. \end{aligned} \tag{20}$$

For the last term in (20), we have

$$2\left|\int_0^T\int_K(\eta_{s,k}-\chi_{[0,s]})(w_1-u_{\varepsilon_m})\varphi(r_{1,\varepsilon_m}+r_{2,\varepsilon_m})\ dxdt\right|$$

$$\leq 2\left(\int_0^T\left|\eta_{s,k}-\chi_{[0,s]}\right|^{q'}dt\right)^{1/q'}$$

$$\times\left(\int_0^T\left(\int_K|(w_1-u_{\varepsilon_m})\varphi(r_{1,\varepsilon_m}+r_{2,\varepsilon_m})|\ dx\right)^q dt\right)^{1/q}$$

$$\leq 2C\left(\int_0^T\left|\eta_{s,k}-\chi_{[0,s]}\right|^{q'}dt\right)^{1/q'}\to 0\quad\text{as } k\to\infty,$$

where $C>0$ is an upper bound for

$$\sup_{m\in\mathbb{N}}\left(\int_0^T\left(\int_K|(w_1-u_{\varepsilon_m})\varphi(r_{1,\varepsilon_m}+r_{2,\varepsilon_m})|\ dx\right)^q dt\right)^{1/q}.$$

Thus, we can switch the limiting processes of $k\to\infty$ and $m\to\infty$ and we obtain, using $r_{1,\varepsilon_m}\to 0$ in $L^1((0,T)\times K)$ as $m\to\infty$

$$\lim_{k\to\infty}\langle\sigma_\rho,\varphi\eta_{s,k}\rangle=\lim_{m\to\infty}\lim_{k\to\infty}2\int_0^T\int_K(w_1-u_{\varepsilon_m})r_{2,\varepsilon_m}\varphi\eta_{s,k}\ dxdt$$

$$=\lim_{m\to\infty}\lim_{k\to\infty}\int_0^T\int_K(w_2-2w_1u_{\varepsilon_m}+u_{\varepsilon_m}^2)(\varphi\eta'_{s,k}+\eta_{s,k}(\partial_t\varphi+b\cdot\nabla\varphi))\ dxdt$$

$$+\lim_{m\to\infty}\lim_{k\to\infty}\int_K\eta_{s,k}(0)\varphi(0,\cdot)\left(u_0^2-2u_{\varepsilon_m}(0,\cdot)u_0+(u_{\varepsilon_m}(0,\cdot))^2\right)\ dx$$

$$+\lim_{m\to\infty}\lim_{k\to\infty}\int_0^T\eta_{s,k}\int_K\varphi(w_4-2w_3u_{\varepsilon_m}+u_{\varepsilon_m}^2\operatorname{div}b)\ dxdt$$

$$= \lim_{m\to\infty} \left[\int_K \varphi(0,\cdot) \left(u_0^2 - 2u_0 u_{\varepsilon_m}(0,\cdot) + (u_{\varepsilon_m}(0,\cdot))^2 + w_2(0,\cdot) \right.\right.$$

$$\left. -2w_1(0,\cdot) u_{\varepsilon_m}(0,\cdot) + (u_{\varepsilon_m}(0,\cdot))^2 \right) \, dx$$

$$\left. - \int_K \varphi(s,\cdot) \left(w_2(s,\cdot) - 2w_1(s,\cdot) u_{\varepsilon_m}(s,\cdot) + (u_{\varepsilon_m}(s,\cdot))^2 \right) \, dx \right]$$

$$+ \lim_{m\to\infty} \left[\int_0^s \int_K (w_2 - 2w_1 u_{\varepsilon_m} + u_{\varepsilon_m}^2)(\partial_t \varphi + b \cdot \nabla \varphi) \, dxdt \right.$$

$$\left. + \varphi(w_4 - 2w_3 u_{\varepsilon_m} + u_{\varepsilon_m}^2 \operatorname{div} b) \, dxdt \right]$$

$$= \int_0^s \int_K (w_2 - 2w_1 u + u^2)(\partial_t \varphi + b \cdot \nabla \varphi) + \varphi(w_4 - 2w_3 u + u^2 \operatorname{div} b) \, dxdt$$

$$- \int_K \varphi(s,\cdot) \left(w_2(s,\cdot) - 2w_1(s,\cdot) u + u(s,\cdot)^2 \right) \, dx$$

since

$$\begin{aligned} & w_2(0,\cdot) - 2w_1(0,\cdot) u_{\varepsilon_m}(0,\cdot) + (u_{\varepsilon_m}(0,\cdot))^2 \\ = {} & u_0^2 - 2u_0 u_{\varepsilon_m}(0,\cdot) + (u_{\varepsilon_m}(0,\cdot))^2 \rightharpoonup 0 \quad \text{in } L^2(\Omega). \end{aligned}$$

From the above equation and the preceding estimates and equations we obtain the following information: if we omit $\eta_{s,k}$ at the beginning and just test with φ, we see that the measure σ_ρ is given by

$$\sigma_\rho = -\partial_t(w_2 - 2w_1 u + u^2) - \operatorname{div}(b(w_2 - 2w_1 u + u^2)) + (w_4 - 2w_3 u + u^2 \operatorname{div} b)$$

and thus, it is independent of the mollifier ρ. Therefore, we call σ_ρ just σ in the following. Furthermore, if we restrict σ to the set $[0,s] \times K$ and denote the restriction σ_s, we obtain from the above equation for any $\varphi \in C_c([0,T] \times K)$:

$$\begin{aligned}
&\int_{[0,s]} \int_K \varphi \, d\sigma_s = \int_{[0,T]} \int_K \chi_{[0,s]} \varphi \, d\sigma \\
&= \lim_{k\to\infty} \int_{[0,T]} \int_K \varphi(\chi_{[0,s]} - \eta_{s,k}) \, d\sigma + \lim_{k\to\infty} \int_{[0,T]} \int_K \varphi \eta_{s,k} \, d\sigma \\
&= -\int_K \varphi(s,\cdot)(w_2(s,\cdot) - 2w_1(s,\cdot)u + (u(s,\cdot))^2) \, dx \\
&+ \int_0^s \int_K (w_2 - 2w_1 u + u^2)(\partial_t \varphi + b \cdot \nabla \varphi) \\
&+ \varphi(w_4 - 2w_3 u + u^2 \operatorname{div} b) \, dxdt,
\end{aligned}$$

i.e. the restriction $2[(w_1 - u_{\varepsilon_m}) r_{2,\varepsilon_m}]|_{[0,s]\times K} \mathcal{L}^1 \otimes \mathcal{L}^N$ converges weakly* to

$$\begin{aligned}
\sigma_s &= -\partial_t \left((w_2 - 2w_1 u + u^2)|_{[0,s]\times K} \right) - \operatorname{div} \left(b(w_2 - 2w_1 u + u^2)|_{[0,s]\times K} \right) \\
&\quad + (w_4 - 2w_3 u + u^2 \operatorname{div} b)|_{[0,s]\times K}.
\end{aligned}$$

