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Growth control of cracks under contact conditions based on the topological derivative of the Rice's integral*

by

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Dedicated to Günter Leugering on the occasion of his 65th birthday

Abstract: In the present paper we propose a simple method for dealing with growth control of cracks under contact type boundary conditions on their lips. The aim is to find a mechanism for decreasing the energy release rate of cracked components, which means increasing their fracture toughness. The method consists in minimizing a shape functional defined in terms of the Rice's integral, with respect to the nucleation of hard and/or soft inclusions, according to the information provided by the associated topological derivative. Based on Griffith's energy criterion, this simple strategy allows for an increase in fracture toughness of the cracked component. Since the problem is non-linear, the domain decomposition technique, combined with the Steklov-Poincaré pseudo-differential boundary operator, is used to obtain the sensitivity of the associated

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shape functional with respect to the nucleation of a small circular inclusion with different material property from the background. Then, the obtained topological derivatives are used to indicate the regions, where the controls should be positioned in order to solve the minimization problem we are dealing with. Finally, a numerical example is presented showing the applicability of the proposed methodology.

Keywords: Rice's integral, Griffith's criterion, Eshelby's tensor, topological derivative

1. Introduction

In materials science, toughness is an intrinsic property of components, which is used to describe their capability to resist fracture. In particular, when the original component is already partially cracked, this property is called fracture toughness and represents the ability of materials in resisting the activation of the crack propagation mechanism. The fracture toughness of a component is related to its energy release rate, which is defined as the variation of the strain energy stored in the body with respect to the crack growth. More specifically, based on Griffith's energy criterion (Griffith, 1921), the lower is the energy release rate of the cracked component, the higher is its fracture toughness. Following this idea, different strategies, meant to reduce the energy release rate of the components, have been proposed in the literature. See, for instance, Destuynder (1989), Hild, Münch and Ousset (2008), Khludnev, Leugering and Specovious-Neugebauer (2012), Münch and Pedregal (2010) and related works, Homberg and Khludniev (2002), Saliba et al. (2005), Saurin (2000).

This paper deals with crack growth control problems by using the concept of topological derivative, see Khludnev, Leugering and Specovious-Neugebauer (2012), Kovtunenko and Leugering (2016), Leugering, Sokołowski and Żochowski (2015), Sokołowski, Leugering and Żochowski (2014, 2016). Following the original ideas, presented in Xavier, Novotny and Sokołowski (2018), a shape functional defined in terms of the Rice's integral (Rice, 1968) is minimized with respect to the nucleation of hard and/or soft inclusions far from the crack tip. Since the Rice's integral is defined in terms of energy release, based on Griffith's energy criterion, this simple strategy allows for an increase of fracture toughness of the cracked body. However, the methodology referred to was developed over a linear elastic model. One well-known limitation of this class of models is that they are not able to distinguish between traction and compression stress states, so that crack closure phenomenon cannot be captured, for example. Therefore, in this work, an extension of the method, presented in Xavier, Novotny and Sokołowski (2018) to the non-linear case, associated with contact type boundary conditions on the crack lips, is proposed. In particular, the sensitivity of the Rice's integral, with respect to the nucleation of a small circular inclusion, is obtained by using the Domain Decomposition Technique combined with the Steklov-Poincaré pseudo-differential boundary operator (Sokołowski and Żochowski, 2005). As proposed in Xavier, Novotny and Sokołowski (2018), the resulting expression is used to indicate the regions, where the controls (inclusions) should be positioned (nucleated) in order to solve the minimization problem. A numerical example, based on the famous Bittencourt's experiment is presented, showing the effectiveness of the proposed methodology. In fact, a gain of 13% in the fracture toughness of the mechanical component is observed.

The work is organized as follows. The statement of the problem is presented in Section 2. In Section 3, the closed formula of the associated topological derivative is obtained. The numerical experiment is described in Section 4. Finally, some concluding remarks are presented in Section 5.

2. Statement of the problem

2.1. The preliminaries

Let us consider an elastic cracked body, represented by an open and bounded domain $\mathcal{D} \subset \mathbb{R}^2$, with boundary $\partial\mathcal{D} = \Gamma_N \cup \Gamma_D \cup \Gamma_c$, submitted to surface loads on Γ_N, prescribed displacements on Γ_D and a possible contact condition on Γ_c. The contour Γ_c is used to represent the crack inside the body. We assume that the normal vectors on both sides of Γ_c are collinear, allowing us to set just one normal vector field n on the potential contact region. The existing cracks are assumed to be straight lines with length h and direction e, where e is a unit vector aligned with the crack. The notation x^* is used to denote the crack tips. Finally, the cracks Γ_c are free of traction and a control region $\omega^* \subset \mathcal{D}$, containing the crack tip, is considered, see the sketch in Fig. 1. Then, the