□

Proof of Theorem 5.2: We first introduce the set

$$\mathcal{K} := \left\{ \rho \in C_c^\infty(B_1(0)) \text{ such that } \rho \geq 0 \text{ is even, and } \int_{B_1(0)} \rho(x) \, dx = 1 \right\}.$$

So far, we have shown that our limits do not depend on the specific mollifier and we go back to estimate (16). Taking the supremum over $m \in \mathbb{N}$ with $t \in [0,T]$ and $\varphi \equiv 1$ on $[0, \max(t, s^*)] \times K$ yields:

$$\begin{aligned}
&\limsup_{n\to\infty} \left(\int_K |u_n(t,x) - u(t,x)| \, dx \right)^2 \\
&\leq 2C \sup_{m\in\mathbb{N}} \left| \int_0^t \int_K (w_1(s,x) - u_{\varepsilon_m}(s,x)) r_{2,\varepsilon_m}(s,x) \, dxds \right| \\
&+ CC_1 \sup_{m\in\mathbb{N}} R_{\varepsilon_m}(s^*) \\
&= C \left| \sigma_t([0,t] \times K) \right| + CC_1 \left| \sigma_{s^*}([0,s^*] \times K) \right|.
\end{aligned}$$

Now, in the remaining part, we show that $\sigma = 0$. This will work in the same way as it is shown that the limit measure of the commutator is zero in Crippa (2007). The sequence $(|(w_1 - u_{\varepsilon_m}) r_{2,\varepsilon_m}|)$ is bounded in $L^1((0,T) \times K)$

and thus, a subsequence converges weakly* to some measure $\lambda \in \mathcal{M}([0,T]\times K)$. Due to Proposition 1.62 in Ambrosio, Fusco and Pallara (2000) we have that $|\sigma| \leq \lambda$. Hence, restricting to this subsequence we obtain for $\varphi \in C_c([0,T]\times K)$

$$
\begin{aligned}
&\int_{[0,T]}\int_K |\varphi(t,x)|\ d|\sigma|(t,x)\\
&\leq \limsup_{m\to\infty}\int_0^T\int_K |\varphi(t,x)|\,|(w_1(t,x)-u_{\varepsilon_m}(t,x))r_{2,\varepsilon_m}(t,x)|\ dxdt\\
&\leq C\limsup_{m\to\infty}\int_0^T\int_K |\varphi(t,x)|\int_{\mathbb{R}^N}|b_{2,\varepsilon_m,z}(t,x)\cdot\nabla\rho(z)|\ dzdxdt. \qquad (21)
\end{aligned}
$$

Now, upon setting $S := \|\varphi\|_{C([0,T]\times K)}$ and

$$W_{t,y} := \overline{\{x\in K|\ |\varphi|(t,x) > y\}}$$

we rewrite (21) and obtain

$$
\begin{aligned}
C\limsup_{m\to\infty}&\int_0^T\int_0^S\int_{W_{t,y}}\int_{\mathbb{R}^N}|b_{2,\varepsilon_m,z}(t,x)\cdot\nabla\rho(z)|\ dzdxdydt\\
&\leq C\int_0^T\int_0^S\int_{\mathbb{R}^N}\limsup_{m\to\infty}\int_{W_{t,y}}|b_{2,\varepsilon_m,z}(t,x)\cdot\nabla\rho(z)|\ dxdzdydt\\
&\leq C\int_0^T\int_0^S\int_{\mathbb{R}^N}\left|(\nabla\rho(z))^\top(D^sb)(t,\cdot)z\right|(W_{t,y})\ dzdydt\\
&= C\int_0^T\int_K|\varphi(t,x)|\,\Lambda(M_b(t,x),\rho)\ d|D^sb(t,\cdot)|(x)dt.
\end{aligned}
$$

Thus, $|\sigma| \leq C\Lambda(M_b,\rho)\,|D^sb|$, and hence there exists a Borel function f such that $|\sigma| = f\,|D^sb|$ and

$$|f(t,x)| \leq C\Lambda(M_b(t,x),\rho) \qquad \text{for } |D^sb|\text{-a.e. } (t,x).$$

Since $|\sigma|$ does not depend on the mollifier ρ, we deduce with the same argumentation as in Crippa (2007) that

$$|f(t,x)| \leq \inf_{\rho\in\mathcal{K}'} C\Lambda(M_b(t,x),\rho) = \inf_{\rho\in\mathcal{K}} C\Lambda(M_b(t,x),\rho) \qquad \text{for } |D^sb|\text{-a.e. } (t,x),$$

where $\mathcal{K}' \subset \mathcal{K}$ denotes a countable dense subset. Then, the Lemma of Alberti (see Lemma 2.6.6 in Crippa, 2007) yields that

$$|f(t,x)| \leq C\,|\mathrm{trace}(M_b(t,x))| = 0 \qquad \text{for } |D^sb|\text{-a.e. } (t,x),$$

since the singular part of div b is zero. Therefore, we obtain that $\sigma = 0$ and thus for $t \in [0, T)$

$$\limsup_{n\to\infty} \left(\int_{\mathbb{R}^N} |u_n(t,x) - u(t,x)| \ dx \right)^2 = 0.$$

For the subsequence (u_n) being convergent to w_1 in $C([0,T], L^2(\Omega) - w)$, we conclude that $w_1(t,\cdot) = u(t,\cdot)$ for all $t \in [0,T]$. Analogously, we obtain that $w_2(t,\cdot) = u^2(t,\cdot)$ for all $t \in [0,T]$. Using a proof by contradiction as in the case of Theorem 4.1, we obtain that the whole sequence (u_n) converges to u in $C([0,T], L^2(\Omega))$ and using the boundedness of (u_n) in $L^\infty((0,T) \times \Omega)$, we get that the convergence is valid in $C([0,T], L^p(\Omega))$ for any $1 \le p < \infty$. □

6. Predual of the space $BV(\Omega)$

In the space $BV(\Omega)$ an often used topology is the so-called weak–topology. The name of the topology is misleading, since this topology is not the standard weak–topology in functional analysis if $BV(\Omega)$ is seen as a dual space of a separable Banach space. In Remark 3.12 in Ambrosio, Fusco and Pallara (2000) it is mentioned that these two topologies coincide if the domain is sufficiently regular. We will show that Lipschitz regularity for the domain is sufficient. With this result we do not need to distinguish between these two topologies in the subsequent parts, in particular in the case when we consider vector fields as Gelfand integrable functions, where $BV(\Omega)$ is regarded as a dual space with (dual) weak–topology. We also refer to Pełczyński and Wojciechowski (2003) for a related characterization of the predual of $BV(\Omega)$.