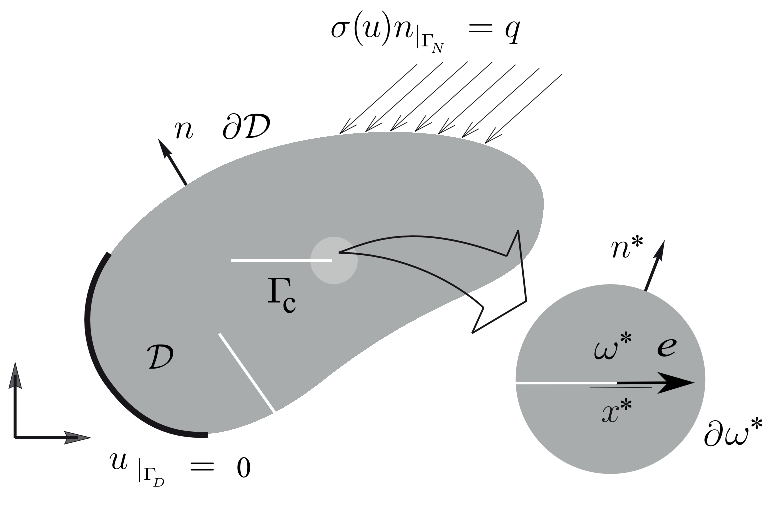


Figure 1. Cracked elastic body

mechanical problem is defined as: Find u, such that

$$\left\{\begin{array}{rcll} \operatorname{div}(\sigma(u)) & = & 0 & \text{in} \quad \mathcal{D}\,, \\ \sigma(u) & = & \mathbb{C}\nabla u^s & \\ u & = & 0 & \text{on} \quad \Gamma_D\,, \\ \sigma(u)n & = & q & \text{on} \quad \Gamma_N\,, \\ \left.\begin{array}{rcl} [\![u]\!]\cdot n & \geq & 0 \\ \sigma^{nn}(u) & \leq & 0 \\ \sigma^{nn}(u)([\![u]\!]\cdot n) & = & 0 \\ \sigma^{n\tau}(u)(u\cdot\tau)+\mu_a|u\cdot\tau| & = & 0 \\ -\mu_a \;\leq\; \sigma^{n\tau}(u) & \leq & \mu_a \end{array}\right\} & & & \text{on} \quad \Gamma_c\,. \end{array}\right. \tag{1}$$

For the purposes of this work, it is necessary to introduce the regularized version of the problem (1). In this case, the total potential energy of the system is given by

$$\mathcal{F}(u) = \frac{1}{2}\int_{\mathcal{D}} \sigma(u)\cdot\nabla u^s - \int_{\Gamma_N} q\cdot u + \mu_a\int_{\Gamma_c}\sqrt{(u\cdot\tau)^2+a} + \mu_c\int_{\Gamma_c} |\,[\![u]\!]\cdot n\,|_+^2\,, \tag{2}$$

where the displacement field u is a solution to the following variational problem: Find $u \in \mathcal{U}$, such that

$$\int_{\mathcal{D}} \sigma(u)\cdot\nabla\eta^s = \\ = \int_{\Gamma_N} q\cdot\eta - \mu_a\int_{\Gamma_c}\frac{(\tau\otimes\tau)u\cdot\eta}{\sqrt{(u\cdot\tau)^2+a}} - 2\mu_c\int_{\Gamma_c}(|\,[\![u]\!]\cdot n\,|_+)n\cdot\eta\,, \forall\eta\in\mathcal{V}. \tag{3}$$

The term $\sigma(u) = \mathbb{C}\nabla u^s$ is the Cauchy stress tensor. We consider isotropic material, so that the elasticity tensor $\mathbb{C}$ can be written as

$$\mathbb{C} = 2\mu\mathbb{I} + \lambda(\mathrm{I}\otimes\mathrm{I})\,, \tag{4}$$

where I and $\mathbb{I}$ are the second and fourth order identity tensors, respectively, and μ and λ are the Lamé's coefficients. In particular, we have

$$\mu = \frac{E}{2(1+\nu)}\,, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \lambda^* = \frac{\nu E}{1-\nu^2}\,, \tag{5}$$

where λ and λ^* are associated with plane strain and plane stress assumptions, respectively. In addition, E is the Young's modulus and ν the Poisson's ratio. The strain tensor is defined as

$$\nabla\varphi^s := (\nabla\varphi)^s = \frac{1}{2}(\nabla\varphi + (\nabla\varphi)^\top)\,. \tag{6}$$

In what follows, the term $q \in H^{\frac{1}{2}}(\Gamma_N;\mathbb{R}^2)$ is a given boundary traction, μ_a is a known friction coefficient, τ denotes the tangential vector field on Γ_c and $a \in \mathbb{R}^+$ is a regularization parameter. The operator $|\,[\![\varphi]\!]\cdot n\,|_+^2$, defined as

$$|\,[\![\varphi]\!]\cdot n\,|_+^2 := \left\{\begin{array}{cl} 0 & \text{if} \quad [\![\varphi]\!]\cdot n > 0\,, \\ ([\![\varphi]\!]\cdot n)^2 & \text{if} \quad [\![\varphi]\!]\cdot n \leq 0\,, \end{array}\right. \tag{7}$$

is introduced to impose the non-interpenetration condition through the penalty parameter μ_c. Finally, the set $\mathcal{U}$ and the space $\mathcal{V}$ are defined as

$$\mathcal{V} := \mathcal{U} := \left\{ \varphi \in H^1(\mathcal{D}) : \varphi_{|\Gamma_D} = 0 \right\} . \tag{8}$$

Since we are considering a cracked domain, the propagation mechanism may be activated according to some dissipation criterion (Griffith, 1921). As mentioned before, the aim is to find a way to retard or even avoid the triggering of such mechanism by minimizing a shape functional, written down in terms of the Rice's integral with respect to the nucleation of circular inclusions far from the crack tip.

2.2. Rice's integral

The Rice's integral, denoted by $\mathcal{J}(u)$, is defined as

$$\mathcal{J}(u) := -\frac{d}{dh}\mathcal{W}(u) , \tag{9}$$

where $\mathcal{W}(u)$ is the energy released (Rice, 1968). By taking the strain energy to compute the energy release rate, i.e., taking $\mathcal{W}(u) = -\mathcal{F}(u)$, we obtain

$$\mathcal{J}(u) = e\cdot \int_{\partial\omega^*} \Sigma(u) n^* + \mu_a \int_{\Gamma_c} \partial_\tau V^\tau \sqrt{(u\cdot\tau)^2 + a} + \mu_c \int_{\Gamma_c} \partial_\tau V^\tau |\, [\![u]\!] \cdot n\, |_+^2, \tag{10}$$

where e is the direction of the crack growth, n^* is the outward unit vector normal to $\partial\omega^*$, and $\Sigma(u)$, defined as

$$\Sigma(u) = \frac{1}{2}(\sigma(u)\cdot\nabla u^s)\mathrm{I} - \nabla u^\top \sigma(u) , \tag{11}$$

is the *Eshelby energy-momentum tensor* introduced in Eshelby (1975). In the sequence, V^τ is the tangential component of the shape change velocity field V, which, in the present case, is defined as

$$V \in C^\infty(\mathcal{D}) : V = e \ \text{ in } \ \omega^*, \tag{12}$$

with compact support in ω^*.

For the purposes of this work, it is necessary to introduce a representation of $\mathcal{J}(u)$ as an integral over the cracked domain. Alternative representations of $\mathcal{J}(u)$ can be found in Fancello, Taroco and Feijóo (1993), Saurin (2000), or Van Goethem and Novotny (2010), for instance. For a more general expression of $\mathcal{J}(u)$ in three spatial dimensions see Feijóo and al. (2000). According to Van Goethem and Novotny (2010), the derivative of $\mathcal{F}(u)$, with respect to the crack length h, can also be written down as

$$\begin{aligned}\frac{d}{dh}\mathcal{F}(u) &= \\ &= \int_{\mathcal{D}} \Sigma(u)\cdot\nabla V + \mu_a \int_{\Gamma_c} \partial_\tau V^\tau \sqrt{(u\cdot\tau)^2 + a} + \mu_c \int_{\Gamma_c} \partial_\tau V^\tau |\, [\![u]\!] \cdot n\, |_+^2 .\end{aligned} \tag{13}$$

Therefore, the following equivalent form for the Rice's integral $\mathcal{J}(u)$ holds true

$$\begin{aligned}&\mathcal{J}(u) =\\ &= \int_{\mathcal{D}} \Sigma(u) \cdot \nabla V + \mu_a \int_{\Gamma_c} \partial_\tau V^\tau \sqrt{(u \cdot \tau)^2 + a} + \mu_c \int_{\Gamma_c} \partial_\tau V^\tau | [\![u]\!] \cdot n|_+^2,\end{aligned} \tag{14}$$

where $\Sigma(u)$ is the Eshelby tensor, defined by (11). The proof of the equivalence between the different representations of the Rice's integral, given by (10) and (14), can be found in details in Van Goethem and Novotny (2010), for instance.

2.3. Topology optimization problem

The topology optimization problem is based on Griffith's energy criterion for crack propagation (Griffith, 1921). This criterion can be written down in terms of the Rice's integral in the following way:

$$\mathcal{J}(u) + G_s \begin{cases} < & 0 \quad \text{the crack is unstable;} \\ = & 0 \quad \text{the crack is in equilibrium;} \\ > & 0 \quad \text{the crack is stable,} \end{cases} \tag{15}$$

where $G_s > 0$ is used to denote the Griffith's surface energy.

Since G_s is a positive number and taking into account that $\mathcal{J}(u)$ is a negative quantity, the less negative is $\mathcal{J}(u)$ the higher is the fracture toughness of the mechanical component. Therefore, by avoiding trivial solution, which consists in rounding the crack tip, the idea is to maximize $\mathcal{J}(u)$ with respect to the nucleation of hard and/or soft inclusions far from the crack tip. Thus, the optimization problem we are dealing with can be formulated as follows:

$$\underset{\Omega \subset \mathcal{D}}{\text{Minimize}} \; \{-\mathcal{J}(u)\}, \text{ subject to (3)} , \tag{16}$$

where $\Omega := \mathcal{D} \setminus \omega^*$ and $\mathcal{J}(u)$ is the Rice's integral, defined through (14). Here, the domain Ω, which is free of geometrical singularities, produced by the crack tip, is assumed to be smooth, with Lipschitz boundary $\partial\Omega$.