In Remark 3.12 in Ambrosio, Fusco and Pallara (2000), a sketch for constructing the predual of $BV(\Omega)$ is given. In the following, we call $\Gamma(\Omega)$ the predual of $BV(\Omega)$ and we give a precise construction of $\Gamma(\Omega)$: we set $X := C_0(\Omega)^{N+1}$ and

$$\begin{aligned} E := \big\{ &\Phi = (\Phi_0, \ldots, \Phi_N) \in X, \varphi = (\Phi_1, \ldots, \Phi_N) \in C_c^\infty(\Omega)^N \\ &\text{such that } \operatorname{div} \varphi = \Phi_0 \big\}. \end{aligned}$$

Then, E is a subspace of X and we set Y as the closure of E with respect to $\|\cdot\|_{C(\Omega)^{N+1}}$. Now, Remark 3.12 in Ambrosio, Fusco and Pallara (2000) yields that the map T given by

$$T : BV(\Omega) \to \mathcal{M}(\Omega)^{N+1}, \qquad u \mapsto (u\mathcal{L}^N, \partial_{x_1} u, \ldots, \partial_{x_N} u)$$

is an isomorphism between $BV(\Omega)$ and $T(BV(\Omega))$ with

$$\|u\|_{BV(\Omega)} \le 2 \|T(u)\|_{\mathcal{M}(\Omega)^{N+1}} \le 2 \|u\|_{BV(\Omega)}.$$

Furthermore, for all $\Phi \in E$ and $u \in BV(\Omega)$ we have that

$$\begin{aligned}
&(T(u), \Phi)_{(\mathcal{M}(\Omega)^{N+1}, C_0(\Omega)^{N+1})} \\
&= \left(u\mathcal{L}^N, \Phi_0\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} + \sum_{k=1}^{N} (\partial_{x_k} u, \Phi_k)_{(\mathcal{M}(\Omega), C_0(\Omega))} \\
&= \left(u\mathcal{L}^N, \operatorname{div} \varphi\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} + \sum_{k=1}^{N} (\partial_{x_k} u, \Phi_k)_{(\mathcal{M}(\Omega), C_0(\Omega))} \\
&= \left(u\mathcal{L}^N, \operatorname{div} \varphi\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} - \left(u\mathcal{L}^N, \operatorname{div} \varphi\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} = 0.
\end{aligned} \tag{22}$$

Hence, we obtain that $(T(u), y) = 0$ for all $u \in BV(\Omega)$ and all $y \in Y$. This means that $T(BV(\Omega)) \subset Y^\circ$, the annihilator of Y, which is the set of linear functionals $L \in X'$ such that Y lies in the kernel of L. By using the following result we conclude that $Y^\circ = T(BV(\Omega))$.

Lemma 6.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\mu, \nu_i \in \mathcal{M}(\Omega)$ for $i = 1, \ldots, N$ such that*

$$\int_\Omega \partial_{x_i} \varphi(x)\ d\mu(x) = -\int_\Omega \varphi(x)\ d\nu_i(x) \qquad \forall\ \varphi \in C_c^1(\Omega),\ i = 1, \ldots, N.$$

Then, there exists a unique $u \in BV(\Omega)$ such that $\mu = u\mathcal{L}^N$.

Proof: The proof can be found in Lemma 4.1.1 in Jarde (2018). □

Hence, Theorem III.1.10 from Werner (2011) yields that $Y^\circ \simeq (X/Y)'$ and an isomorphism is given by

$$T_1 : Y^\circ \to (X/Y)', \qquad y \mapsto T_1(y)$$

with

$$T_1(y) : X/Y \to \mathbb{R}, \quad [w] \mapsto \langle T_1(y), [w] \rangle_{((X/Y)', X/Y)} = \langle y, w \rangle_{(X', X)}$$

which is well-defined, due to (22). Hence, $BV(\Omega)$ is isomorphic to $(X/Y)'$ via $T_1 \circ T$ and we can identify the predual $\Gamma(\Omega)$ with X/Y. Now, for some $u \in BV(\Omega)$, we define

$$\langle u, [w] \rangle_{(BV(\Omega), \Gamma(\Omega))} = \left(u\mathcal{L}^N, w_0\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} + \sum_{k=1}^{N} (\partial_{x_k} u, w_k)_{(\mathcal{M}(\Omega), C_0(\Omega))} \tag{23}$$

for all $[w] \in \Gamma(\Omega)$ with $w \in X$ and $w = (w_0, w_1, \ldots, w_N)$. Therefore, we conclude for a sequence $(u_n) \subset BV(\Omega)$ and some $u \in BV(\Omega)$ (we use the notation $\overset{*}{\rightharpoonup}$ for the standard weak–topology in functional analysis and $\overset{**}{\rightharpoonup}$ for the usually used weak–topology in $BV(\Omega)$) that

$$\begin{aligned} u_n \overset{*}{\rightharpoonup} u &\Leftrightarrow \langle u_n - u, [w] \rangle_{(BV(\Omega),\Gamma(\Omega))} && \forall\ [w] \in \Gamma(\Omega) \\ &\Leftrightarrow u_n \mathcal{L}^N \overset{*}{\rightharpoonup} u \mathcal{L}^N && \text{in } \mathcal{M}(\Omega) \text{ and} \\ &\quad\ \partial_{x_i} u_n \overset{*}{\rightharpoonup} \partial_{x_i} u && \text{in } \mathcal{M}(\Omega) \quad \forall\ i \in \{1, \ldots, N\} \\ &\Leftrightarrow u_n \to u && \text{in } L^1(\Omega) \text{ and} \\ &\quad\ \partial_{x_i} u_n \overset{*}{\rightharpoonup} \partial_{x_i} u && \text{in } \mathcal{M}(\Omega) \quad \forall\ i \in \{1, \ldots, N\} \\ &\Leftrightarrow u_n \overset{**}{\rightharpoonup} u. \end{aligned}$$

In the third equivalence relation we used the fact that for domains with compact Lipschitz boundary $BV(\Omega)$ is compactly embedded in $L^1(\Omega)$ (see Proposition 3.21 and Corollary 3.49 in Ambrosio, Fusco and Pallara, 2000). Hence, for Lipschitz regular and bounded domains, these two topologies coincide and in the following we will use the term weak* and the notation $\overset{*}{\rightharpoonup}$ for both topologies.