A natural approach to deal with such a minimization problem consists in applying the concept of topological derivative (see Novotny and Sokołowski, 2013; Sokolowski and Żochowski, 1999). Therefore, in order to simplify further the analysis, we introduce the following adjoint state: Find $v \in \mathcal{V}$, such that

$$\begin{aligned}&\int_{\mathcal{D}} \sigma(v) \cdot \nabla \eta^s = \langle D_u \mathcal{J}(u), \eta \rangle =\\ &= \int_{\mathcal{D}} \mathrm{tr}(\nabla V) \sigma(u) \cdot \nabla \eta^s - \int_{\mathcal{D}} \sigma(\eta) \cdot (\nabla u \nabla V) - \int_{\mathcal{D}} \sigma(u) \cdot (\nabla \eta \nabla V)\\ &+ \mu_a \int_{\Gamma_c} \partial_\tau V^\tau \frac{(\tau \otimes \tau) u \cdot \eta}{\sqrt{(u \cdot \tau)^2 + a}} + 2\mu_c \int_{\Gamma_c} \partial_\tau V^\tau (| [\![u]\!] \cdot n |_+) n \cdot \eta \; , \; \forall \, \eta \in \mathcal{V} \; ,\end{aligned} \tag{17}$$

where V is the shape change velocity field defined in (12).

3. Topology optimization method

The methodology, proposed in Xavier, Novotny and Sokołowski (2018), is based on the fact that the introduction of an inclusion at the region, where the topological derivative is negative, allows for decreasing the values of the associated shape functional. Therefore, the topological derivative of the shape functional, defined by (14), with respect to the nucleation of a small circular inclusion, is obtained. Then, the resulting expression will be used to indicate the regions, where the inclusions should be nucleated in order to solve the minimization problem (16). Since the domain of analysis contains a singularity, it is necessary first to apply the Domain Decomposition Technique, combined with the Steklov-Poincaré pseudo-differential boundary operator, in order to evaluate the associated topological derivative.

3.1. Domain decomposition method

Let us decompose $\mathcal{D}$ into two subdomains, namely, $\omega^* \subset \mathcal{D}$ and $\Omega := \mathcal{D} \setminus \omega^*$, such that ω^* is the region, which contains the singularity produced by the crack tip. In addition, we consider an intact domain ω of the form $\omega := \omega^* \cup \Gamma_c$ as sketched in Fig. 2. Then, the following boundary value problem is considered: Find w, such that

$$\left\{ \begin{array}{rcll} \operatorname{div}\sigma(w) & = & 0 & \text{in} \quad \omega^* \, , \\ \sigma(w) & = & \mathbb{C}\nabla w^s \, , & \\ \sigma(w)n & = & g(w) & \text{on} \quad \Gamma_c \, , \\ w & = & \varphi & \text{on} \quad \partial\omega \, . \end{array} \right. \tag{18}$$

where the vector function g is given by

$$g(w) = -\mu_a \frac{(\tau \otimes \tau)w}{\sqrt{(w \cdot \tau)^2 + a}} - 2\mu_c(| \, [\![w]\!] \cdot n \, |_+)n \, . \tag{19}$$

Therefore, by using (18), we can define the Steklov-Poincaré pseudo-differential boundary operator $\mathcal{S}$ as follows

$$\begin{array}{rcl} \mathcal{S} : H^{\frac{1}{2}}(\partial\omega) & \to & H^{-\frac{1}{2}}(\partial\omega) \\ \varphi & \mapsto & \sigma(w)n^* \, , \end{array} \tag{20}$$

where n^* is the outward normal vector to the boundary $\partial\omega$. Therefore, the following variational problem is considered over the uncracked domain Ω: Find $u \in \mathcal{U}$, such that

$$\int_\Omega \sigma(u) \cdot \nabla \eta^s + \int_{\partial\omega} \mathcal{S}(u) \cdot \eta = \int_{\Gamma_N} q \cdot \eta \, , \; \forall \eta \in \mathcal{V} \, . \tag{21}$$

Note that, by setting $\varphi = (u)_{|\partial\omega}$, we have $w = (u)_{|\omega^*}$.