7. Closedness of bounded sets of time dependent vector fields

In this section, we take a closer look at the norm bounded sets of vector fields. In the main theorem we will prove that sequences $(b_n) \subset \mathrm{V}^q$, which are bounded with respect to some norm, contain subsequences, which are convergent in a weak sense and whose limits are again vector fields with the same temporal and spatial regularities. The statement will play a crucial role in the next section: in the proof of existence of minima, the result of this section will give us a limit, for which it can be shown that it represents a minimum. We start with the definition of K-convergence for vector-valued functions.

Definition 7.1 (Komlós convergence (K-convergence)) *Let X be a separable Banach space. A sequence of functions $f_n : (0,T) \to X'$ is said to be K-convergent to a mapping $f : (0,T) \to X'$ if for every subsequence (n_k) of (n)*

$$\frac{1}{n} \sum_{k=1}^{n} f_{n_k}(t) \overset{*}{\rightharpoonup} f(t)$$

for almost all $t \in (0,T)$.

This type of convergence plays an important role in the proof of the following main result of this section, which is based on results of Cornet and Martins da Rocha (2004).

Theorem 7.1 *Let $q \in (1,\infty)$ and let $(b_n) \subset \mathrm{V}^q$ be a sequence. If (b_n) is bounded, i.e.*

$$\sup_{n\in\mathbb{N}} \|b_n\|_{L^q((0,T),BV(\Omega))^N} \le C < \infty$$

for some $C > 0$, then there exists a subsequence (b_{n_k}) and a function $b \in \mathrm{V}^q$ such that the following properties are satisfied:

(i) $b(t) \in \overline{\operatorname{conv}(\overline{\{b_n(t) | n \in \mathbb{N}\}}^{w^*})}^{w^*}$ *for almost all $t \in (0,T)$,*

(ii) for any measurable set $B \in \mathcal{B}((0,T))$

$$\int_B b_n(t,\cdot)dt \overset{*}{\rightharpoonup} \int_B b(t,\cdot)dt \quad \text{in } BV(\Omega)^N,$$

(iii) for any measurable set $B \in \mathcal{B}((0,T))$ and any monotonically increasing, convex function $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $g(x) \in \mathcal{O}(|x|)$ (for $|x| \to \infty$)

$$\int_B g\left(\|Db(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) dt \leq \liminf_{n\to\infty} \int_B g\left(\|Db_n(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) dt,$$

(iv) $b_n \rightharpoonup b$ *in $L^p((0,T)\times\Omega)^N$ as $n \to \infty$ for any $p \in [1, \min(q, N/(N-1)))$.*

Proof: We first show that for any $[w] \in \Gamma(\Omega)^N$ the set of functions

$$t \mapsto \langle b_n(t,\cdot), [w]\rangle_{(BV(\Omega)^N, \Gamma(\Omega)^N)} \tag{24}$$

is uniformly integrable in $n \in \mathbb{N}$. Then, results from Cornet and Martins da Rocha (2004) will yield most of our statements. Let $[w] \in \Gamma(\Omega)^N$. We take a fixed representative $w \in C_0(\Omega)^{N\times(N+1)}$ and estimate for any measurable set $B \subset (0,T)$

$$\int_B \left|\langle b_n(r,\cdot), [w]\rangle_{(BV(\Omega)^N,\Gamma(\Omega)^N)}\right| \, dr \leq \sum_{i=1}^N \int_B \left|\langle b_{i,n}(r,\cdot)\mathcal{L}^N, w_{i,1}\rangle\right| \, dr \tag{25}$$

$$+ \sum_{i=1}^N \sum_{j=1}^N \int_B \left|\langle \partial_{x_j} b_{i,n}(r,\cdot), w_{i,j+1}\rangle\right| \, dr. \tag{26}$$

Now, we have a closer look at the terms (25) and (26). For term (25) we obtain

$$\sum_{i=1}^N \int_B \left|\langle b_{i,n}(r,\cdot)\mathcal{L}^N, w_{i,1}\rangle\right| \, dr \leq |B|^{1/q'} C_1 \sum_{i=1}^N \|w_{i,1}\|_{C(\Omega)} \tag{27}$$

for some $C_1 > 0$ independent of $n \in \mathbb{N}$. For the second term, (26), we estimate

$$\sum_{i=1}^N \sum_{j=1}^N \int_B \left|\langle \partial_{x_j} b_{i,n}(r,\cdot), w_{i,j+1}\rangle\right| \, dr \leq |B|^{1/q'} C_2 \sum_{i=1}^N \sum_{j=1}^N \|w_{i,j+1}\|_{C(\Omega)} \tag{28}$$

for some $C_2 > 0$ independent of $n \in \mathbb{N}$. The uniform integrability of the functions in (24) follows directly from estimates (25)-(28). Now, Theorem 3.1

(b) in Cornet and Martins da Rocha (2004) yields that there exists a subsequence (labeled by n again) and a Gelfand integrable function $b \in L^1((0,T), BV(\Omega))^N$ such that

$$\left\langle \int_B b(t,\cdot)dt, [w] \right\rangle = \int_B \langle b(t,\cdot), [w]\rangle \, dt \le \liminf_{n\to\infty} \int_B \langle b_n(t,\cdot), [w]\rangle \, dt$$
$$= \liminf_{n\to\infty} \left\langle \int_B b_n(t,\cdot)dt, [w] \right\rangle$$

for any $[w] \in \Gamma(\Omega)^N$ and for any measurable $B \in \mathcal{B}((0,T))$. Since the above inequality is satisfied both for $[w]$ and $-[w]$, we conclude that

$$\int_B b_n(t,\cdot)dt \stackrel{*}{\rightharpoonup} \int_B b(t,\cdot)dt \quad \text{in } BV(\Omega)^N \tag{29}$$

for any $B \in \mathcal{B}((0,T))$. Due to Proposition 3.1 in Cornet and Martins da Rocha (2004) we can choose the subsequence (b_n) such that it is K-convergent to b. Furthermore, part (c) of Theorem 3.1 in Cornet and Martins da Rocha (2004) yields point (i). Since $BV(\Omega)$ is compactly embedded in $L^p(\Omega)$ for any $p < N/(N-1)$, (29) yields that