By using the Domain Decomposition Technique, the cracked domain $\mathcal{D}$ is decomposed, so that the singularity, produced by the crack tip, is absorbed by the

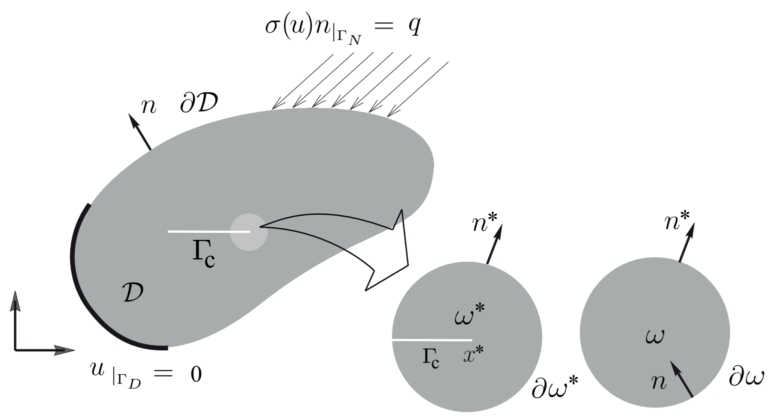


Figure 2. Truncated domain

auxiliary problem (18), defined over the cracked subdomain ω^*. Consequently, the remaining subdomain Ω becomes smooth, which allows us to evaluate the associated topological derivative by using results known from the literature. For more details on the domain decomposition method, see Amigo et al. (2016), Lopes et al. (2017), or Sokołowski and Żochowski (2005), for instance.

Now, in order to apply the concept of topological derivative (Novotny and Sokołowski, 2013), let us introduce the topologically perturbed counterpart of the problem (21). The idea consists in nucleating a circular inclusion, denoted by $B_\varepsilon(\hat{x})$, of radius ε and center at the arbitrary point $\hat{x} \in \Omega$, such that $\overline{B_\varepsilon(\hat{x})} \subset \Omega$, see the sketch in Fig. 3. More precisely, we define a piecewise constant function of the form

$$\gamma_\varepsilon = \gamma_\varepsilon(x) := \begin{cases} 1 & \text{if } x \in \Omega \setminus \overline{B_\varepsilon(\hat{x})}\ ; \\ \gamma & \text{if } x \in B_\varepsilon(\hat{x})\ , \end{cases} \tag{22}$$

where $\gamma = \gamma(x)$ is the contrast in the material properties. The variational formulation, associated with the topologically perturbed problem, is stated as: Find $u_\varepsilon \in \mathcal{U}$, such that

$$\int_\Omega \sigma_\varepsilon(u_\varepsilon) \cdot \nabla\eta^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon) \cdot \eta = \int_{\Gamma_N} q \cdot \eta \quad \forall \eta \in \mathcal{V}\ , \tag{23}$$

where $\sigma_\varepsilon(u_\varepsilon) = \gamma_\varepsilon \sigma(u_\varepsilon)$. Note that, by setting $\varphi = (u_\varepsilon)_{|\partial\omega}$, we have $w = (u_\varepsilon)_{|\omega^*}$.

3.2. Existence of the topological derivative

The existence of the associated topological derivative is ensured by the following result:

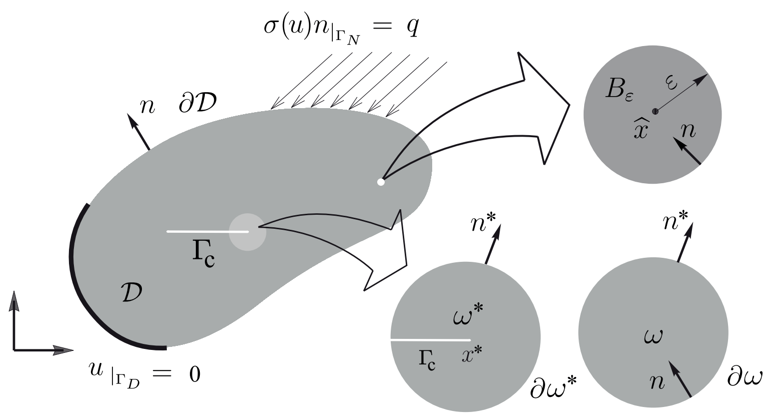


Figure 3. The perturbed problem

Lemma 1 *Let* u_ε *and* u *be solutions of problems* (23) *and* (21)*, respectively. Then, the following estimate holds true:*

$$\|u_\varepsilon - u\|_{H^1(\Omega)} \leq C\varepsilon \,, \tag{24}$$

where C *is a constant independent of the small parameter* ε.

Proof Let us subtract (21) from (23). Then, from the definition for the contrast (22), we obtain

$$\begin{aligned} 0 &= \int_\Omega (\sigma_\varepsilon(u_\varepsilon) - \sigma(u)) \cdot \nabla\eta^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u) \cdot \eta \\ &= \int_{\Omega\setminus B_\varepsilon} (\sigma(u_\varepsilon) - \sigma(u)) \cdot \nabla\eta^s + \int_{B_\varepsilon} (\gamma\sigma(u_\varepsilon) - \sigma(u)) \cdot \nabla\eta^s \\ &\quad + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u) \cdot \eta \,. \end{aligned}$$

After adding and subtracting the term

$$\int_{B_\varepsilon} \gamma\sigma(u) \cdot \nabla\eta^s$$

in the above expression, we have

$$\int_\Omega \sigma_\varepsilon(u_\varepsilon - u) \cdot \nabla\eta^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u) \cdot \eta = \int_{B_\varepsilon} (1-\gamma)\sigma(u) \cdot \nabla\eta^s \,. \tag{25}$$