$$\int_B b_n(t,\cdot)dt \to \int_B b(t,\cdot)dt \quad \text{in } L^p(\Omega)^N$$

for any $B \in \mathcal{B}((0,T))$ and any $p < N/(N-1)$. Now, Theorem 10.4 (i) in Schweizer (2013) yields that for $p \in (1, \min(q, N/(N-1)))$ and for $h \in L^{p'}((0,T)\times\Omega)^N$ with $1/p' + 1/p = 1$, there is a sequence $(h_k) \subset L^{p'}((0,T), L^{p'}((\Omega))^N$ of simple functions such that $h_k \to h$ in $L^{p'}((0,T), L^{p'}(\Omega))^N$. Denote by $A_{k,i} \subset (0,T)$, $i = 1, \ldots, K(k)$ the different measurable subsets where h_k is constant with value $h_{k,i} \in L^{p'}(\Omega)$. Then, we conclude that

$$|\langle h, b_n - b\rangle| \le \sum_{i=1}^{K(k)} \left| \langle h_{k,i}, \int_{A_{k,i}} b_n(t,\cdot) - b(t,\cdot)dt\rangle \right| + C \, \|h_k - h\|_{L^{p'}((0,T),L^{p'}(\Omega))^N}$$

for some $C > 0$, since (b_n) is bounded in $L^p((0,T)\times\Omega)^N$. This yields that $|\langle h, b_n - b\rangle| \to 0$ as $n \to \infty$. Thus, $b_n \rightharpoonup b$ in $L^p((0,T)\times\Omega)^N$ and hence in $L^1((0,T)\times\Omega)^N$. It remains to show that $b \in L^q((0,T), BV(\Omega))^N$ and point (iii) holds. We consider the sequence $(Db_n) \subset L^q((0,T), \mathcal{M}(\Omega)^{N\times N})$. For this sequence we do the same steps as in the proof of Theorem 3.1 (a) in Cornet and Martins da Rocha (2004), but with some differences: due to the boundedness of $\left(\int_0^T \|Db_n(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}} \, dt\right)$ and $g(x) \in \mathcal{O}(|x|)$, we obtain that

$\sup_{n\in\mathbb{N}} \int_0^T g\left(\|Db_n(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) dt < \infty$. Thus,

$$A := \liminf_{n\to\infty} \int_0^T g\left(\|Db_n(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) dt < \infty$$

and we choose a convergent subsequence (labeled by n again), such that

$$A = \lim_{n\to\infty} \int_0^T g\left(\|Db_n(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) dt.$$

Then, as in the above mentioned proof we construct a subsequence (Db_{n_k}), which is K-convergent to some $f \in L^1((0,T), \mathcal{M}(\Omega)^{N\times N})$. On the other hand, we already know that the whole sequence (Db_n) is K-convergent to Db. Thus, we conclude that $Db = f$ and we have, as in Cornet and Martins da Rocha (2004):

$$\begin{aligned}\|Db(t,\cdot)\|_{\mathcal{M}(\Omega)^{N\times N}} &\le \liminf_{n\to\infty} \left\| \frac{1}{n}\sum_{i=1}^n Db_i(t,\cdot)\right\|_{\mathcal{M}(\Omega)^{N\times N}} \\ &\le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \|Db_i(t,\cdot)\|_{\mathcal{M}(\Omega)^{N\times N}}\end{aligned}$$

for almost all $t \in (0,T)$. Since $x \mapsto |x|^q$ is convex and continuous, while g is monotonically increasing and convex, we deduce that

$$g\left(\|Db(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n g\left(\|Db_i(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right)$$

for almost all $t \in (0,T)$. In addition, due to $g(x) \in \mathcal{O}(|x|)$, the above expressions are integrable over measurable sets $B \subset (0,T)$. Fatou's lemma for positive functions then yields

$$\begin{aligned}\int_B g\left(\|Db(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) dt &\le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \int_B g\left(\|Db_i(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) dt \\ &= \liminf_{n\to\infty} \int_B g\left(\|Db_n(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) dt\end{aligned}$$

for any $B \in \mathcal{B}((0,T))$. The boundedness of (b_n) in $L^q((0,T), BV(\Omega))^N$ and the choice of $g(x) = x$ finally yields that $b \in L^q((0,T), BV(\Omega))^N$. □

In addition to this result for Gelfand integrable functions, we need the following result for Bochner integrable functions in the subsequent section.

Lemma 7.1 *Let $l \in \mathbb{N}$, $g : \mathbb{R} \to \mathbb{R}_0^+$ be a monotonically increasing and convex function with $g \in \mathcal{O}(x)$ and let $(f_n) \subset L^2((0,T), L^2(\Omega))^l$ be a bounded sequence. Then, there exists a subsequence (f_{n_k}) and some $f \in L^2((0,T), L^2(\Omega))^l$ such that*

$$\int_0^T g\left(\|f(t,\cdot)\|^2_{L^2(\Omega)^l}\right) dt \le \liminf_{n\to\infty} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right) dt.$$

Proof: Due to the boundedness of (f_n) in $L^2((0,T), L^2(\Omega))^l$, there exists a subsequence (labeled by n again) and some $f \in L^2((0,T), L^2(\Omega))^l$ such that $f_n \rightharpoonup f$ in $L^2((0,T), L^2(\Omega))^l$. Furthermore, due to the properties of g, we have

$$\sup_{n\in\mathbb{N}} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right) dt < \infty$$

and thus, we can choose a subsequence (f_n) (labeled by n again) such that

$$\liminf_{n\to\infty} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right) dt = \lim_{n\to\infty} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right) dt$$

holds. By applying Theorem 2.1 from Diestel, Ruess and Schachermeyer (1993), we then obtain that there is a sequence $(h_n) \subset L^2((0,T), L^2(\Omega))^l$ with $h_n \in \operatorname{conv}(\{f_k |\ k \ge n\})$ for $n \in \mathbb{N}$ such that $(h_n(t,\cdot))$ is convergent to some $h(t,\cdot) \in L^2(\Omega)^l$ for almost all $t \in (0,T)$, i.e.

$$h_n = \sum_{i=n}^{N(n)} \lambda_{n,i} f_i \quad \text{with } 0 \le \lambda_{n,i} \le 1 \quad \text{for } n \le i \le N(n) \in \mathbb{N} \quad \text{and} \quad \sum_{i=n}^{N(n)} \lambda_{n,i} = 1$$

for all $n \in \mathbb{N}$. We assume that $h(t,\cdot) \ne f(t,\cdot)$ for $t \in B \subset (0,T)$ with $\mathcal{L}^1(B) > 0$. Then, we have for $\varphi \in L^2(\Omega)^l$

$$\int_0^T |\langle h_n(t,\cdot), \varphi\rangle|^2\, dt \le \|\varphi\|^2_{L^2(\Omega)^l} \sup_{n\in\mathbb{N}} \int_0^T \|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\, dt < \infty.$$

Due to Theorem 1.35 from Ambrosio, Fusco and Pallara (2000) we obtain that

$$[t \mapsto \langle h_n(t,\cdot), \varphi\rangle] \rightharpoonup [t \mapsto \langle h(t,\cdot), \varphi\rangle] \quad \text{in } L^2((0,T)).$$

Hence, we conclude for $\psi \in L^2(B)$ that

$$\begin{aligned} \int_B \int_\Omega \psi(t)\varphi(x)h(t,x)\ dxdt &\leftarrow \int_B \int_\Omega \psi(t)h_n(t,x)\varphi(x)\ dxdt \\ &\to \int_B \int_\Omega \psi(t)\varphi(x)f(t,x)\ dxdt, \end{aligned}$$

i.e. $\langle h(t,\cdot),\varphi\rangle = \langle f(t,\cdot),\varphi\rangle$ for almost all $t \in B$. Since $\varphi \in L^2(\Omega)^l$ can be arbitrarily chosen, we obtain that $h(t,\cdot) = f(t,\cdot)$ in $L^2(\Omega)^l$ for almost all $t \in B$. But this is a contradiction to our assumption, and thus $h = f$ in $L^2((0,T),L^2(\Omega))^l$. Consequently, we obtain

$$\begin{aligned} g\left(\|f(t,\cdot)\|^2_{L^2(\Omega)^l}\right) &= \lim_{n\to\infty} g\left(\|h_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right) \\ &\le \liminf_{n\to\infty} \sum_{i=n}^{N(n)} \lambda_{n,i} g\left(\|f_i(t,\cdot)\|^2_{L^2(\Omega)^l}\right) \end{aligned}$$

for almost all $t \in (0,T)$. Thus, Fatou's lemma finally yields

$$\begin{aligned} \int_0^T g\left(\|f(t,\cdot)\|^2_{L^2(\Omega)^l}\right) dt &\le \liminf_{n\to\infty} \sum_{i=n}^{N(n)} \lambda_{n,i} \int_0^T g\left(\|f_i(t,\cdot)\|^2_{L^2(\Omega)^l}\right) dt \\ &= \liminf_{n\to\infty} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right) dt. \end{aligned}$$

□

8. Existence of minima of the optimal control problems

In this last section, we apply the results of the previous sections to prove the existence of minimizing points for optimal control problems with the transport equation as a constraint. We start with the optimal control problems and the admissible sets and finish the section with the existence result.

8.1. Optimal control problems

We consider the following type of optimal control problems

$$\begin{aligned} &\min_{u,b} J(u,b) \\ &= \frac{1}{2}\sum_{k=2}^{K} \Upsilon_k\left(\|u(t_k,\cdot)-Y_k\|^2_{L^2(\Omega)}\right) + \frac{\alpha}{2}\int_0^T \Gamma_1\left(\|Db(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\right) dt \quad (30) \\ &+R(b) \quad (31) \end{aligned}$$

with regularization parameter $\alpha > 0$, functions $\Upsilon_k, \Gamma_1 : \mathbb{R} \to \mathbb{R}$, $k = 2,\dots,K$ and constraints

$$\begin{aligned} u_t + \operatorname{div}(bu) - u\operatorname{div}(b) &= 0 \quad \text{in } (0,T]\times\Omega, && (32) \\ u(0,\cdot) &= Y_1 \quad \text{in } \Omega, && (33) \\ b &= 0 \quad \text{on } (0,T)\times\partial\Omega, && (34) \end{aligned}$$

where $Y_k \in L^\infty(\Omega),\ k = 1, \dots, K$ are given. The term R denotes additional regularization terms and we will cover the following ones in our investigations:

(i) $R_1(b) \equiv 0$,

(ii) $R_2(b) = \frac{\beta}{2} \int_0^T \Gamma_2 \left(\|\partial_t b(t,\cdot)\|^2_{L^2(\Omega)^N} \right) dt$,

(iii) $R_3(b) = \frac{\gamma}{2} \int_0^T \Gamma_3 \left(\|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)} \right) dt$,

(iv) $R_4(b) = R_2(b) + R_3(b)$,

where $\beta, \gamma > 0$ are regularization parameters and $\Gamma_2, \Gamma_3 : \mathbb{R} \to \mathbb{R}$ are given functions. In the first two cases, we will additionally distinguish between two further subcases: the set of constraints given by (32)-(34) and the same set plus the additional constraint

$$\operatorname{div} b = 0 \qquad \text{in } (0,T) \times \Omega. \tag{35}$$

For the functions Υ_k, $k = 2, \dots, K$ and Γ_i, $i = 1, 2, 3$ we assume the following:

(a) the functions $\Upsilon_k : \mathbb{R} \to \mathbb{R}_0^+$ are lower semi-continuous,

(b) the functions $\Gamma_i : \mathbb{R} \to \mathbb{R}_0^+$ are convex, monotonically increasing, in $\mathcal{O}(x)$ and $\lim_{x\to\infty} \Gamma_i(x) = \infty$.

In this case, the regularization terms in (8.1) and in (ii)-(iv) are well-defined.

8.2. Admissible sets

Before we can introduce a setting for an admissible set, we have a closer look at the BV-regularity for our considered vector fields. So far, we have the obvious setting

$$\begin{aligned} & b \in \mathrm{V}^2 \\ & = \left\{ b \in L^\infty((0,T)\times\Omega)^N \cap L^2((0,T), BV(\Omega))^N \;\middle|\; \operatorname{div} b \in L^2((0,T), L^\infty(\Omega)) \right\}. \end{aligned}$$

For the existence and uniqueness of solutions we need vector fields b, which have zero trace at the boundary of the spatial domain. The demand of $b \in L^2((0,T), BV_0(\Omega))$ would not be enough, since the trace operator is not continuous with respect to the weak*-convergence, but with respect to the strict convergence in $BV(\Omega)$. As we will get at best weak*-convergence for a subsequence of a minimizing sequence, the weak*-limit would not need to have zero trace at $\partial\Omega$ for almost all $t \in (0,T)$. This means that we need some control of behavior of our BV-functions close to the boundary in order to ensure that limits of weakly*-convergent sequences of BV-functions with zero boundary trace do have zero boundary trace. Therefore, we introduce the following setting. Given some $\varepsilon > 0$, we define for an open bounded set $\mathcal{O} \subset \mathbb{R}^N$ with Lipschitz boundary

$$\mathcal{O}_\varepsilon = \{ x \in \mathcal{O} |\, \operatorname{dist}(x, \partial\mathcal{O}) \le \varepsilon \}.$$