By taking $\eta = u_\varepsilon - u$ as test function in (25) we obtain the following equality

$$\begin{aligned} \int_\Omega \sigma_\varepsilon(u_\varepsilon - u) \cdot \nabla(u_\varepsilon - u)^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u) \cdot (u_\varepsilon - u) \\ = \int_{B_\varepsilon} \mathrm{T}(u) \cdot \nabla(u_\varepsilon - u)^s \,, \end{aligned} \tag{26}$$

where we have introduced the notation

$$\mathrm{T}(u) = (1-\gamma)\sigma(u)\ . \tag{27}$$

From the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned}\int_{B_\varepsilon} \mathrm{T}(u)\cdot\nabla(u_\varepsilon - u)^s &\leq \|\mathrm{T}(u)\|_{L^2(B_\varepsilon)}\|\nabla(u_\varepsilon - u)\|_{L^2(B_\varepsilon)}\\ &\leq C_0\varepsilon\|\nabla(u_\varepsilon - u)\|_{L^2(B_\varepsilon)}\\ &\leq C_1\varepsilon\|u_\varepsilon - u\|_{H^1(\Omega)}.\end{aligned} \tag{28}$$

By coercivity of the bilinear form on the left-hand side of (29) we have

$$c\|u_\varepsilon - u\|^2_{H^1(\Omega)} \leq \int_\Omega \sigma_\varepsilon(u_\varepsilon - u)\cdot\nabla(u_\varepsilon - u)^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u)\cdot(u_\varepsilon - u), \tag{29}$$

which leads to the result with $C = C_1/c$ independent of the small parameter ε. □

3.3. Topological derivative formula

Since the topological perturbation is nucleated far from the control region ω^* and taking into account the definition of the shape change velocity field V from (12), the Rice's integral (14) becomes concentrated over the fixed domain ω^*. In this particular case, the topological derivative can be adapted from Amstutz (2006). For the general case, associated with singular domain perturbations, which is much more complicated from the mathematical point view, see, for instance, Nazarov, Sokołowski and Specovious-Neugebauer (2010), or Sokołowski and Żochowski (2005). See also Ammari et al. (2002) for the complete topological asymptotic expansion of solutions governed by the elasticity system.

Theorem 1 *The topological derivative of the shape functional $\{-\mathcal{J}(u)\}$, where $\mathcal{J}(u)$ is the Rice's integral given by* (14)*, with respect to the nucleation of a small circular inclusion endowed with contrast γ, can be formulated in terms of the solutions to the direct* (21) *and adjoint* (17) *problems, namely:*

$$\mathcal{T}(x) = \mathbb{P}_\gamma\sigma(u)(x)\cdot\nabla v^s(x), \quad \forall x\in\Omega\ , \tag{30}$$

where the polarization tensor $\mathbb{P}_\gamma$ is given by a fourth order isotropic tensor as follows

$$\mathbb{P}_\gamma = -\frac{1-\gamma}{1+\beta\gamma}\left((1+\beta)\mathbb{I} + \frac{1}{2}(\alpha-\beta)\frac{1-\gamma}{1+\alpha\gamma}\mathrm{I}\otimes\mathrm{I}\right)\ , \tag{31}$$

with the coefficients α and β defined as

$$\alpha = \frac{\mu+\lambda}{\mu} \quad \textit{and} \quad \beta = \frac{3\mu+\lambda}{\mu+\lambda}\ . \tag{32}$$

Corollary 1 *The following limit cases for the contrast parameter γ can be formally obtained from Theorem 1, whose rigorous mathematical justification can be found in Ammari et al. (2013), for instance:*

Case 1. *Contrast parameter going to zero* ($\gamma \to 0$),

$$\mathcal{T}_0(x) = \mathbb{P}_0 \sigma(u)(x) \cdot \nabla v^s(x) \ , \tag{33}$$

where the polarization tensor $\mathbb{P}_0$ *is given by*

$$\mathbb{P}_0 = -\frac{4\mu + 2\lambda}{\mu + \lambda} \left(\mathbb{I} - \frac{\mu - \lambda}{4\mu} \mathrm{I} \otimes \mathrm{I} \right) . \tag{34}$$

Case 2. *Contrast parameter going to infinity* ($\gamma \to \infty$),

$$\mathcal{T}_\infty(x) = \mathbb{P}_\infty \sigma(u)(x) \cdot \nabla v^s(x) \ , \tag{35}$$

with the polarization tensor $\mathbb{P}_\infty$ *given by*

$$\mathbb{P}_\infty = \frac{4\mu + 2\lambda}{3\mu + \lambda} \left(\mathbb{I} + \frac{\mu - \lambda}{4(\mu + \lambda)} \mathrm{I} \otimes \mathrm{I} \right) . \tag{36}$$

4. Bittencourt's experiment

In this section a well known numerical experiment is presented in order to illustrate some preliminary results. As mentioned before, the obtained topological derivatives will be used to indicate the regions, where the controls should be positioned. Then, a combination of such indications is performed in order to verify the effects caused by the topological changes. The mechanical problem is solved by using the Finite Element Method with linear triangular elements only.