Then, we set for $\delta \geq 0$ and $\varepsilon > 0$

$$\mathrm{W}_{\varepsilon,\delta}(\mathcal{O}) := \left\{ w \in L^1(\mathcal{O}) |\ |w(x)| \leq \delta \operatorname{dist}(x, \partial\mathcal{O}) \text{ for almost all } x \in \mathcal{O}_\varepsilon \right\}, \tag{36}$$

and obtain the following result:

Lemma 8.1 *Let $\mathcal{O} \subset \mathbb{R}^N$ be open and bounded with Lipschitz boundary $\partial\mathcal{O}$ and let $\varepsilon > 0$ and $\delta \geq 0$. Then, any $f \in BV(\mathcal{O})$, satisfying $f \in \mathrm{W}_{\varepsilon,\delta}(\mathcal{O})$, lies in $BV_0(\mathcal{O})$.*

Proof: The proof can be easily deduced by using properties of BV-functions and is presented in Lemma 4.2.1 in Jarde (2018). □

Lemma 8.2 *Let $\mathcal{O} \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary $\partial\mathcal{O}$ and let $\varepsilon > 0$ and $\delta \geq 0$. Furthermore, let $(f_n) \subset L^1(\mathcal{O})$ be convergent to $f \in L^1(\mathcal{O})$ with $f_n \in \mathrm{W}_{\varepsilon,\delta}(\mathcal{O})$ for all $n \in \mathbb{N}$. Then $f \in \mathrm{W}_{\varepsilon,\delta}(\mathcal{O})$.*

Proof: The proof can be found in Lemma 4.2.2 in Jarde (2018). □

With this technical assumption we define the set of admissible vector fields S_{ad} for the various optimal control problems. We take fixed $M > 0$, $\delta \geq 0$ and $\varepsilon > 0$ and we consider vector fields $b : (0,T) \times \Omega \to \mathbb{R}^N$ with

$$b \in \mathrm{S}_{ad}^{\varepsilon,\delta} := \left\{ b \in \mathrm{V}^2 |\ b(t,\cdot) \in \mathrm{W}_{\varepsilon,\delta}(\Omega) \text{ for almost all } t \in (0,T) \right\}$$

and define the admissible set for M, ε and δ

$$\mathrm{S}_{ad}^{M,\varepsilon,\delta} := \left\{ b \in \mathrm{S}_{ad}^{\varepsilon,\delta} \ \middle|\ \|b\|_{L^\infty((0,T)\times\Omega))^N} + \|\operatorname{div} b\|_{L^2((0,T),L^\infty(\Omega))} \leq M \right\}. \tag{37}$$

Obviously, we have that $\mathrm{S}_{ad}^{\varepsilon,\delta} \subset \mathrm{V}_0^2$. Furthermore, for the case of the additional constraint $\operatorname{div} b \equiv 0$, we define the set

$$\mathrm{S}_{ad,0}^{M,\varepsilon,\delta} := \left\{ b \in \mathrm{S}_{ad}^{M,\varepsilon,\delta} |\ \operatorname{div} b \equiv 0 \right\} \tag{38}$$

and in the case of time regularization

$$\mathrm{S}_{ad,\partial_t}^{M,\varepsilon,\delta} := \left\{ b \in \mathrm{S}_{ad}^{M,\varepsilon,\delta} |\ \partial_t b \in L^2((0,T) \times \Omega)^N \right\}. \tag{39}$$

The previous sections yield that there is a well-defined solution operator

$$S : L^\infty(\Omega) \times \mathrm{V}_0^1 \to C([0,T], L^\infty(\Omega) - w^*), \quad (u_0, b) \mapsto S(u_0, b).$$

Based on this solution operator we define the control-to-state operator L_{Y_1} as

$$L_{Y_1} : \mathrm{V}_0^1 \to C([0,T], L^\infty(\Omega) - w^*), \quad b \mapsto L_{Y_1}(b) = S(Y_1, b) \tag{40}$$

and its restriction to $\mathrm{S}_{ad}^{M,\varepsilon,\delta}$ as $L_{Y_1,ad}$. We abbreviate the terms $\mathrm{S}_{ad}^{M,\varepsilon,\delta}$, $\mathrm{S}_{ad,0}^{M,\varepsilon,\delta}$ and $\mathrm{S}_{ad,\partial_t}^{M,\varepsilon,\delta}$ to S_{ad}, $\mathrm{S}_{ad,0}$ and $\mathrm{S}_{ad,\partial_t}$, respectively, if it is clear which constants M, ε and δ are used in the current setting. Incorporating these control-to-state mappings into the objective function J leads to various reduced objective functions F_i for our considered cases: we define

in the case	the reduced objective function $J(L_{Y_1,ad}(\cdot),\cdot)$ as	with admissible set
$R = R_1$	F_1	S_{ad}
$R = R_1$	$F_{1,0}$	$\mathrm{S}_{ad,0}$
$R = R_2$	F_2	$\mathrm{S}_{ad,\partial_t}$
$R = R_2$	$F_{2,0}$	$\mathrm{S}_{ad,0} \cap \mathrm{S}_{ad,\partial_t}$
$R = R_3$	F_3	S_{ad}
$R = R_4$	F_4	$\mathrm{S}_{ad,\partial_t}$

For these reduced objective functions we show in the subsequent theorem that they attain their infima on their admissible sets, i.e. there are minima within the admissible sets for each optimal control problem.

8.3. Existence of minima

THEOREM 8.1 (EXISTENCE OF MINIMA OF THE OPTIMAL CONTROL PROBLEMS) *Let $M > 0$, $\varepsilon > 0$ and $\delta \geq 0$ be fixed chosen. Then, the reduced objective functions F_i, $i \in \{1, \ldots, 4\}$ and $F_{j,0}$, $j = 1, 2$ attain their minima on their admissible sets.*

PROOF: We just show the statement for the objective function F_4, since the proof works in the same way for the other problems.