This example, called Bittencourt's experiment (see Bittencourt et al., 1996), has been proposed in Ingraffea and Grigoriu (1990). The geometry and boundary conditions can be seen in details in Fig. 4. A concentrated load $q = -(0, 10^4)$ lbf is applied at the middle point of the top face. In particular, we highlight the three holes, located between the load and the initial crack. In addition, the control region ω^* is given by a circle centered at the crack tip with radius $r^* = 0.5$ in. It is assumed that the structure is under plane strain assumption. The remaining parameters are shown in Table 1.

The obtained topological derivatives in the neighborhood of the control region ω^* (see Fig. 4 for details) are presented in Fig. 5. In particular, the limit Cases 1 and 2, according to (33) and (35) in Corollary 1, are presented in Figs 5(a) and 5(b), respectively. Note that, as indicated in Fig. 5(a), two soft inclusions should be nucleated at both sides of the crack tip. Now, taking into account the result shown in Fig. 5(b), a hard inclusion should be nucleated in front of the crack in the direction of the applied load.

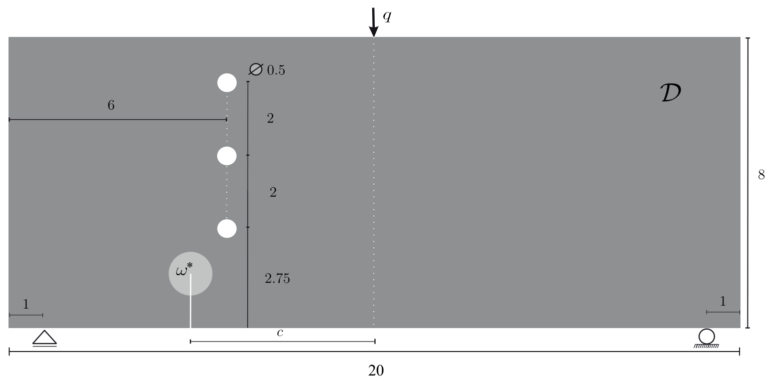


Figure 4. Bittencourt's experiment. Geometry and boundary conditions

Table 1. Bittencourt's experiment. Parameters

Parameter	Value
E	4.5×10^5 psi
ν	0.35
e	$(0, 1)$
h	1.5 in
c	5 in

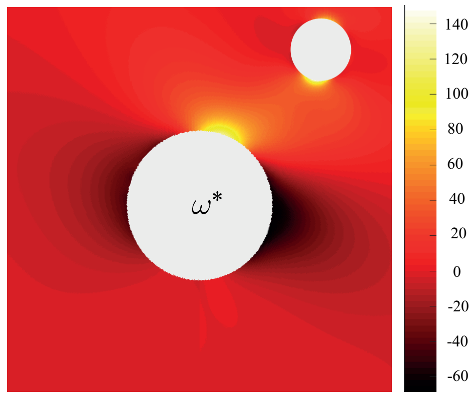
In order to verify the effects, caused by the nucleation of such inclusions, the following four cases are considered. Case A: no inclusions are nucleated; Case B: a hard inclusion is nucleated at the point (5.4125, 2.25) since $\mathcal{T}_\infty(x) < 0$; Case C: two soft inclusions are nucleated at the points (4.125, 1.75) and (5.87, 1.125) since $\mathcal{T}_0(x) < 0$; Case D: the cases B and C are considered simultaneously. In all cases the radius of the inclusion is $r = 0.25$ in, see Fig. 6 for details, where white/black circles represent soft/hard inclusions.

The obtained results are presented in Table 2, and they are also presented in Fig. 7 after normalization with respect to the first obtained value of $-\mathcal{J}(u)$.

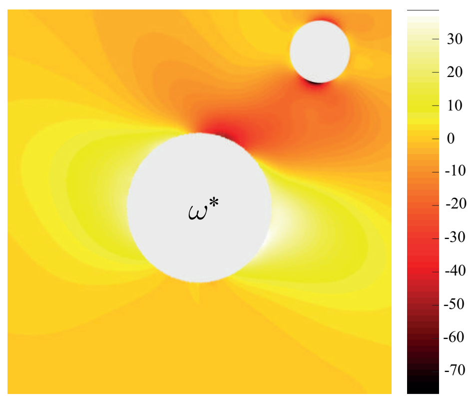
Table 2. Bittencourt's experiment. Obtained results

Cases	A	B	C	D
$-\mathcal{J}(u)$	101.5803	97.291	92.1696	88.3871

Note that the values of the associated shape functional decrease after introducing the topology changes according to the signal of the topological derivative. In the last case, for example, a gain of approximately 13% is observed.



(a) Case 1: $\mathcal{T}_0(x)$



(b) Case 2: $\mathcal{T}_\infty(x)$

Figure 5. Bittencourt's experiment. Topological derivatives in the neighborhood of the control region ω^*, centered at the crack tip (see Figure 4 for details)

5. Conclusions

In this paper, an extension of the methodology proposed in Xavier, Novotny and Sokołowski (2018) to deal with crack growth control problems for the non-linear case by considering contact type boundary conditions on the crack lips, is proposed. The main idea consists in minimizing a shape functional defined in terms of the Rice's integral, by nucleating hard and/or soft inclusions far from the crack tip, according to the information provided by the topological derivative. In particular, the Domain Decomposition Technique, combined with the Steklov-Poincaré pseudo-differential boundary operator, is used to obtain the sensitivity of the associated shape functional with respect to the nucleation of a small circular inclusion with different material property from the background. Then, the resulting expression is used to indicate the regions, where

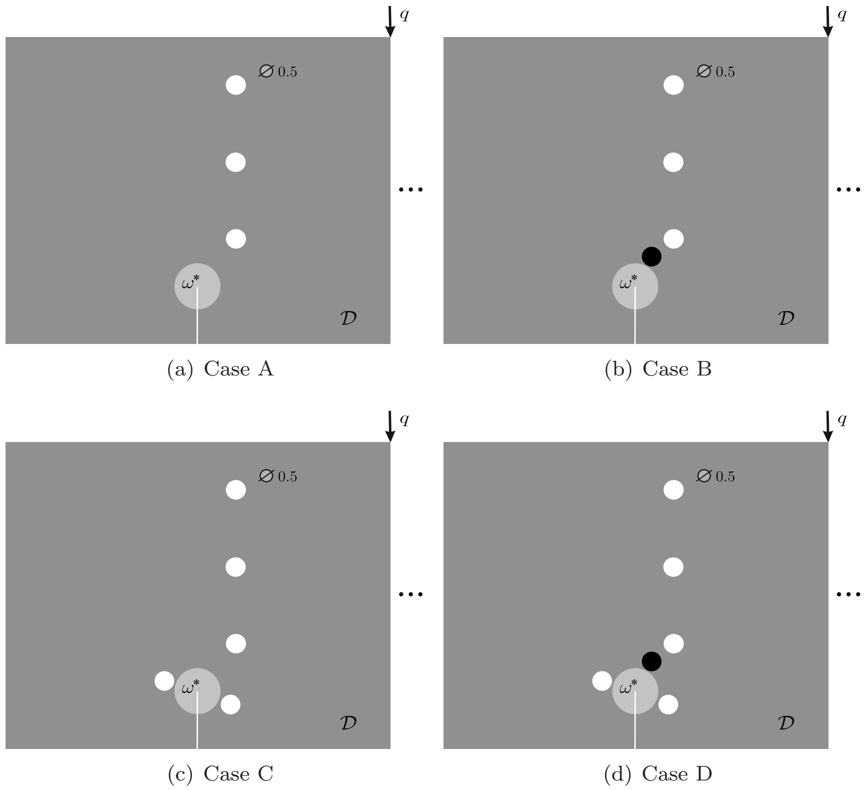


Figure 6. Cases A, B, C, and D

the controls (inclusions) should be positioned (nucleated) in order to solve the minimization problem. According to the Griffith's energy criterion, this procedure allows for an increase of the fracture toughness of the cracked component. The well known Bittencourt's experiment is presented to illustrate the applicability of the method in the case of pure traction. In fact, this example shows that a gain of 13% in the fracture toughness of the mechanical component can be obtained by applying the proposed method. Finally, it should be emphasized that the numerical example can be seen as a preliminary result only. Actually, further studies, related to the implementation of the numerical treatment of the problems with the non-interpenetration conditions, are now being carried out.

In addition, the numerical experiments only show a tendency concerning the behavior of the shape function after nucleation of inclusions according to the value of the topological derivative. Solving a topology optimization problem in the strict sense by using the derived theoretical results is the subject of future work.

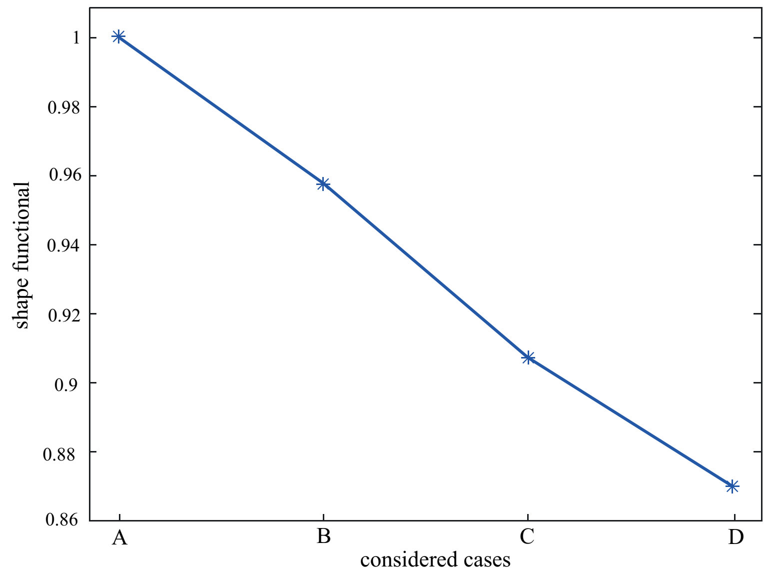


Figure 7. Bittencourt's experiment. Obtained results

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