The objective function F_4 has a finite infimum in $\mathrm{S}_{ad,\partial_t}$ since $F_4(b) \geq 0$ for all $b \in \mathrm{S}_{ad,\partial_t}$. Now, let $(b_n) \subset \mathrm{S}_{ad,\partial_t}$ be a minimizing sequence, i.e.

$$F_4(b_n) \geq F_4(b_{n+1}) \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} F_4(b_n) = \inf_{\tilde{b}\in \mathrm{S}_{ad,\partial_t}} F_4(\tilde{b}).$$

The sequence (b_n) is bounded in $L^2((0,T), BV(\Omega))^N$:

$$F_4(b_1) \geq F_4(b_n) \geq \frac{T\alpha}{2}\Gamma_1\left(\frac{1}{T}\int_0^T \|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\, dt\right) \quad \forall\, n \in \mathbb{N}$$

and thus,

$$\sup_{n\in\mathbb{N}} \int_0^T \|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\, dt < \infty.$$

In addition, $\|b_n\|_{L^\infty((0,T)\times\Omega)^N} \leq M$ for all $n \in \mathbb{N}$, and hence (b_n) is also bounded in $L^2((0,T), L^1(\Omega))^N$. Using Theorem 7.1, we obtain that there exists a subsequence (b_n) (which is labeled by n again) and some $b \in L^2((0,T), BV(\Omega))^N$ such that

$$\int_0^T \Gamma_1\left(\|Db(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\right) dt \leq \liminf_{n\to\infty} \int_0^T \Gamma_1\left(\|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\right) dt \quad (41)$$

and $b_n \rightharpoonup b$ in $L^1((0,T)\times\Omega)^N$. For the limit b we have that $b(t,\cdot) \in \mathrm{W}_{\varepsilon,\gamma}(\Omega)$ for almost all $t \in (0,T)$: denote

$$\mathcal{N}_n := \{t \in (0,T), b_n(t,\cdot) \notin BV(\Omega)^N\} \cup \{t \in (0,T), b_n(t,\cdot) \notin \mathrm{W}_{\varepsilon,\delta}(\Omega)^N\}$$

and

$$\mathcal{N} := \{t \in (0,T), b(t,\cdot) \notin BV(\Omega)^N\}.$$

Then, $\mathcal{N}_n$ and $\mathcal{N}$ are null sets and

$$\mathcal{W} = \mathcal{N} \cup \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$$

is also a null set as a countable union of null sets. Furthermore, due to Lemma 8.2 we conclude that for any $t \in (0,T)\backslash\mathcal{W}$

$$g \in \overline{\{b_n(t,\cdot)|\ n \in \mathbb{N}\}}^{w^*} \Rightarrow g \in \mathrm{W}_{\varepsilon,\delta}(\Omega)^N$$

is satisfied. Consequently, in the same way we conclude that for any $t \in (0,T)\backslash\mathcal{W}$

$$g \in \overline{\mathrm{conv}\left(\overline{\{b_n(t,\cdot)|\ n \in \mathbb{N}\}}^{w^*}\right)}^{w^*} \Rightarrow g \in \mathrm{W}_{\varepsilon,\delta}(\Omega)^N$$

is also satisfied. Thus, $b(t,\cdot) \in \mathrm{W}_{\varepsilon,\delta}(\Omega)^N$ for almost all $t \in (0,T)$. In addition, since (b_n), $(\partial_t b_n)$ and $(\operatorname{div} b_n)$ are bounded sequences in $L^\infty((0,T)\times\Omega)^N$, in $L^2((0,T)\times\Omega)^N$ and in $L^2((0,T), L^\infty(\Omega))$, respectively, we conclude, using standard arguments, that $b_n \overset{*}{\rightharpoonup} b$ in $L^\infty((0,T)\times\Omega)^N$, $\partial_t b_n \rightharpoonup \partial_t b$ in $L^2((0,T)\times\Omega)^N$ and $\operatorname{div} b_n \rightharpoonup \operatorname{div} b$ in $L^2((0,T)\times\Omega)$ with $\operatorname{div} b \in L^2((0,T), L^\infty(\Omega))$ for some subsequences. Due to Lemma 7.1, we know that each of these subsequences contains a subsequence (labeled by n again) such that

$$\int_0^T \Gamma_2\left(\|\partial_t b(t,\cdot)\|^2_{L^2(\Omega)^N}\right) dt \leq \liminf_{n\to\infty} \int_0^T \Gamma_2\left(\|\partial_t b_n(t,\cdot)\|^2_{L^2(\Omega)^N}\right) dt$$

and

$$\int_0^T \Gamma_3\left(\|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)}\right) dt \leq \liminf_{n\to\infty} \int_0^T \Gamma_3\left(\|\operatorname{div} b_n(t,\cdot)\|^2_{L^2(\Omega)}\right) dt$$

hold. We restrict ourselves to those subsequences. Summing up, we have shown that $b \in \mathrm{S}_{ad,\partial_t}$. Finally, using Theorem 5.2, we obtain that

$$L_{Y_1,ad}(b_n) \to L_{Y_1,ad}(b) \quad \text{in } C([0,T], L^r(\Omega)) \quad \text{for } 1 \le r < \infty$$

and thus we get for all $2 \le k \le K$

$$L_{Y_1,ad}(b_n)(t_k,\cdot) - Y_k \to L_{Y_1,ad}(b)(t_k,\cdot) - Y_k \qquad \text{in } L^2(\Omega) \quad \text{as } n \to \infty.$$

In total, we obtain with estimate (41):

$$\begin{aligned}
&F_4(b)\\
&= \frac{1}{2}\sum_{k=2}^{K} \Upsilon_k \left(\|L_{Y_1,ad}(b)(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)} \right) + \frac{\alpha}{2}\int_0^T \Gamma_1 \left(\|Db(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}} \right) dt\\
&+ \frac{\beta}{2}\int_0^T \Gamma_2 \left(\|\partial_t b(t,\cdot)\|^2_{L^2(\Omega)} \right) dt + \frac{\gamma}{2}\int_0^T \Gamma_3 \left(\|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)} \right) dt\\
&\le \liminf_{n\to\infty} \Bigg[\frac{1}{2}\sum_{k=2}^{K} \Upsilon_k \left(\|L_{Y_1,ad}(b_n)(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)} \right)\\
&+ \frac{\alpha}{2}\int_0^T \Gamma_1 \left(\|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}} \right) dt\\
&+ \frac{\beta}{2}\int_0^T \Gamma_2 \left(\|\partial_t b_n(t,\cdot)\|^2_{L^2(\Omega)} \right) dt + \frac{\gamma}{2}\int_0^T \Gamma_3 \left(\|\operatorname{div} b_n(t,\cdot)\|^2_{L^2(\Omega)} \right) dt \Bigg]\\
&= \liminf_{n\to\infty} F_4(b_n) = \inf_{\tilde{b}\in \mathrm{S}_{ad,\partial_t}} F_4(\tilde{b}).
\end{aligned}$$

Thus, the infimum is attained and F_4 has a minimum in $\mathrm{S}_{ad,\partial_t}$. □

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